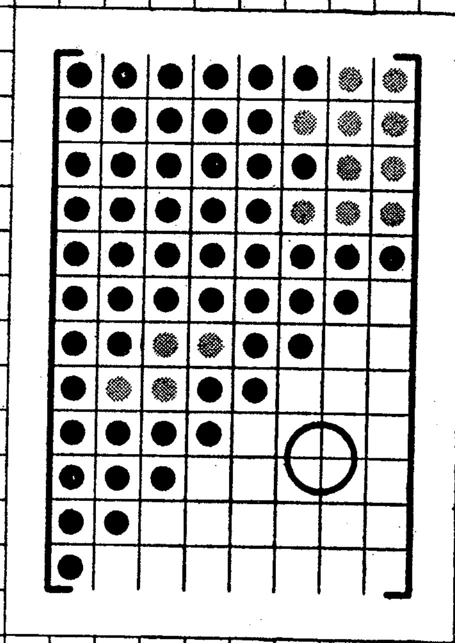
ADVANCED STRUCTURAL ANALYSIS
WITH COMPUTER APPLICATIONS

ASHOK K. JAIN



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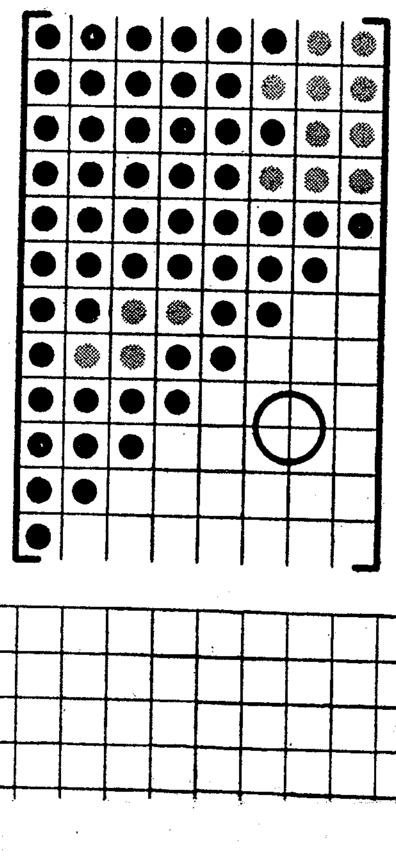

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DEDICATED TO MY MOTHER

of the two programs are provided so that the reader may have a direct exposure to the computer applications and develops confidence. Earlier it was planned that the source listing of the two programs STAP-3D and CABLE-3D will be included in the book, but later it was decided to make it available on a floppy so that the reader does not have to go through the drudgery of feeding into a computer and then checking the same till full confidence is developed. Those who have gone through this exercise will appreciate this decision. The floppy is available with the publishers of this book at a nominal price.

Although written mainly for the undergraduate students, the practicing engineers will find it equally useful. The intricacies of structural analysis have been explained with the help of over one hundred and fifty solved examples. Over one hundred and fifty problems are included at the end of the chapters and answers of most of them are given at the end of each chapter.

I am grateful to my students who were taught from the manuscript over the past fifteen years for inadvertently providing helpful comments, for including typical examples and improving its contents. The critical comments offered by my colleagues have been utilized in the final text. Dr. P.N. Godbole deserves a special mention who gladly provided a copy of his cable program for use in this book. Thanks to Shri I.P.S.Verma for meticulously and neatly drawing the tracings. I am especially grateful to my wife Sarita and children Payal and Gaurav for their patience and encouragement in writing another last book.

December 1, 1995

Ashok K. Jain

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Preface

The aim of this book is to present up-to-date methodologies in the analysis of statically indeterminate structures. This book is a companion of my earlier book entitled *Elementary Structural Analysis* which deals with statically determinate structures. Thus this set of two books completes the wide spectrum of structural analysis. The methods of structural analyses are classified into the flexibility methods and stiffness methods. In the present book, both the classical methods and matrix based methods are discussed in detail. The attention is devoted entirely to develop understanding of the behaviour of statically indeterminate structures. There is strong emphasis throughout the text on the use of computers.

Each chapter begins with the introduction and develops the algorithms along with suitable sign convention which is peculiar to each classical method. The examples are chosen and solutions arranged so that the fine points of structural analysis are clearly brought out. They serve to amplify and supplement the theory.

Chapter 1 provides an overview of basic concepts for the analysis of statically indeterminate structures. Chapter 2 deals with the elements of matrix algebra. Now onwards the book is divided in two parts : Flexibility methods and stiffness methods. Part 1 covers the method of consistent deformations, three moment equation, strain energy method, column analogy method, influence coefficient method , influence lines and arches. The last two chapters do not form part of the flexibility methods but are included here for the sake of completeness. Part 2 covers the slope deflection method, moment distribution method, direct stiffness method : 2-D elements, and 3-D elements , and salient features of STructural Analysis Program STAP-3D. A key feature of this book is a comprehensive treatment of non-linear analysis of structures covering theory of plastic analysis, material as well as geometric non-linear problems. The concepts of hysteresis models, unbalanced load vector, updating of stiffness matrix, ductility and incremental and iterative methods of solution are introduced. The sequence of formation of plastic hinges is explained through examples. The salient features of CABLE-3D program are also discussed. Several listings of sample input data and output

CHAPTER
one

BASIC CONCEPTS

1.1 INTRODUCTION

A framed structure is a network of a number of units known as members or elements. A floor may consist of beams and slabs. A building may consist of beams, columns, floors and foundations. Similarly a roof truss is made up of top chords, bottom chords, diagonal members, purlins, ties and cover sheets. A bridge may consist of longitudinal beams, transverse beams, deck slabs and even cables. Such components are referred to as members or elements.

A beam or a column element, and a truss or a cable element are most frequently employed in framed structures. These are one dimensional (1-D) elements since their cross-sectional dimensions are very small as compared to their lengths. The scope of this book is restricted to the study of various analytical methods employed for the analysis of various 2-D and 3-D framed structures consisting of these 1-D elements. There are three basic requirements for a unique structural analysis :

1. stress-strain relationship of the materials in the structure,
2. equations of static equilibrium, and
3. conditions of compatibility or kinematics.

These were discussed in detail in chapter 2 of volume 1 of this book.

1.2 STATICALLY DETERMINATE VS. INDETERMINATE STRUCTURES

A single span beam or a continuous beam is a 1-D structure carrying transverse loads. It is completely analyzed when the shear force and bending moment diagrams are drawn, and slopes and deflections are known. A rigid frame consists of flexural members (beams and columns) connected by rigid or semi - rigid joints. It is completely

consists of only axial members, that is, pin-connected bars. A structure whether a truss, a continuous beam or a rigid frame is either stable or unstable, and either statically determinate or statically indeterminate depending upon the number and arrangement of members, internal joints and external supports. A structure may be externally indeterminate, or internally indeterminate or both. Idealization of internal joints and external supports, determinacy and stability of structures were discussed in volume 1 of this book. Figure 1.1 shows typical statically determinate beams while Fig.1.2 shows typical statically indeterminate beams.

Similarly, typical statically determinate and statically indeterminate frames are shown in Figs. 1.3 and 1.4 respectively; and typical statically determinate and indeterminate trusses are shown in Figs. 1.5a and 1.5b respectively. The statically determinate structures can be fully analyzed by using the equations of static equilibrium:

$$\Sigma F_x = 0,\ \Sigma F_y = 0,\ \Sigma M_z = 0 \qquad \text{(2-D structure)} \qquad (1.1a)$$

$$\Sigma F_x = 0,\ \Sigma F_y = 0,\ \Sigma F_z = 0,\ \Sigma M_x = 0,\ \Sigma M_y = 0,\ \Sigma M_z = 0 \qquad \text{(3-D structure)} \qquad (1.1b)$$

The statically indeterminate structures cannot be analyzed directly and need additional equations due to conditions of compatibility.

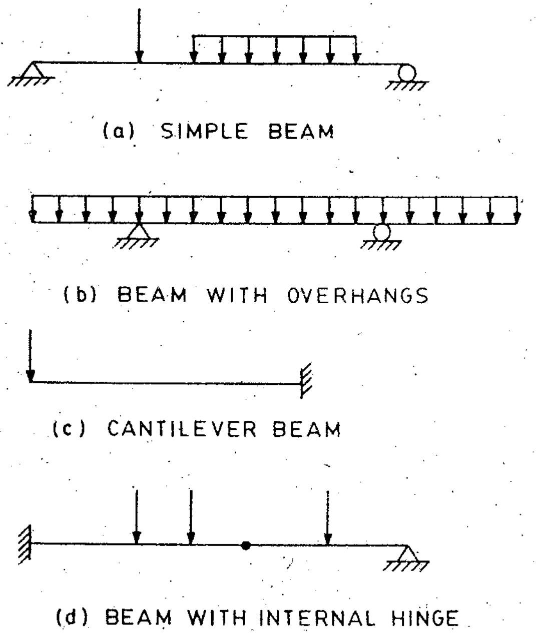


Fig. 1. 1 Statically determinate beams

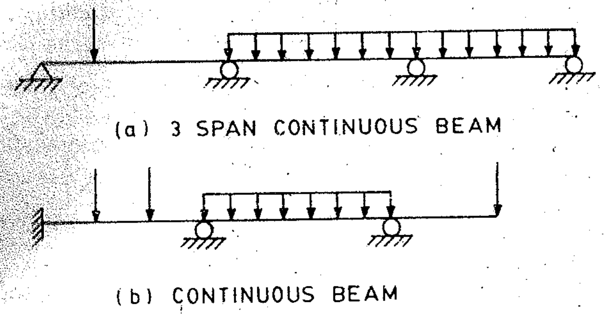


Fig. 1.2 Statically indeterminate beams

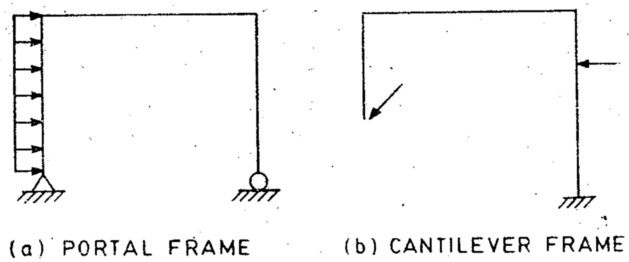


Fig. 1.3 Statically determinate frames

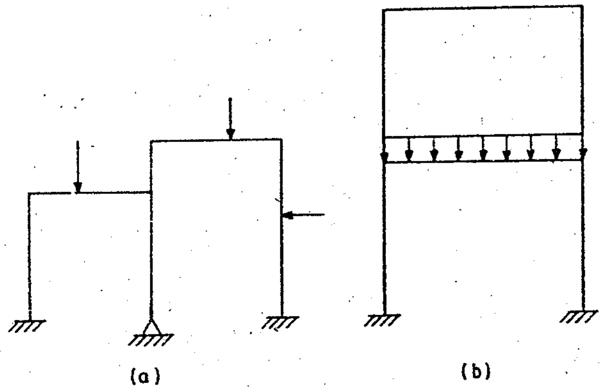


Fig. 1.4 Statically indeterminate frames

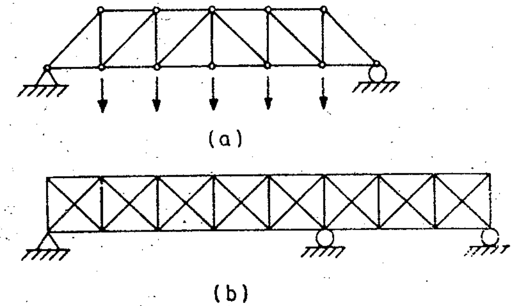


Fig. 1.5 (a) Statically determinate truss (b) Statically indeterminate truss

In practice statically indeterminate structures are frequently encountered because of better strength, stiffness and economy. Therefore, it is essential to study the development of various methods of analyzing such structures. One of the important conditions for the analysis of structures is that the principle of superposition is valid. It means that the structure is linear and deflections are small. The stresses in the structure are below the proportional limit. The first fourteen chapters are devoted to the analysis of linear statically indeterminate structures. Practical considerations require that there is considerable strength in the nonlinear region of a structure which should be fully exploited. The non-linearity may be due to material or geometry. Material nonlinear analysis is discussed in chapter 15, while geometrical nonlinear analysis is discussed in chapter 16.

1.3 FLEXIBILITY METHOD

The most powerful method for the analysis of statically indeterminate structures is the *method of consistent deformations*. It is also called as the *flexibility method* or the *force method* or the *compatibility method*. The application of the flexibility method requires that first the structure be reduced to a stable, statically determinate system. The problem then reduces to establishing a set of independent, simultaneous equations in terms of unknown forces or *redundant actions* so that they may be evaluated and analysis of the indeterminate structure completed.

When the statically determinate or the *released structure* is subjected to the applied loads, it will undergo deformations that are inconsistent with the behaviour of the original structure. However, by applying forces equal to the released actions, the deformations of the released structure are made consistent with those of the original structure. In analyzing the released structure, the displacement at the point of application and along the line of action of each redundant must be evaluated. Although the magnitude and direction of the redundant actions are unknown at this stage, the released structure can be analyzed due to the application of an assumed unit value of each redundant, successively. Finally, considering displacement of released structure at the point of application of each redundant action, due to both the applied loads and the individual unit values of the redundants, a set of compatibility equations can be written. These equations describe the actual unknown displacement of the original structure at

each point of the application of the redundant force. The solution of these linear simultaneous equations provide the magnitude and direction of each redundant necessary to maintain compatibility of the system. Finally, the member end forces, end displacements and support reactions can be determined.

Consider a three span continuous beam shown in Fig. 1.6a. The total number of reactions is six. Therefore, the degree of statical indeterminacy is three. The structure can be made statically determinate in several ways:

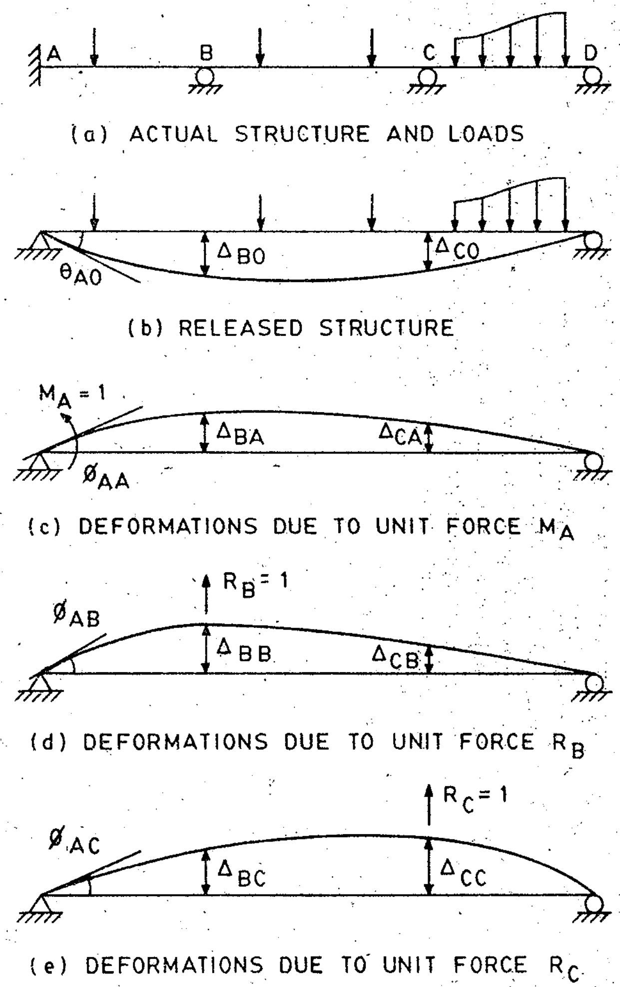


Fig. 1.6 Development of flexibility method

1. By removing the supports at B, C and D, it reduces to a determinate cantilever beam.
2. By removing the support A, it reduces to an unstable continuous beam, hence it is not a feasible solution.
3. By removing the restraint at A, and the supports B and C, it reduces to a simply supported and stable beam.

Let us adopt the third option as shown in Fig. 1.6b. The simply supported beam undergoes rotation θ_{AO} at A, and deflections Δ_{BO} and Δ_{CO} at B and C, respectively. There are three unknown actions or redundants, M_A, R_B and R_C. Their magnitudes and directions are not known. Let us assume that the moment M_A is anti-clockwise and the reactions R_B and R_C are acting upward. Let us apply unit redundants on the structure, one by one, and determine the deformations at the points of application and direction of each redundant as shown in Figs. 1.6c, 1.6d and 1.6e. The slopes and deflections can be determined by using any one of the several methods (viz. moment-area method, conjugate beam method, or unit load method) as discussed in volume 1 of this book. The number of equations is equal to the number of redundants.

The compatibility condition requires that

$$\theta_A = 0,\ \Delta_B = 0, \text{ and } \Delta_C = 0 \tag{1.2}$$

that is,

$$\begin{aligned}
-\theta_{AO} + \phi_{AA} M_A + \phi_{AB} R_B + \phi_{AC} R_C &= \theta_A = 0 \\
-\Delta_{BO} + \Delta_{BA} M_A + \Delta_{BB} R_B + \Delta_{BC} R_C &= \Delta_B = 0 \\
\text{and} \quad -\Delta_{CO} + \Delta_{CA} M_A + \Delta_{CB} R_B + \Delta_{CC} R_C &= \Delta_C = 0
\end{aligned} \tag{1.3}$$

in matrix notation

$$(-)\begin{Bmatrix} \theta_{AO} \\ \Delta_{BO} \\ \Delta_{CO} \end{Bmatrix} + \begin{bmatrix} \phi_{AA} & \phi_{AB} & \phi_{AC} \\ \Delta_{BA} & \Delta_{BB} & \Delta_{BC} \\ \Delta_{CA} & \Delta_{CB} & \Delta_{CC} \end{bmatrix} \begin{Bmatrix} M_A \\ R_B \\ R_C \end{Bmatrix} = \begin{Bmatrix} \theta_A \\ \Delta_B \\ \Delta_C \end{Bmatrix} = 0 \tag{1.4}$$

or

$$-\{\Delta_{ro}\} + [F]\{R\} = \{\Delta_r\}$$

or

$$-\mathbf{\Delta_{ro}} + \mathbf{F\,R} = \mathbf{\Delta_r} \tag{1.5}$$

where $\{\Delta_{ro}\}$ = deformation vector due to the actual applied loads; the first subscript 'r' represents the location of the redundant where the displacement is computed and the second subscript 'o' represents the applied loads.

[F] = the elements f_{ij} of this matrix represent the displacement Δ_i in the direction and at the location of the redundant R_i caused by a unit action redundant R_j acting at the location j. This is called *flexibility matrix.*

{R} = redundant force vector

$\{\Delta_r\}$ = total known displacements at the location and direction of the redundants in the actual structure. Some of these displacements may not be zero as in the case of settlement of supports.

In case vector $\{\Delta_r\}$ is zero, Eq. 1.5 can be written as:

$$[F]\{R\} = \{\Delta_{ro}\} \tag{1.6a}$$

or

$$\mathbf{F\,R} = \mathbf{\Delta} \tag{1.6b}$$

This equation represents the deformation - force relationship. For a single element, the term **F** represents *deformation per unit force* and is called as *flexibility.* Knowing Δ and **F**, **R** can be evaluated which leads to the solution of the complete structure.

The method of consistent deformation is one of the earliest methods in vogue (Circa 1860). The other methods that can fall in this category are three moment equation, strain energy method, column analogy method and influence coefficient method. These will be discussed in detail in the subsequent chapters.

1.4 STIFFNESS METHOD

Another approach for the analysis of statically indeterminate structures is the *stiffness method.* It is also called as the *displacement method* or the *equilibrium method.* Let us reconsider the three span continuous beam which was used to develop the flexibility method. The application of the stiffness method requires that a statically indeterminate structure be first reduced to a *kinematically determinate system.* A kinematically determinate system is the one whose end displacements are known. *Kinematics* relates the deformations and displacements of elements. The first step is to identify the degree of kinemetic indeterminacy and, therefore, the unrestrained joint displacements. Now, the corresponding artificial restraints must be introduced so as to make it a kinematically determinate structure, that is, a *restrained structure.*

This structure must now be analyzed for the actual loading imposed on the original structure. The support reactions of the restrained structure for any loading condition are simply the action required to constrain the various joint displacements. At each joint, each restraining reaction is equal to the sum of

(a) the fixed end forces required to constrain the ends of the members that frame into that joint, and

(b) the action equal and opposite to the force acting directly at the joint.

The restrained structure also needs to be analyzed for displacements of the artificially restrained joints. But magnitude and direction of displacements of the unrestrained joints are unknown. It is, therefore, convenient to assume unit value of each of the artificially restrained joint displacements and analyze the restrained structure for each displacement individually. Since, the artificial restraining support actions do not exist in the original structure, a set of independent linear simultaneous equations can be written in terms of unknown joint displacements. The number of equations is equal to the number of constraints required to make the displacements zero. The solution of this system of equations gives the magnitude and direction of the joint displacements necessary to maintain equilibrium of the system. Finally, member end forces and support reactions can be determined.

Consider the three span continuous beam shown in Fig. 1.7a. The unrestrained displacements are shown in Fig. 1.7b and the corresponding restrained structure is shown in Fig. 1.7c. The unrestrained displacements are θ_B, θ_C and θ_D. The fixed end actions due to the applied loading on the original structure are shown in Fig. 1.7d. These are required to constrain the joint displacements, and the net support reactions are shown in Fig. 1.7e.

Now remove the applied loading and impose unit joint displacement, successively, and evaluate the support reactions as shown in Figs. 1.7f, 1.7g and 1.7h. K_{ij} represents moment at joint i due to a unit displacement imposed at joint j. Thus, K_{ij} is force per unit displacement and is called stiffness or *stiffness coefficient*. Similarly, R_{ij} represents vertical reaction at joint i due to unit displacement imposed at joint j. The equilibrium equation can be written at each joint,
that is,

$$\Sigma M_B = 0, \quad \Sigma M_C = 0 \quad \text{and} \quad \Sigma M_D = 0 \tag{1.7}$$

or,

$$\begin{aligned} M_{BO} + K_{BB}\,\theta_B + K_{BC}\,\theta_C + K_{BD}\,\theta_D &= 0 \\ M_{CO} + K_{CB}\,\theta_B + K_{CC}\,\theta_C + K_{CD}\,\theta_D &= 0 \\ \text{and} \quad M_{DO} + K_{DB}\,\theta_B + K_{DC}\,\theta_C + K_{DD}\,\theta_D &= 0 \end{aligned} \tag{1.8}$$

in matrix notation

$$\begin{Bmatrix} M_{BO} \\ M_{CO} \\ M_{DO} \end{Bmatrix} + \begin{bmatrix} K_{BB} & K_{BC} & K_{BD} \\ K_{CB} & K_{CC} & K_{CD} \\ K_{DB} & K_{DC} & K_{DD} \end{bmatrix} \begin{Bmatrix} \theta_B \\ \theta_C \\ \theta_D \end{Bmatrix} = 0 \tag{1.9}$$

or

$$\{P\} + [K]\,\{\Delta'\} = \{0\} \tag{1.10a}$$

or, $\{P\} = [K]\,\{\Delta\}$ if $\Delta = -\Delta'$ for convenience

or,

$$\mathbf{P} = \mathbf{K}\,\Delta \tag{1.10b}$$

where, $\{P\}$ = equivalent nodal load vector due to the actual applied loads; the first subscript represents location of the joint and second subscript 'o' represents the applied loads.

$[K]$ = elements of this matrix k_{ij} represent the force M_i at joint i in the direction of the constraint due to a unit displacement applied at joint j. This is called *stiffness matrix*.

$\{\Delta\}$ = unknown displacement vector

The equation $\mathbf{P} = \mathbf{K}\Delta$ represents the force-deformation relationship. For a single element the term $\mathbf{K}$ represents *force per unit deformation* and is called as *stiffness*. Knowing $\mathbf{P}$ and $\mathbf{K}$, Δ can be evaluated which leads to solution of the complete structure.

The slope-deflection method and moment distribution method are in vogue since 1915 and 1930, respectively. The other method that falls in this category is the direct stiffness method. A relatively recent method, it has revolutionized the concept of

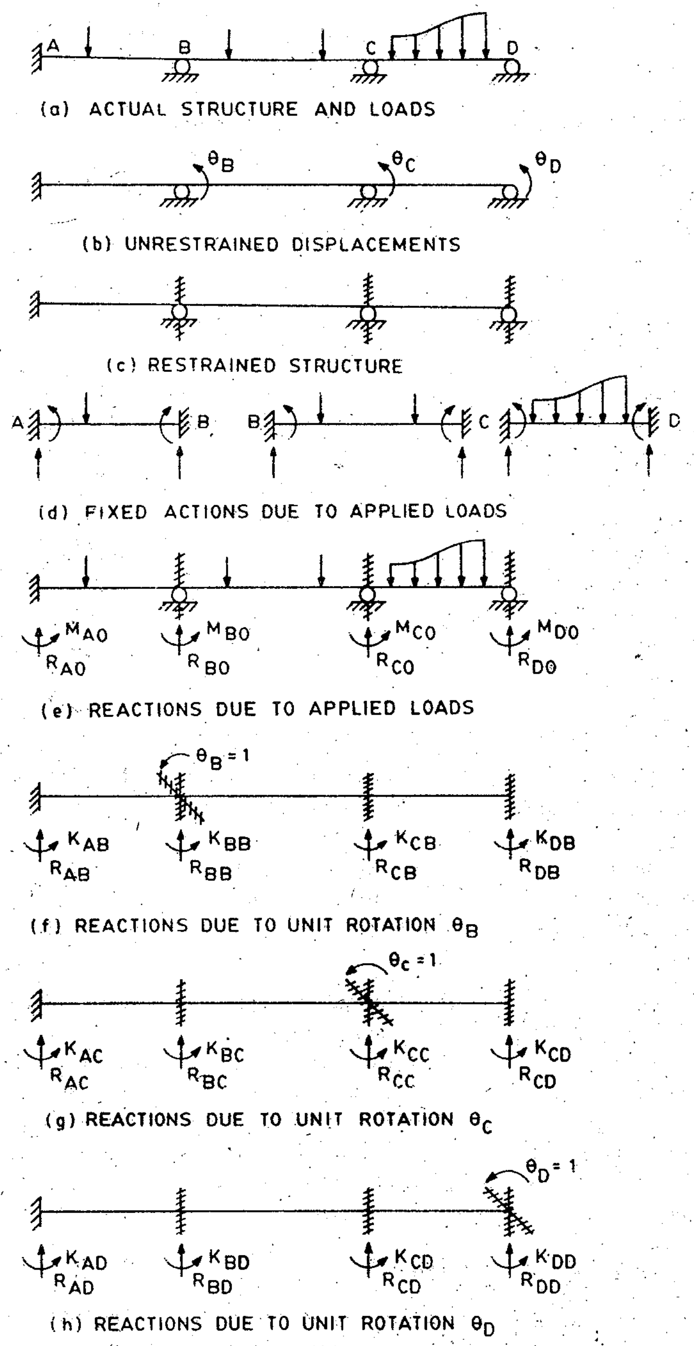


Fig. 1.7 Development of stiffness method

structural analysis due to the simultaneous development of powerful electronic computers. These methods will be discussed in detail in the subsequent chapters.

1.5 SYSTEM APPROACH VS. ELEMENT APPROACH

Various methods of structural analysis were developed and perfected over a number of years during the mid-nineteenth century. There were no computers at that time, therefore, methods of analysis had to be based on the physical reasoning of the structure. Although the concept of matrix algebra was very well known, but it could not attract the attention of the analysts for obvious reasons. Methods such as the consistent deformation method, strain energy method, slope-deflection method and moment distribution method considered the structure as a whole.

With the advent of digital computers around the middle of twentieth century, matrix approach became attractive and the stiffness methods and flexibility methods were developed. The **F** and **K** matrices for the entire structure were developed through the respective matrices for the constitutive elements. This approach came to be known as *element approach*. It is more appropriate for an automatic analysis by a digital computer.

Having understood the concept of flexibility and stiffness methods, the analysts took another look at the conventional methods. It was a pleasant surprise to discover that the earlier methods could be classified either as a flexibility or as a stiffness method. Since they considered the structure as a whole, this approach was called as *system approach*. Thus the classification of system or element approach is a relatively new term and quite convenient.

The flexibility method is also known as the force method because the forces are treated as unknowns. Since the condition of compatibility of displacements is imposed to generate the final equation, it is also known as the compatibility method. Similarly, the stiffness method is also known as the displacement method because the displacements are treated as unknowns. Since the condition of equilibrium of forces is imposed to generate the final equation, it is also known as the equilibrium method. It is necessary to understand that both these approaches make use of equilibrium as well as compatibility conditions.

It is convenient to write a general purpose computer programme using the stiffness method for static or dynamic analysis of any framed structure. It is also very efficient for the non-linear analysis of such structures. It is referred to as the *user friendly method*. The flexibility method cannot be generalised due to several possibilities of selecting proper redundants and the associated difficulties in writing simple and straight forward algorithms. Hence, it is not much in use. Nevertheless, the classical flexibility methods provide a deep insight into the physical understanding of the structural behaviour and are necessary to study.

1.6 CHOICE OF A METHOD

A structure can be analyzed using the force approach or the displacement approach. The question is how to decide which method is better. As discussed earlier, the number

of equations in the force method is equal to the number of redundants, whereas, the number of equations in the displacement method is equal to the number of constraints required to make the displacements zero. In other words, size of the flexibility matrix is equal to the degree of statical indeterminacy (α_S), whereas, size of the stiffness matrix is equal to the degree of kinematic indeterminacy (α_k). Obviously, a method which leads to a fewer number of equations is preferable. Thus, first step is to determine the degree of statical indeterminacy as well as the degree of kinematic indeterminacy and the method of analysis can be selected accordingly keeping in view the convenience involved. It is also possible to adopt a *mixed approach* in certain types of problems. The mixed approach is beyond the scope of the present text book.

1.7 DEGREE OF STATIC INDETERMINACY

If all unknown reactions in a structure can be uniquely determined with the help of equations of static equilibrium, the reactions of the structure are referred to as statically determinate. Otherwise, the reactions of the structure are referred to as statically indeterminate. The degree of indeterminacy is equal to the number by which the unknowns exceed the available equations of statics. The total degree of static indeterminacy of a structure is considered as sum of the following two types of indeterminacies:

(a) degree of external indeterminacy

and (b) degree of internal indeterminacy

The external indeterminacy is related to the number and type of supports. The internal indeterminacy is concerned with the determination of all member forces knowing the support reactions. In a pin-jointed truss, if the number of members meeting at a joint is just sufficient to preserve its geometry, the truss is *internally determinate*, otherwise the truss is *internally indeterminate*. Similarly, a rigid- jointed frame is internally determinate if its members form an open configuration, that is, there are no closed cells, otherwise the rigid-jointed frame is internally indeterminate. Alternatively, if more member actions are present than can be solved for from statics , the frame is *internally indeterminate*.

For a 2 - D pin-jointed truss,

$$\alpha_S = (m + r) - 2j \tag{1.11a}$$

For a 3 - D pin-jointed truss,

$$\alpha_S = (m + r) - 3j \tag{1.11b}$$

For a 2 - D rigid-jointed frame,

$$\alpha_S = (3m + r) - 3j \tag{1.12a}$$

For a 3 - D rigid-jointed frame,

$$\alpha_S = (6m + r) - 6j \tag{1.12b}$$

The total static indeterminacy is given by the above relations. The degree of external indeterminacy is easy to calculate. The total number of external reactions minus the number of available equation of static equilibrium gives the degree of external indeterminacy. Thus, the degree of internal indeterminacy can be computed. Often a structure is said simply to be statically indeterminate without stating whether it is indeterminate internally, externally or in both manners.

1.8 DEGREE OF KINEMATIC INDETERMINACY

When determining deformations, there are no equations which are analogous to the equations of static equilibrium. Therefore, all structures, with certain exceptions, are *kinematically indeterminate*. If deformations of the ends of a member are known, then the deformation at any other point in the member can be easily determined. A fixed-end beam is kinematically determinate and a simple supported beam is kinematically indeterminate. The degree of kinematic indeterminacy of a structure is equal to the number of independent displacement components. In a pin-jointed truss, the member of independent translations at each joint is 2 or 3 depending upon whether it is a plane truss or a space truss. In a rigid-jointed frame, the number of independent joint displacements is 3 or 6 depending upon whether it is a plane frame or a space frame. The number of independent joint displacements is also called as the *degree of freedom* of the joint or of the node. If the number of fully restraint support displacements is S_1 and known support displacements is S_2 the degree of kinematic indeterminacy can be computed by the following equations:

For a 2 - D pin-jointed truss,

$$\alpha_k = 2j - S_1 - S_2 \tag{1.13a}$$

For a 3 - D pin-jointed truss,

$$\alpha_k = 3j - S_1 - S_2 \tag{1.13b}$$

For a 2 - D rigid-jointed frame,

$$\alpha_k = 3j - S_1 - S_2 \tag{1.14a}$$

For a 3 - D rigid-jointed frame,

$$\alpha_k = 6j - S_1 - S_2 \tag{1.14b}$$

In most practical situations, the supports are unyielding and, hence, $S_2 = 0$.

Under certain conditions, in rigid-jointed plane as well as space frames, it is possible to reduce the degree of kinematic indeterminacy by ignoring the axial deformations. Thus, computations in the classical methods of structural analysis were simplified to a great extent.

The number of independent joint translation in a plane frame is given by the equation:

$$\alpha_{ku} = 2j - (2S_f + 2S_h + S_r + m) \tag{1.15}$$

where,
S_f = number of fixed supports
S_h = number of hinged supports
S_r = number of roller supports
m = number of members

If the number of joint rotations is $\alpha_{k\theta}$ the total degree of kinematic indeterminacy of a plane rigid-jointed frame is given by

$$\alpha_k = \alpha_{ku} + \alpha_{k\theta} \tag{1.16}$$

1.9 ILLUSTRATIVE EXAMPLES

The following examples illustrate the concept of static and kinematic indeterminacy, and the applications of stiffness and flexibility methods in simple cases.

Example 1.1

Determine the degree of static indeterminacy of plane frames shown in Fig. 1.4, and the pin-jointed trusses shown in Fig. 1.5.

Solution

Plane frame (Fig. 1.4a)

Total number of independent external reactions $r = 3 + 3 + 2 = 8$
number of joints $j = 7$, number of members $m = 6$
using Eq. 1.12a,

$$\text{Total statical indeterminacy} \quad \alpha_s = (3m + r) - 3j = 3 \times 6 + 8 - 3 \times 7 = 5$$

Degree of external indeterminacy $= r - 3 = 3 + 3 + 2 - 3 = 5$
$\therefore$ Degree of internal indeterminacy $= \alpha_s - 5 = 5 - 5 = 0$

Plane frame (Fig. 1.4b)

Total number of external reactions $r = 2 \times 3 = 6$
Total number of members $m = 6$
Total number of joints $j = 6$
Hence, $\alpha_s = 3 \times 6 + 6 - 3 \times 6 = 6$

Degree of external indeterminacy $= r - 3 = 2 \times 3 - 3 = 3$
$\therefore$ Degree of internal indeterminacy $= \alpha_s - 3 = 6 - 3 = 3$

Plane truss (Fig. 1.5a)

Total members $m = 21$, joints $j = 12$, reactions $r = 3$
Eq. 1.11a gives,

$$\alpha_s = (m + r) - 2j = 21 + 3 - 2 \times 12 = 0$$

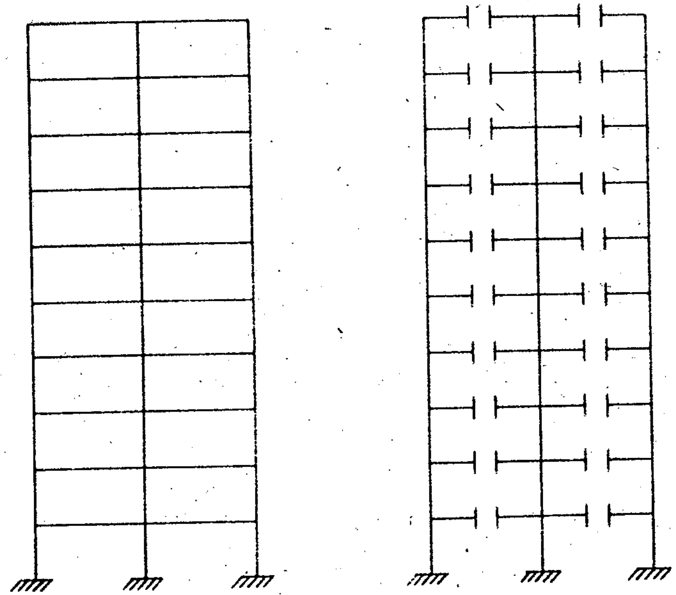
The truss is statically determinate.

Plane truss (Fig. 1.5b)

Total members $m = 41$, joints $j = 18$, reactions $r = 4$
$\therefore$ Total statical indeterminacy $\alpha_s = 41 + 4 - 2 \times 18 = 9$
Degree of external indeterminacy $= r - 3 = 2 + 1 + 1 - 3 = 1$
$\therefore$ Degree of internal indeterminacy $= 9 - 1 = 8$

Example 1.2

Determine the degrees of static and kinematic indeterminacies for the multistoreyed frame shown in Fig. 1.8a. Assume that axial effects, that is, changes in member lengths may be ignored.



(a) (b)

Fig. 1.8

Solution

Static indeterminacy

Total number of external reactions $= 3 \times 3 = 9$

$\therefore$ Degree of external static indeterminacy $= r - 3 = 9 - 3 = 6$

If a structure is cut at any point, three forces or actions must be applied to the left end of the cut and three equal and opposite member forces must be applied at the opposite end of the cut to maintain continuity. These three forces are : an axial force, a shear force and a bending moment.

This is a convenient method to determine the degree of static indeterminacy. Simply introduce as many cuts as necessary to reduce a structure to be statically determinate but ensure that each cut portion is stable.

In the present frame, introduce a cut in each beam. The sturcture is reduced to three independent vertical cantilevers. There are 20 cuts, and 3 unknowns per cut. Thus, total degree of static indeterminacy is $20 \times 3 = 60$. Therefore, the degree of internal static indeterminacy is $60 - 6 = 54$.

The total degree of indeterminacy can also be found using Eq. 1.12a.

Total members $m = 50$, Total joints $j = 33$, Total reactions $r = 9$

$$\therefore \quad \alpha_s = (3m + r) - 3j = 3 \times 50 + 9 - 3 \times 33 = 60 \qquad \text{O.K.}$$

Kinematic indeterminacy

Total number of joints $= 33$

Degree of freedom per joint $= 3$

Known zero support displacements $S_f = 3 \times 3 = 9$

$\therefore$ Degree of kinematic indeterminacy $\alpha_k = 3 \times 33 - 9 = 90$

If axial deformation is ignored,

Lateral displacement at each floor $= 1$

$\therefore$ Total lateral displacements $= 10 \times 1 = 10$

Vertical displacement at each joint $= 0$

Rotation at each joint $= 1$

$\therefore$ Total rotations in the frame $= 30 \times 1 = 30$

$\therefore$ Total unknown displacements in the frame $= 10 + 30 = 40$

or Degree of kinematic indeterminacy $\alpha_k = 40$

Thus, the degree of kinematic indeterminacy α_k is reduced by 50 by neglecting the axial deformations in the members.

Alternatively,

Kinematic indeterminacy can also be determined using Eq.1.16.

$j = 33, \quad S_f = 3, \quad S_h = 0, \quad S_r = 0, \quad m = 50$

$\therefore \quad \alpha_{ku} = 2 \times 33 - (2 \times 3 + 0 + 0 + 50) = 10$

$\alpha_{k\theta} = 30$

$\therefore \quad \alpha_k = \alpha_{ku} + \alpha_{k\theta} = 10 + 30 = 40$ O. K.

Example 1.3

Two springs are connected in series and are subjected to an axial load P_o as in Fig.1.9a. Determine the elongation of each spring and their equivalent stiffness.

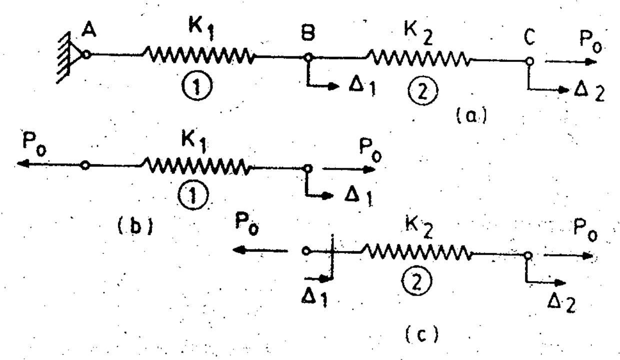


Fig. 1.9

Solution

Let point B elongates by Δ_1 and point C elongates by Δ_2. The free body diagrams of the two springs are shown in Figs 1.9b and 1.9c. The force in each spring remains P_0. Let us write the force-deformation relations:

Force = stiffness × displacement (i)

for spring 1, $P_0 = K_1 \Delta_1$

or $\Delta_1 = \dfrac{P_0}{K_1}$

if $P_0 = 10$ kN, $K_1 = 100$ kN/m and $K_2 = 150$ kN/m

$$\Delta_1 = \frac{10}{100} = 0.1\,\text{m}$$

for spring 2,

The net elongation in spring 2 is $(\Delta_2 - \Delta_1)$,

$$\therefore \quad P_0 = K_2(\Delta_2 - \Delta_1) \tag{ii}$$

or $\Delta_2 = \dfrac{P_0}{K_2} + \Delta_1$

or $\Delta_2 = \dfrac{10}{150} + 0.1 = 0.167\,\text{m}$

To obtain the equivalent spring stiffness, let us rearrange Eqs. (i) and (ii),

$$\Delta_2 = \frac{P_0}{K_1} + \frac{P_0}{K_2}, \quad \text{and} \quad \Delta_2 = \frac{P_0}{K_{eq}}$$

$$\therefore \quad \frac{P_0}{K_{eq}} = \frac{P_0}{K_1} + \frac{P_0}{K_2}$$

$$\text{or,} \quad K_{eq} = \frac{K_1 K_2}{K_1 + K_2} \tag{iii}$$

Thus, springs 1 and 2 can be replaced with a single spring, whose stiffness is given by Eq.(iii), which will give the same displacement.

Example 1.4

A two springs system shown in Fig. 1.10a is subjected to an axial load P_0. Determine the forces produced in the two springs and displacement of joint B.

Solution

Let forces produced in the two springs be F_1 and F_2. The free body diagrams of the springs are shown in Figs.1.10b and c. The force-deformation relations can be written for the two springs:

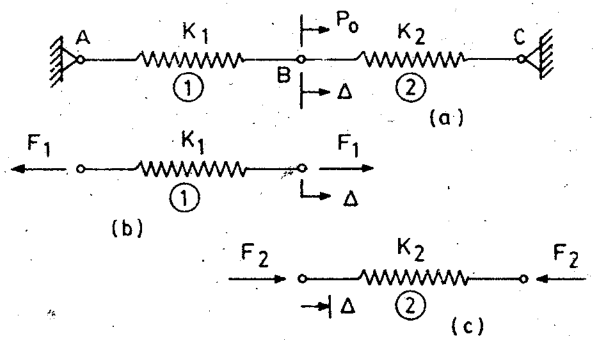


Fig. 1.10

$$F_1 = K_1 \Delta \quad \text{and} \quad F_2 = K_2 \Delta \tag{i}$$

The equilibrium of joint B gives,

$$F_1 + F_2 = P_0 \tag{ii}$$

It may be seen that spring 1 elongates by Δ, whereas spring 2 shortens by the same amount. Substituting the values of F_1 and F_2 in Eq. (ii) gives,

$$(K_1 + K_2)\Delta = P_0$$

or $\Delta = \dfrac{P_0}{K_1 + K_2}$

if $P_0 = 10$ kN, $K_1 = 100$ kN/m, $K_2 = 150$ kN/m,

$$\therefore \quad \Delta = \frac{10}{100 + 150} = 0.04\,\text{m}$$

$F_1 = 100 \times 0.04 = 4$ kN tension

and $F_2 = 150 \times (-0.04) = -6$ kN compression.

Example 1.5

A four springs system shown in Fig 1.11a is subjected to axial loads P_1 and P_2. Determine the displacements Δ_1 and Δ_2, and spring forces, if they are connected through a weightless rigid body Q.

Solution

Since the springs are interconnected through a rigid body Q, the compatibility condition requires that springs 1, 4 and 3 must elongate by Δ_2 each. The free body diagrams of each of the four springs are shown in Figs.1.11b to d.

Force-deformation relations give

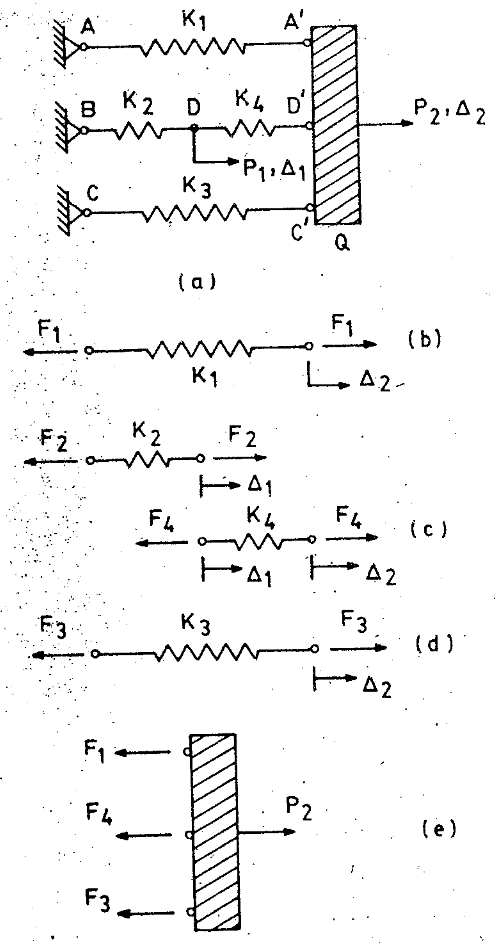


Fig. 1.11

$$F_1 = K_1\Delta_2, \quad F_2 = K_2\Delta_1, \quad F_3 = K_3\Delta_2, \quad F_4 = K_4(\Delta_2 - \Delta_1) \tag{i}$$

Joint equilibrium equations give,

$$F_2 - F_4 = P_1 \tag{ii}$$

and $$F_1 + F_4 + F_3 = P_2 \qquad \text{(Fig. 1.11e)} \tag{iii}$$

There are six relations and six unknowns F_1, F_2, F_3, F_4, Δ_1 and Δ_2. Substituting the values of F_1 to F_4 in Eqs. (ii) and (iii) give,

$$K_2\Delta_1 - K_4(\Delta_2 - \Delta_1) = P_1$$

and $$K_1\Delta_2 + K_4(\Delta_2 - \Delta_1) + K_3\Delta_2 = P_2$$

or $$\begin{bmatrix} (K_2+K_4) & -K_4 \\ -K_4 & (K_1+K_3+K_4) \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \tag{iv}$$

Eq.(iv) gives the values of Δ_1 and Δ_2. The spring forces can be obtained using Eqs. (i).

Numerical Example

Let $K_1 = 50$ kN/m, $K_2 = 75$ kN/m, $K_3 = 50$ kN/m, $K_4 = 60$ kN/m, $P_1 = 20$ kN and $P_2 = 40$ kN.

Eq. (iv) gives, $$\begin{bmatrix} 135 & -60 \\ -60 & 160 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 40 \end{Bmatrix}$$

or $$\begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} 0.311 \\ 0.367 \end{Bmatrix} \text{m}$$

Eqs. (i) give $$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} 18.33 \\ 23.34 \\ 18.33 \\ 3.34 \end{Bmatrix} \text{kN}$$

Example 1.6

A weightless bar ABCD is supported on two springs as shown in Fig.1.12a. The spring flexibilities are f_1 and f_2. Determine the spring forces and rotation of the bar if it is subjected to a load P at point D.

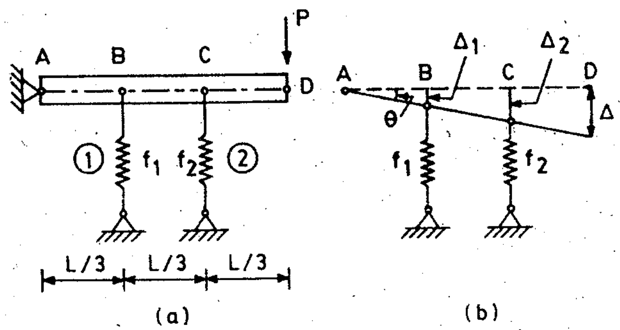


Fig. 1.12

Solution

If bar AD rotates by θ, (Fig. 1.12b), the compatibility condition requires that

$$\Delta = L\theta, \quad \Delta_1 = \frac{L}{3}\theta \quad \text{and} \quad \Delta_2 = \frac{2L}{3}\theta \tag{i}$$

Spring 1 compresses by Δ_1 and spring 2 compresses by Δ_2. The deformation-force relation gives,

$$\Delta_1 = f_1 R_1, \quad \text{and} \quad \Delta_2 = f_2 R_2 \tag{ii}$$

where f_1, f_2 = spring flexibilities,
R_1, R_2 = spring forces

The moment equilibrium equation gives,

$$\Sigma M_A = 0$$

or

$$PL = R_1 \frac{L}{3} + R_2 \frac{2L}{3} \tag{iii}$$

Among θ, Δ_1, Δ_2 and Δ, θ is the only independent variable. Substituting Eqs. (ii) in Eq. (iii),

$$P = \frac{\Delta_1}{3f_1} + \frac{2}{3}\frac{\Delta_2}{f_2} \tag{iv}$$

Substituting the values of Δ_1 and Δ_2 from Eqs. (i) in Eq. (iv),

$$P = \frac{L}{9}\frac{\theta}{f_1} + \frac{4L}{9}\frac{\theta}{f_2}$$

or

$$\theta = \frac{9 f_1 f_2}{(4f_1 + f_2)}\left(\frac{P}{L}\right) \tag{v}$$

Numerical Example

Let $P = 20$ kN, $L = 6$ m, $f_1 = 0.005$ m/kN, $f_2 = 0.010$ m/kN.

$\therefore$ $\theta = 0.05$ radian $= 2.86°$

$\Delta_1 = 0.10$ m, $\Delta_2 = 0.20$ m, $R_1 = 20$ kN and $R_2 = 20$ kN

Example 1.7

A weightless rigid bar is supported on four springs as shown in Fig. 1.13a. Determine the spring forces and rotation of the bar if it is subjected to a vertical load P.

Solution

If bar AE rotates by θ, the compatibility condition requires that: (Fig. 1.13b)

$$\Delta_1 = \frac{L}{4}\theta, \quad \Delta_2 = \frac{L}{2}\theta, \quad \Delta_3 = \frac{3L}{4}\theta \quad \text{and} \quad \Delta_4 = L\theta \tag{i}$$

Springs 1 and 3 compress by Δ_1 and Δ_3 whereas, Springs 2 and 4 elongate by Δ_2 and Δ_4. The force-deformation relations give,

$$R_1 = K_1\Delta_1, \quad R_2 = K_2\Delta_2, \quad R_3 = K_3\Delta_3 \quad \text{and} \quad R_4 = K_4\Delta_4 \tag{ii}$$

where, K_1, K_2, K_3, K_4 = spring stiffnesses
R_1, R_2, R_3, R_4 = spring forces

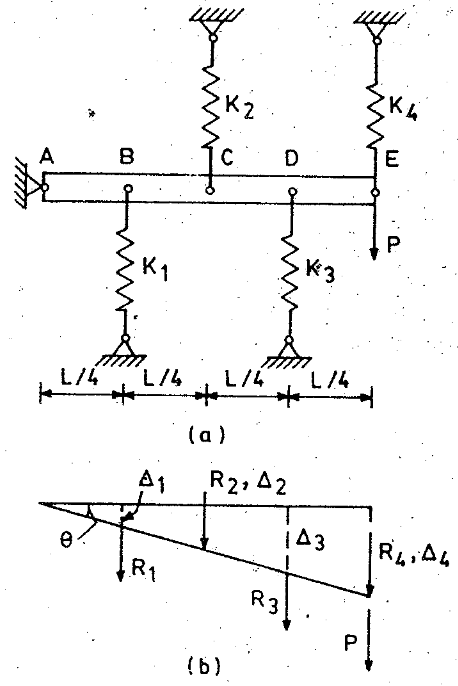


Fig. 1.13

The moment equilibrium equation gives,

$$\Sigma M_A = 0$$

or

$$PL = R_1 \times \frac{L}{4} + R_2 \times \frac{L}{2} + R_3 \times \frac{3L}{4} + R_4 \times L \tag{iii}$$

Substituting Eqs. (i) and (ii) in Eq. (iii),

$$P = \left[K_1\left(\frac{1}{4}\right)^2 + K_2\left(\frac{1}{2}\right)^2 + K_3\left(\frac{3}{4}\right)^2 + K_4\right]L\theta$$

or

$$\theta = \frac{P/L}{\left[K_1\left(\frac{1}{4}\right)^2 + K_2\left(\frac{1}{2}\right)^2 + K_3\left(\frac{3}{4}\right)^2 + K_4\right]} \tag{iv}$$

Numerical Example

Let $K_1 = 30$ kN/m, $K_2 = 40$ kN/m, $K_3 = 50$ kN/m, $K_4 = 60$ kN/m,

$P = 20$ kN and $L = 6$ m

$\therefore$

$\theta = 0.033$ radian $= 1.91°$, $\quad \Delta_4 = 0.198$ m

$R_1 = -1.485$ kN compression, $\quad R_2 = 3.96$ kN tension

$R_3 = -7.425$ kN compression, $\quad R_4 = 11.88$ kN tension

Example 1.8

A pin-jointed three bar truss is shown in Fig. 1.14a. Determine the member forces and joint displacements.

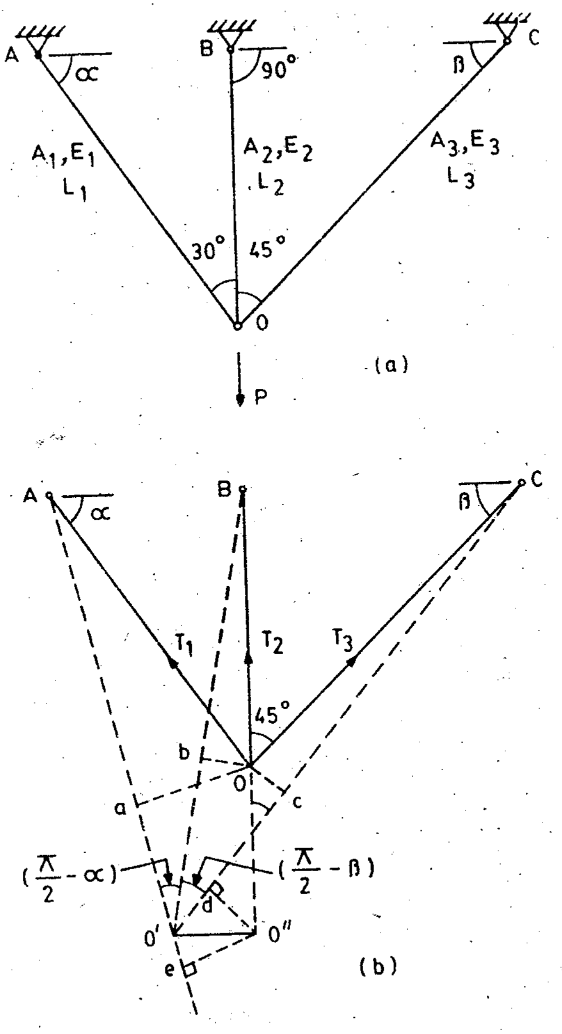


Fig. 1.14 Pin-jointed three bar truss

Solution

The joint o occupies position o' under the application of vertical load W as shown in Fig. 1.14b. Displacements are small.

Let $\quad o'o'' = u, \quad oo'' = v$

Considering the equilibrium of joint o,

$$\Sigma F_x = 0, \quad T_1 \cos\alpha = T_3 \cos\beta \tag{i}$$

$$\Sigma F_y = 0, \quad T_1 \sin\alpha + T_2 + T_3 \sin\beta = W \tag{ii}$$

Eqs. (i) and (ii) can be arranged in the matrix form :

$$\begin{Bmatrix} 0 \\ W \end{Bmatrix} = \begin{bmatrix} -\cos\alpha & 0 & \cos\beta \\ \sin\alpha & 1 & \sin\beta \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} \tag{iii}$$

or, $\quad \mathbf{P} = \mathbf{R}^T \mathbf{T}$

where $\quad T = \{T_1, T_2, T_3\}$ is the vector of internal forces

$P = \{0, W\}$ is the vector of joint loads

R^T = equilibrium matrix

There are three unknown forces but only two equilibrium equations, hence compatibility conditions are required to obtain a solution. Considering the compatibility conditions,

Member 1 elongates by Δ_1

$$\therefore \quad \Delta_1 = a\,o' = a\,e - e\,o' \tag{iv}$$

or $\quad \Delta_1 = v \sin\alpha - u \cos\alpha$

Member 2 elongates by Δ_2

$$\text{or} \quad \Delta_2 = o\,o'' = v \tag{v}$$

Member 3 elongates by Δ_3

or $\quad \Delta_3 = c\,o' = c\,d + d\,o'$

$$\text{or} \quad \Delta_3 = v \sin\beta + u \cos\beta \tag{vi}$$

Eqs. (iv), (v) and (vi) can be arranged in the matrix form:

$$\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{Bmatrix} = \begin{bmatrix} -\cos\alpha & \sin\alpha \\ 0 & 1 \\ \cos\beta & \sin\beta \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\text{or,} \quad \Delta = \mathbf{R}\,\mathbf{U} \tag{vii}$$

Here **R** is the kinematic matrix relating the member elongations vector Δ to the joint displacement vector **U**. The transpose of the equilibrium matrix is the kinematic matrix. This can be proved using the principle of virtual work.

Force – deformation relations give

$$\Delta_1 = \frac{T_1 L_1}{A_1 E_1}$$

or, $$T_1 = \frac{A_1E_1}{L_1}\Delta_1 \qquad \text{(viii)}$$

Similarly, $$T_2 = \frac{A_2E_2}{L_2}\Delta_2 \qquad \text{(ix)}$$

and $$T_3 = \frac{A_3E_3}{L_3}\Delta_3 \qquad \text{(x)}$$

These in matrix notation are

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{bmatrix} \frac{A_1E_1}{L_1} & 0 & 0 \\ 0 & \frac{A_2E_2}{L_2} & 0 \\ 0 & 0 & \frac{A_3E_3}{L_3} \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{Bmatrix} \qquad \text{(xi)}$$

or, $$\mathbf{T} = \mathbf{k}\,\Delta \qquad \text{(xii)}$$

or $$\mathbf{T} = \mathbf{k\,R\,U} \quad \text{from Eq.(vii)} \qquad \text{(xiii)}$$

By premultiplying both sides by $\mathbf{R^T}$ gives,

$$\mathbf{R^T\,T} = \mathbf{R^T\,k\,R\,U} \qquad \text{(xiv)}$$

or, $$\mathbf{P} = \mathbf{K\,U} \qquad \text{(xv)}$$

where $$\mathbf{K} = \mathbf{R^T\,k\,R} \qquad \text{(xvi)}$$

Knowing u and v from Eq. (xv), member forces can be obtained from Eq. (xi) or (xiii). This method is referred to as the displacement or stiffness method since displacements are the unknowns.

Numerical Example

Let $A_1 = A_2 = A_3 = 6\ \text{cm}^2$, $L_2 = 300$ cm, $\alpha = 60°$ $\beta = 45°$
$E_1 = E_3 = 8 \times 10^3\ \text{kN/cm}^2$, $E_2 = 2 \times 10^4\ \text{kN/cm}^2$,
$W = 100$ kN

$$\frac{A_1E_1}{L_1} = 138.56\ \text{kN/cm}, \quad \frac{A_2E_2}{L_2} = 400\ \text{kN/cm}, \quad \frac{A_3E_3}{L_3} = 113.15\ \text{kN/cm}$$

Eq. (vii) gives,

$$\mathbf{R} = \begin{bmatrix} -0.5 & 0.866 \\ 0 & 1 \\ 0.707 & 0.707 \end{bmatrix} \quad \text{or,} \quad \mathbf{R^T} = \begin{bmatrix} -0.5 & 0 & 0.707 \\ 0.866 & 1 & 0.707 \end{bmatrix}$$

$$\mathbf{k} = \begin{bmatrix} 138.56 & 0 & 0 \\ 0 & 400 & 0 \\ 0 & 0 & 113.15 \end{bmatrix} \quad \text{from Eq. (xi)}$$

$$\therefore \quad \mathbf{K} = \mathbf{R^T\,k\,R} = \begin{bmatrix} 91.200 & -3.420 \\ -3.420 & 560.522 \end{bmatrix}$$

K is square and symmetric matrix

$$\mathbf{P} = \mathbf{K\,U}$$

or, $$\begin{Bmatrix} 0 \\ 100 \end{Bmatrix} = \begin{bmatrix} 91.200 & -3.420 \\ -3.420 & 560.522 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

or, $$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} 0.0067 \\ -0.1784 \end{Bmatrix} \text{cm}$$

and Eq. (xiii) gives $$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 20.95 \\ 71.36 \\ 14.80 \end{Bmatrix} \text{kN}$$

CHAPTER
two

MATRIX ALGEBRA

2.1 INTRODUCTION

Matrices have a special importance in structural analysis. They are user friendly as well as computer friendly. The memory capacity of a computer is usually limited and unless program space is minimized, and data space is curtailed, large problems cannot be solved. Too much algebra normally leads to too many arithmetic operations making the calculations too unwieldy. To avoid getting lost in a labyrinth of algebra, numerous variables entering a problem should be treated in groups rather than individually. This is possible by making use of *matrices*.

From the point of view of generalizing the solution of problems, the matrix approach offers many advantages. Normally an algorithm written using matrices need not be changed just because the size of the problem is increased. A program written to analyze a 5 - storey frame can be easily used even when the number of stories is increased to 50. In combination with DO loops and subscripted variables of FORTRAN or C language, the matrix approach helps in the production of efficient and general solutions to large size problems in structural engineering.

The basic principles of matrix algebra required to understand the structural analysis procedures described in this book are now explained.

2.2 DEFINITIONS

(a) A matrix is a rectangular array or table of numerical quantities or mathematical symbols represented by a single symbol. Thus

$$\mathbf{A}_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots \\ \cdots & \cdots & a_{ij} & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad (2.1)$$

means that **A** is the symbol selected to represent the m × n array of elements where element $\mathbf{A}_{ij}$ is the element in the i th row and j th column. Note that element $\mathbf{A}_{mn}$ may be distinguished from matrix $\mathbf{A}_{m \times n}$ by the times (×) sign between the subscripts. Where ambiguity may occur a comma is used to separate numerical i, j values. Thus $a_{12,\,4}$ indicates the element of **A** in the 12th row and 4th column, while $\mathbf{A}_{12 \times 4}$ denotes a matrix of 12 rows and 4 columns containing 12 × 4 = 48 elements.

The subscripts m, n indicate that matrix **A** contains m columns and n rows and **A** is said to be of order m by n or m × n. The subscripts are often omitted and the matrix symbol may be included in square brackets. Thus **A**, **[A]**, $[\mathbf{A}]_{m \times n}$, and $\mathbf{A}_{m \times n}$ are equivalent ways of representing the matrix given by Eq. 2.1

(b) A square matrix is a rectangular matrix for which m = n.
(c) A row vector is defined by m = 1.
$\mathbf{X}_{1 \times 3} = \{x_1 \; x_2 \; x_3\}_{1 \times 3}$
(d) A column vector is defined by n = 1.

$$\mathbf{P}_{3 \times 1} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}_{3 \times 1}$$

In this text-book a set of curly brackets {} is reserved for row or column vectors.
(e) A diagonal matrix is a square matrix with all off-diagonal terms equal to 0.

Thus $$\mathbf{D} = \begin{bmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ 0 & 0 & 0 & d_{44} \end{bmatrix}$$

is a diagonal matrix of order four by four.
(f) A unit or identity matrix is a diagonal matrix with all diagonal terms equal to 1 and is written with the symbol **I**. Thus

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an identity matrix of order three by three.
(g) A null matrix contains all zeros and is written with the symbol **0** or [0].
(h) A symmetric matrix is one in which each $A_{ij} = A_{ji}$.
(i) A skew symmetric (or anti symmetric) matrix occurs if each $A_{ij} = -A_{ji}$ and each $A_{ii} = 0$
(j) A skew matrix occurs if each $A_{ij} = -A_{ji}$ and not all $A_{ii} = 0$.

(k) *Linear Simultaneous Equations* A set of linear simultaneous equations expresses a relation between an array of dependent variables and an array of independent variables, that is ,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots + a_{2m}x_m &= b_2 \\ \dots\dots\dots \\ \dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots\dots + a_{nm}x_m &= b_n \end{aligned} \tag{2.2}$$

The m independent variables are $x_1, x_2, x_3, \dots\dots x_m$ and n dependent variables are $b_1, b_2, b_3, \dots\dots b_n$. The terms $a_{11}, a_{12} \dots$ represent the constant coefficients. These linear algebraic equations may be written in matrix notation as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & : & a_{1m} \\ a_{21} & a_{22} & a_{23} & : & a_{2m} \\ : & : & : & : & : \\ a_{n1} & a_{n2} & a_{n3} & : & a_{nm} \end{bmatrix}_{n\times m} \begin{Bmatrix} x_1 \\ x_2 \\ : \\ x_m \end{Bmatrix}_{m\times 1} = \begin{Bmatrix} b_1 \\ b_2 \\ : \\ b_n \end{Bmatrix}_{n\times 1} \tag{2.3}$$

The coefficients 'a's are enclosed in a pair of square brackets. The independent variables x's and the dependent variables b's are enclosed within a pair of curly brackets. The rectangular block of elements is called a *matrix*. It may be a rectangular matrix of size (m × n) or a square matrix of size (n × n). The vertical array of elements is called a *vector*. It may be a column vector of size (m × 1) or a row vector of size (1 × m). The first letter represents the number of rows and the second letter represents the number of columns.

Equation 2.3 can be written in an abbreviated form as

$$[A]_{n\times m}\{x\}_{m\times 1} = \{b\}_{n\times 1} \tag{2.4a}$$

or

$$\mathbf{A}\ \mathbf{x} = \mathbf{b} \tag{2.4b}$$

where,

$$\mathbf{A} = [A]_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & a_{1m} \\ a_{21} & a_{22} & a_{23} & : & a_{2m} \\ : & : & : & : & : \\ a_{n1} & a_{n2} & a_{n3} & : & a_{nm} \end{bmatrix}$$

$$\mathbf{x} = \{x\}_{m\times 1} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ : \\ x_m \end{Bmatrix}, \quad \mathbf{b} = \{b\}_{n\times 1} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ : \\ : \\ b_n \end{Bmatrix}$$

2.3 MATRIX ALGEBRA

Addition

Addition of two matrices is performed by adding corresponding terms in each matrix.

Thus $\mathbf{A} = \mathbf{B} + \mathbf{C}$

implies that $a_{ij} = b_{ij} + c_{ij}$

for all i and j. Obviously, **A**, **B**, and **C** must be of the same order.

Example 2.1

Find the sum **B** + **C**.

$$\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 5 & -4 \\ 7 & 0 \end{bmatrix}$$

Solution

$$\mathbf{A} = \mathbf{B} + \mathbf{C} = \begin{bmatrix} (1+5) & (2-4) \\ (3+7) & (4+0) \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 10 & 4 \end{bmatrix}$$

Subtraction

Subtraction of two matrices is performed by subtracting corresponding elements in each matrix. Thus

$\mathbf{D} = \mathbf{E} - \mathbf{F}$

implies that $d_{ij} = e_{ij} - f_{ij}$ for all i and j.

Commutative Law

The commutative law for addition or subtraction states that

$$\mathbf{B} + \mathbf{C} = \mathbf{C} + \mathbf{B}$$

and

$$\mathbf{E} - \mathbf{F} = -\mathbf{F} + \mathbf{E}$$

Associative Law

The associative law for addition or subtraction states that

$$A + (B + C) = (A + B) + C$$

Transposition

The transpose of $A_{m \times n}$ is $A^T_{n \times m}$ (read "A transpose") where superscript T indicates the transpose and each $a_{ji}^{\ T} = a_{ij}$ and $i = 1, 2........m$, $j = 1, 2.......n$.

Example 2.2

Find the transpose of **A**.

Solution

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \qquad \text{therefore,} \quad \mathbf{A}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$$

The transpose of a symmetric matrix equals the matrix. Thus $\mathbf{A}^T = \mathbf{A}$ if **A** is symmetric.

Multiplication

The product **C** of **A** times **B** is

$$\mathbf{C}_{m \times p} = \mathbf{A}_{m \times n} \times \mathbf{B}_{n \times p} \tag{2.5a}$$

where each C_{ij} term is given by

$$c_{ij} = \sum_{s=1}^{n} a_{is} \times b_{sj} \tag{2.5b}$$

and $i = 1, 2.......m$, $j = 1, 2.........p$. Thus a total of $m \times n \times p$ multiplications must be made in computing $C_{m \times p}$. The number of columns (n) in **A** must equal to the number of rows (n) in **B** for **A** and **B** to be conformable for multiplication. The s subscripts on **A** and **B** in Eq. 2.5b are the inner subscripts. The outer subscripts (i on **A** and j on **B**) denote the location of element $c_{i\,j}$ in the **C** matrix. In Eq.2.5a, **B** is said to be premultiplied by **A** while **A** is said to be postmultiplied by **B**.

Matrices can be multiplied by vectors as long as the multiplication conditions are satisfied. The order of products can be predicted without actually carrying out the operations. Thus

scalar	×	matrix	=	matrix
matrix	×	column	=	column
row	×	matrix	=	row
row	×	column	=	scalar
column	×	row	=	matrix

Other multiplications like pre multiplication of a matrix by a column are not possible.

Example 2.3

Find the product **A** × **B**.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -5 \\ 2 & 4 & 6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 0 \\ 3 & 6 & 3 \end{bmatrix}$$

Solution

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \quad \text{or,} \quad \begin{bmatrix} -8 & -11 & -13 \\ 28 & 53 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 0 \\ 3 & 6 & 3 \end{bmatrix}$$

Element $c_{1,2}$ is computed as the product of each element in row 1 of **A** times the associated elements in column 2 of **B**. Thus

$$C_{1,2} = [(1 \times 4) + (3 \times 5) + (-5 \times 6)] = -11$$

The associative law in multiplication gives

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$$

The distributive law in multiplication gives

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

The commutative law generally does not hold. Thus

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$$

except for some special cases such as **A** or **B** = **I**

Matrix Partitioning

Partitioning is performed by dividing a matrix into two or more sub matrices.

Example 2.4

Indicate one possible partition of **A**.

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & : & A_{13} & A_{14} \\ A_{21} & A_{22} & : & A_{23} & A_{24} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{31} & A_{32} & : & A_{33} & A_{34} \\ A_{41} & A_{42} & : & A_{43} & A_{44} \\ A_{51} & A_{52} & : & A_{53} & A_{54} \end{bmatrix}$$

Partitioning of A is performed by drawing lines as shown above to produce the desired sub matrices. The partitioned matrix may be written as

$$\mathbf{A} = \left[\begin{array}{c:c} \mathbf{A}'_{11} & \mathbf{A}'_{12} \\ \hdashline \mathbf{A}'_{21} & \mathbf{A}'_{22} \end{array}\right]$$

where,

$$\mathbf{A}'_{11} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad \mathbf{A}'_{12} = \begin{bmatrix} A_{13} & A_{14} \\ A_{23} & A_{24} \end{bmatrix}$$

$$A'_{21} = \begin{bmatrix} A_{31} & A_{32} \\ A_{41} & A_{42} \\ A_{51} & A_{52} \end{bmatrix} \qquad A'_{22} = \begin{bmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \\ A_{53} & A_{54} \end{bmatrix}$$

Partitioned matrices may be added, subtracted, multiplied, and transposed keeping in mind that each "*element*" of a partitioned matrix is in turn a sub matrix which must satisfy the rules of matrix operations.

Example 2.5

Find the product $\mathbf{A} \times \mathbf{B}$ making use of partitioning of matrices.

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & : & \mathbf{A}_{12} \\ \cdots & : & \cdots \\ \mathbf{A}_{21} & : & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 2 & : & -3 & 4 \\ 5 & 6 & : & 7 & 8 \\ .. & .. & .. & .. & .. \\ 9 & 1 & : & -2 & 4 \\ 4 & 0 & : & 2 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & : & \mathbf{B}_{12} \\ \cdots & : & \cdots \\ \mathbf{B}_{21} & : & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} 4 & 2 & : & 1 \\ 3 & 0 & : & 0 \\ .. & .. & : & .... \\ 2 & 1 & : & 7 \\ 5 & 0 & : & -3 \end{bmatrix}$$

Solution

$$\mathbf{C} = \mathbf{A}\,\mathbf{B} = \begin{bmatrix} \mathbf{C}_{11} & : & \mathbf{C}_{12} \\ .. & : & .. \\ \mathbf{C}_{21} & : & \mathbf{C}_{22} \end{bmatrix} = \begin{bmatrix} [\mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21}] & : & [\mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22}] \\ \cdots\cdots & : & \cdots\cdots \\ [\mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21}] & : & [\mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22}] \end{bmatrix}$$

The detailed computations for $[C_{12}]$ are

$$[C_{12}] = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 & 4 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 33 \\ 25 \end{bmatrix} = \begin{bmatrix} -32 \\ 30 \end{bmatrix}$$

The final $\mathbf{C}$ matrix is

$$\mathbf{C} = \begin{bmatrix} 24 & -1 & : & -32 \\ 92 & 17 & : & 30 \\ .... & ..... & : & ..... \\ 55 & 16 & : & -17 \\ 20 & 10 & : & 18 \end{bmatrix}$$

The transpose of $\mathbf{C}$ is
$$\mathbf{C}^T = \begin{bmatrix} C_{11}^T & : & C_{21}^T \\ ..... & : & ..... \\ C_{12}^T & : & C_{22}^T \end{bmatrix} = \begin{bmatrix} 24 & 92 & : & 55 & 20 \\ -1 & 17 & : & 16 & 10 \\ .... & .... & : & .... & .... \\ -32 & 30 & : & -17 & 18 \end{bmatrix}$$

Determinant

A scalar quantity associated with a square matrix is called its *determinant*. Determinant of a second order matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is denoted by } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ and is given by } \det A = ad - bc \tag{2.6}$$

The determinant of a 3×3 matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \text{ is denoted by } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ and is given as}$$

$$a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Determinant of a higher order matrix is obtained by repeatedly using the formula

$$\det\ \mathbf{A} = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13} \ldots\ldots + (-1)^{r+1} a_{1r}A_{1r} + \ldots (-1)^{n+1} a_{1n}A_{1n} \tag{2.7}$$

where,

$a_{i\,j}$ are elements of first row (or any convenient row) and $\mathbf{A}_{i\,j}$ are the determinants of $(n - 1)$th order obtained by suppressing first row and jth column. By continuous reduction of the order of the determinant to final second order determinants, det $\mathbf{A}$ can be calculated as in the following example.

Example 2.6

Find the determinant of $\mathbf{A}_{3 \times 3}$.

Solution

$$\det \quad \mathbf{A} = \det\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \times \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \times \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \times \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= -3 - 2(-6) + 3(-3) = 0$$

Though this method of calculating determinants is convenient for third or fourth order matrices, it is very inconvenient for higher order matrices. Usually, the

determinant is obtained as a by-product while solving a set of simultaneous equations which has the same coefficient matrix as the matrix of the determinant.

Inversion

Matrix division is undefined. Thus if

$$\mathbf{A} \times \mathbf{X} = \mathbf{C} \qquad (2.8)$$

the familiar arithmetic operation

$$\mathbf{X} = \mathbf{C}/\mathbf{A}$$

may not be performed except for the trivial case where **X**, **C**, and **A** are of order one by one. Instead, both sides of Eq. 2.8 must be pre multiplied by the inverse of **A**. The inverse of **A** is denoted by $\mathbf{A}^{-1}$, where $\mathbf{A}^{-1}$ (read "A inverse") is the matrix which when multiplied by **A** gives the identity matrix **I**. Thus

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

Pre multiplying both sides of Eq. 2.8 by $\mathbf{A}^{-1}$ gives

$$\mathbf{A}^{-1}\,\mathbf{A}\,\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{C}$$

or

$$\mathbf{I}\,\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{C}$$

or

$$\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{C}$$

Inverse of A is defined as $\quad \mathbf{A}^{-1} = \dfrac{\text{adjoint A}}{|A|} \qquad (2.9)$

A square matrix is invertible if and only if **A** is non-singular, that is $|A| \neq 0$.

The adjoint of **A** is defined to be the transpose of the matrix A_{ij} and is denoted by adj.**A**, where A_{ij} is cofactor of the element a_{ij} of matrix **A**.

Let **A** be the 2×2 matrix given by

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

then $\quad A^{-1} = \dfrac{1}{A_{11}A_{22} - A_{21}A_{12}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$

Transpose and Inverse of a Product

Let $\quad \mathbf{D} = \mathbf{Z\,B\,C}$

then $\quad \mathbf{D}^T = (\mathbf{Z\,B\,C})^T = \mathbf{C}^T\,\mathbf{B}^T\,\mathbf{Z}^T$

and $\quad \mathbf{D}^{-1} = (\mathbf{Z\,B\,C})^{-1} = \mathbf{C}^{-1}\,\mathbf{B}^{-1}\,\mathbf{Z}^{-1}$

Example 2.7

Find the inverse of **Z**.

Solution

$$Z = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \quad \therefore \quad Z^{-1} = \frac{1}{4+6}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

Check $\quad Z^{-1} \times Z = \dfrac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \dfrac{1}{10}\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad$ O.K.

The procedure for computing the inverse of higher order matrices is much more involved and is not described. Note, however, that inverse of any diagonal matrix Z is another diagonal matrix whose diagonal elements are reciprocals of the Z_{ii} terms.

Example 2.8

Find the inverse of **B**.

Solution

$$\mathbf{B} = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \therefore \quad \mathbf{B}^{-1} = \begin{bmatrix} 1/7 & 0 & 0 & 0 \\ 0 & 1/9 & 0 & 0 \\ 0 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 1/5 \end{bmatrix}$$

Example 2.9

Find the inverse of $\quad \mathbf{A} = \begin{bmatrix} 2 & 1 & 4 \\ 4 & -1 & 0 \\ -6 & 2 & 1 \end{bmatrix}$

Solution

$$|\mathbf{A}| = \begin{vmatrix} 2 & 1 & 4 \\ 4 & -1 & 0 \\ -6 & 2 & 1 \end{vmatrix}$$

$$= 2(-1 - 0) - 1(4 - 0) + 4(8 - 6) = 2$$

Since $|A| \neq 0$, A is invertible.

Now,

$$A_{11} = \text{Cofactor of} \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1, \quad A_{12} = -\begin{vmatrix} 4 & 0 \\ -6 & 1 \end{vmatrix} = -4, \quad A_{13} = \begin{vmatrix} 4 & -1 \\ -6 & 2 \end{vmatrix} = 2$$

$$A_{21} = +7, \quad A_{22} = 26, \quad A_{23} = -10$$

$$A_{31} = 4, \quad A_{32} = +16, \quad A_{33} = -6$$

$$\text{adj } \mathbf{A} = \begin{bmatrix} -1 & 7 & 4 \\ -4 & 26 & 16 \\ 2 & -10 & -6 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{\text{adj}\,\mathbf{A}}{|\mathbf{A}|} = \frac{1}{2}\begin{bmatrix} -1 & 7 & 4 \\ -4 & 26 & 16 \\ 2 & -10 & -6 \end{bmatrix}$$

Check $\quad \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$

$$= \frac{1}{2}\begin{bmatrix} -2\times1-1\times4+4\times2 & +2\times7+1\times26-4\times10 & 2\times4+1\times16-4\times6 \\ -4\times1+1\times4+0 & +4\times7-1\times26+0 & 4\times4-1\times16+0 \\ 6\times1-2\times4+1\times2 & -6\times7+2\times26-1\times10 & -6\times4+2\times16-1\times6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

O. K.

2.4 APPLICATION OF A WORK SHEET

A work sheet or a spread sheet is an electronic sheet consisting of rows and columns. A cell is formed where each column and row intersects. Each cell can hold one piece of information - a number, a formula, or a label. The columns are referenced by letters (A to ZZ), and rows are referenced by numbers. LOTUS 1-2-3 is one of the most popular softwares. With the help of @ FUNCTIONS, most mathematical operations can be conveniently carried out. Data analysis techniques provide matrix commands for multiplication and inversion. The details of the software can be seen in the user's manual supplied by the LOTUS Development Corporation, New York. It is desirable that a student of structural analysis is well versed with the various capabilities of a work sheet software.

The following example illustrates the applications of LOTUS spreadsheet for matrix operations.

Example 2.10

Carryout the following matrix operations on a LOTUS worksheet, and print all input and output matrices.

$$[\mathbf{A}] = \begin{bmatrix} 24 & 15 & 10 \\ 35 & 20 & 0 \\ 56 & -20 & 45 \end{bmatrix}, \quad [\mathbf{B}] = \begin{bmatrix} 34 & -20 & 0 \\ 25 & 45 & -9 \\ 67 & -9 & 22 \end{bmatrix}$$

(i) $A + B = C$ (ii) $A - B = C$

(iii) $A\,B = C$ (iv) $A\,B + B = C$

Solution

Let us enter the matrices **A** and **B** in the LOTUS worksheet cells as indicated:

[**A**] in cells B2 to D4; [**B**] in cells F2 to H4

(i) ADDITION

Step 1 — In cell A6, enter the caption
MATRIX ADDITION [A] + [B] =[C]
which will extend upto cell C6

Step 2 — In cell B8, enter the formula:
+ B2 + F2

Step 3 — Copy this formula in cells B8 to D10 using the copy command.
/ C
copy from cell : B8
copy to cells : B8 . D10

Matrix [C] looks as
$$\begin{bmatrix} 58 & -5 & 10 \\ 60 & 65 & -9 \\ 123 & -29 & 67 \end{bmatrix}$$

(ii) SUBTRACTION

Step 1 — In cell A12, enter the caption
MATRIX SUBTRACTION [A] - [B] = [C]
which will extend upto cell D12

Step 2 — In cell B14, enter the formula: + B2 - F2

Step 3 — Copy this formula in cells B14 to D16 using the copy command.
/ C
copy from cell : B14
copy to cells: B14 . D16

Matrix [C] looks as
$$\begin{bmatrix} -10 & 35 & 10 \\ 10 & -25 & 9 \\ -11 & -11 & 23 \end{bmatrix}$$

(iii) MULTIPLICATION

Step 1 — In cell A18, enter the caption
MATRIX MULTIPLICATION [A] × [B] = [C]

Step 2 — Use DATA MATRIX MULTIPLICATION command as follows:
/ D M M

range of first matrix : B2 . D4
range of second matrix : F2 . H4
range of output matrix : B20 . D22

Matrix [C] looks as $\begin{bmatrix} 1861 & 105 & 85 \\ 1690 & 200 & -180 \\ 4419 & -2425 & 1170 \end{bmatrix}$

(iv) MULTIPLICATION AND ADDITION

Matrix [C] obtained in part (iii) can be added to matrix [B] as explained in part (i), the resulting matrix [D] can be stored in cells B24 to D26.

2.5 SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS

The most commonly used methods of solution of linear simultaneous equations are as follows:

1. Gauss elimination method (without or with partial pivoting)
2. Gauss-Jordan method
3. Cholesky method for symmetric and positive definite matrices. The matrix **A** is decomposed into $\mathbf{LDL}^T$,
   where, **L** = lower triangular matrix
   D = diagonal matrix
4. Successive Over Relaxation method

These are standard methods and are discussed in any book on numerical analysis. FORTRAN77 listings of these programs are given in Appendix A. There are sufficient comment lines that explain the working of the programs. These programs can be executed on any personal computer (PC).

PART 1

FLEXIBILITY METHODS

CHAPTER
three

METHOD OF CONSISTENT DEFORMATIONS

3.1 INTRODUCTION

It has been demonstrated in Chapter 1 that a statically indeterminate structure cannot be analyzed by the laws of statics alone. In these situations the conditions of compatibility are required to be enforced to generate the additional required number of equations. The supports play a very important role. The number of reactions existing at a support depends upon the type of support. A roller support restrains only one translation normal to its surface. A hinged support permits only a rotation and restrains against any translation. Thus, there are two mutually perpendicular reactions at a hinged support. A fixed support does not permit any movement at all and there exist two mutually perpendicular reactions and a moment.

The steps for the analysis of any statically indeterminate structure – whether it be a beam, a rigid frame or a truss were enumerated in Chapter 1. A statically indeterminate structure is made determinate while maintaining its stability by removing all redundant reactions. If a roller support is removed and the reaction is replaced by an unknown reacting force, the compatibility condition requires that the deflection there must be zero. If a fixed support is changed into a hinged support, and the moment reaction is replaced by an unknown reacting moment, the compatibility condition is that the slope there must be zero. If a fixed support is completely removed and the reactions are replaced by two unknown mutually perpendicular reacting forces and a reacting moment, the compatibility condition requires that both the deflections and the slope there must be zero. The unknown redundants can thus be computed. This method of analysis is called the *consistent deformation method.*

The unknown redundant forces together with the forces of the applied loads must cause the structure to deform such that the deformations are consistent with the support conditions. Enforcing the consistent deformations or the compatibility requirements of the structure leads to the governing equations in terms of all the unknown redundants. This means that the action of one redundant will affect the displacements associated with the compatibility equation of another redundant. This is known as coupling of

degree of freedom, that is , the compatibility equations are coupled. Such equations cannot be solved independently but all equations need to be solved simultaneously.

3.2 CHOICE OF REDUNDANTS

Once the degree of static indeterminacy is determined, a released structure must be chosen by identifying suitable redundants. A *released structure* which is also known as the *primary structure* is *statically determinate and stable.* The reactions and forces in excess of those required to make a structure determinate and stable are the redundants. Redundants may be either support reactions or internal actions such as moment, shear or axial force. The question is, how do we identify the redundants or unknown forces of the structure? Generally a structure can be made statically determinate and stable in more than one way. Some indeterminate structures and the corresponding released structures and redundants are shown in Fig. 3.1. The choice of redundants is usually based upon convenience. Some released structures lead to more cumbersome computations than others. By a judicious selection of the redundants, however, numerical computations can be minimized. The following points may be kept in mind while selecting redundants:

1. A released structure should be such that the effect of the various loading conditions is localized as much as possible.
2. Take advantage of symmetry of the structure, if possible .
3. The positive sense of a redundant may be chosen arbitrarily.
4. A statically determinate reaction component must never be chosen as redundant as it may lead to instability.

Sign Convention

Shear force at any transverse cross-section is the algebraic sum of all forces acting transverse to the member on either side of the section. A shear force is said to be positive at a section if the right hand portion of the beam tends to slide upward with respect to the left hand portion. In other words, it is positive if the resultant of forces on the right hand side is upward.

A bending moment is said to be positive if it causes compression in the top of the cross-section. That is, a positive bending moment causes a downward convex curvature. The positive bending moment is also referred to as a sagging moment and the negative bending moment as a hogging moment. The bending moment diagram is plotted on the tension side.

3.3 BEAMS WITH ONE REDUNDANT

Example 3.1

A propped cantilever beam is shown in Fig. 3.2a. Draw shear force and bending moment diagrams.

Solution

The degree of indeterminacy is 1. Choose R_C as redundant, remove the support C

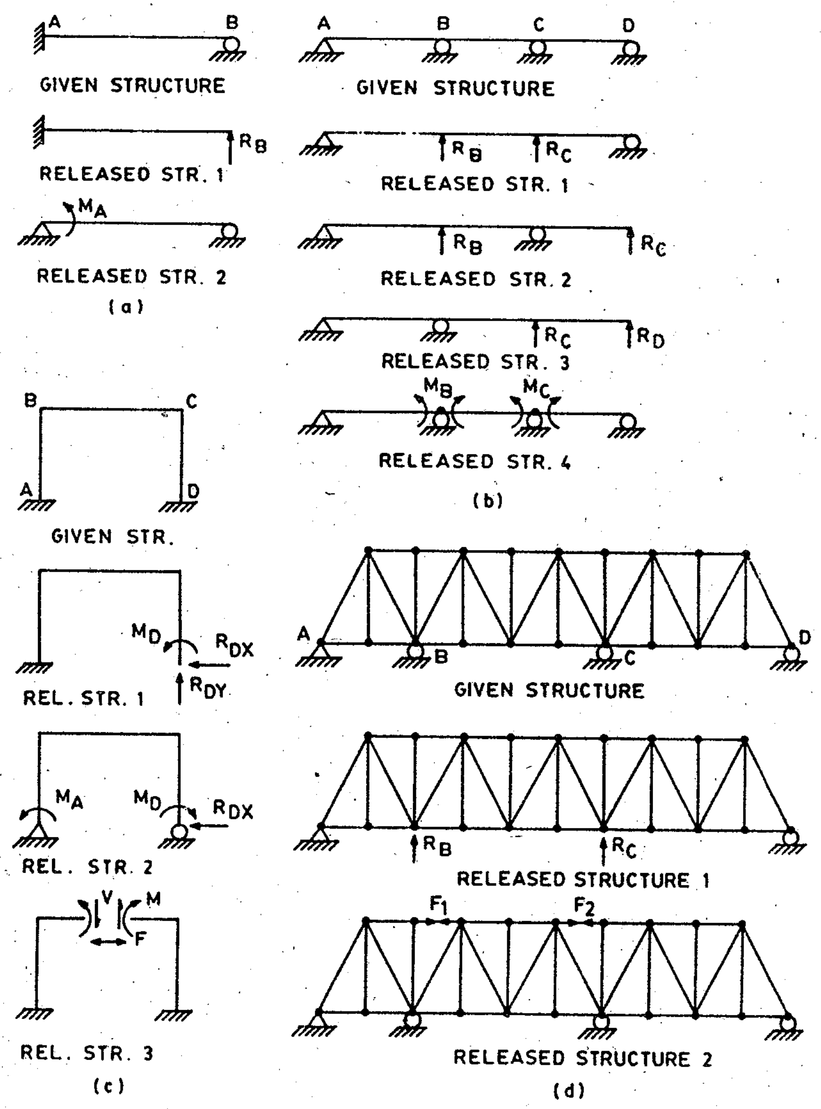


Fig. 3.1 Typical released structures

and obtain a statically determinate beam AC as shown in Fig. 3.2b. The value of R_C will be such that will make the deflection of the beam at C equal to zero.

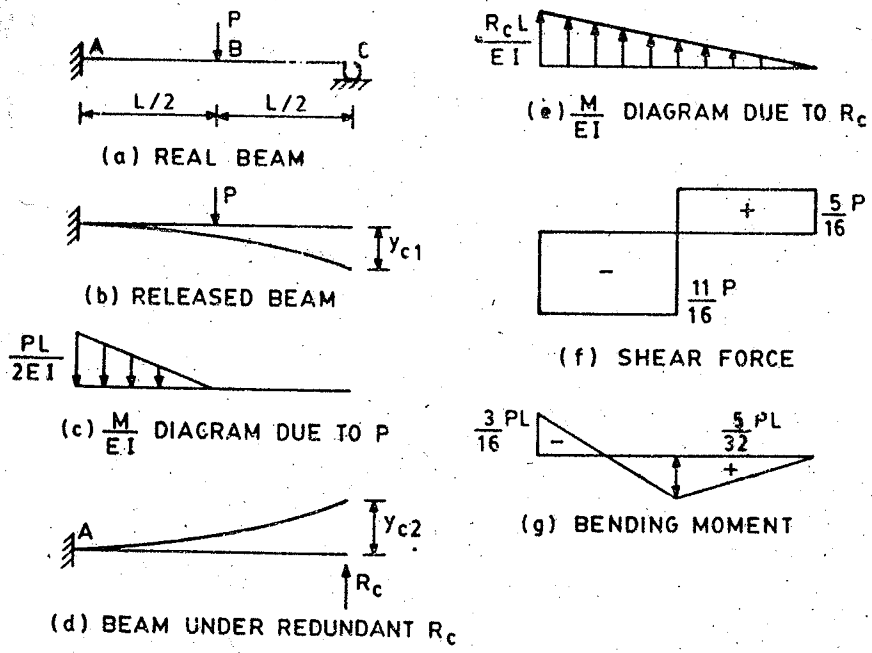


Fig. 3.2 Propped cantilever beam

y_{C1} = moment of the M/EI loading diagram due to the applied load between A and B about C (Fig. 3.2c) *(moment - area theorem)*

$$y_{C1} = \frac{1}{2} \times \frac{PL}{2EI} \times \frac{L}{2} \times \left(\frac{L}{2} + \frac{2}{3}\frac{L}{2}\right)$$

y_{C2} = moment of the M/EI loading diagram due to the redundant R_C (Fig. 3.2e)

$$y_{C2} = \frac{1}{2} \times \frac{R_C L}{EI} \times L \times \frac{2L}{3}$$

For compatibility, $y_{C1} = y_{C2}$

or, $$\frac{PL^3}{8EI} \times \frac{5}{6} = \frac{R_C L^3}{3EI}$$

or, $$R_C = \frac{5}{16} P$$

$$\therefore \quad M_B = \frac{5}{16} P \times \frac{L}{2} = \frac{5}{32} PL \quad (sagging)$$

and $$M_A = -\frac{PL}{2} + \frac{5}{16} PL = -\frac{3}{16} PL \quad (hogging)$$

The resulting shear and moment diagrams are shown in Figs 3.2f and g. A sagging bending moment is taken as positive, whereas a hogging bending moment is taken as negative.

Alternatively

Choose M_A as redundant, introduce moment release at A and obtain a simply supported beam AC as shown in Fig. 3.3a. This beam is also known as a released beam. The value of M_A will be such as to make net slope at A equal to zero.

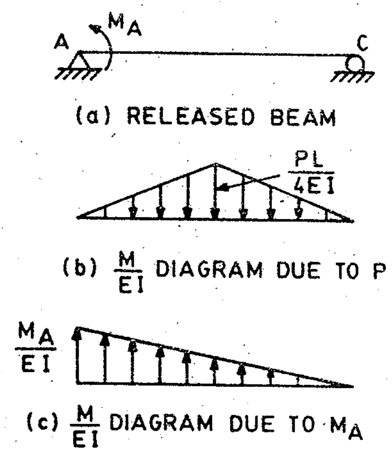


Fig. 3.3

θ_{A1} = shear at A due to M/EI loading diagram due to the applied loading between A and C in the conjugate beam (Fig. 3.3b)

$$= \frac{1}{2} \times \frac{PL}{4EI} \times L \times \frac{1}{2} \qquad (= Reaction)$$

θ_{A2} = shear at A due to M/EI loading diagram due to the redundant (or unknown) M_A (Fig. 3.3c)

$$\theta_{A2} = \frac{1}{2} \times \frac{M_A}{EI} \times L \times \frac{2}{3} \qquad (= Reaction)$$

For compatibility, $\theta_{A1} = \theta_{A2}$

or, $$M_A = \frac{3PL}{16} \quad \text{hogging}$$

O. K.

Example 3.2

Analyze a two span continuous beam using the method of consistent deformations as shown in Fig. 3.4a and draw shear force and bending moment diagrams.

Solution

The degree of static indeterminacy is equal to

$$\alpha_S = 4 - 3 = 1$$

Choose R_B as redundant, remove the support B and obtain a statically determinate beam AC. The value of R_B will be such that will make the deflection of the beam at B equal to zero. The deflection at B can be computed using the conjugate beam method.

y_{B1} = moment of the M/EI loading due to the applied loads between A and B about point B in the conjugate beam (Fig.3.4b)

$$A = \frac{a^2}{12}(3L - 2a), \quad \bar{x} = \frac{a}{2}\left(\frac{4L - 3a}{3L - 2a}\right), \quad a = 5\text{ m}, \; L = 13\text{ m}$$

where A = area of M/EI loading diagram between A and B

$\bar{x}$ = center of gravity of the M//EI loading diagram between A and B from A

$$y_{B1} = \frac{1}{EI}\left\{\frac{1}{2}\times 20 \times \frac{13^3}{12} \times 5 - 20 \times \frac{5^2}{12}(3\times 13 - 2\times 5)\times\left(5 - \frac{5(4\times 13 - 3\times 5)}{2(3\times 13 - 2\times 5)}\right)\right\}$$

$$= 9154 - 1208.34\left[5 - \frac{5\times 37}{2\times 29}\right]$$

$$= \frac{6967}{EI}$$

y_{B2} = moment of the M/EI loading due to the redundant R_B between A and B about point B in the conjugate beam (Fig. 3.4c)

$$y_{B2} = \frac{1}{EI}\left[\frac{1}{2}\times 5 \times R_B \times \frac{5\times 8}{13}\left(13 - \frac{2}{3}\times 5\right) + \frac{1}{2}\times 8 \times R_B \times \frac{5\times 8}{13}\left(\frac{2}{3}\times 8\right)\right]\frac{1}{13}\times 5$$

$$- \frac{1}{EI}\left[\frac{1}{2}\times 5 \times R_B \times \frac{5\times 8}{13}\times\frac{5}{3}\right]$$

$$= \left\{\frac{R_B}{EI}[24.17 + 21.33]\times\frac{5}{13} - 4.17\frac{R_B}{EI}\right\}\times\frac{5\times 8}{13}$$

$$= \frac{41}{EI}R_B$$

Compatibility condition requires, $y_{B1} = y_{B2}$

or, $$\frac{6967}{EI} = \frac{41}{EI}R_B$$

or, $$R_B = 169.9 \text{ kN}$$

Reactions Taking moment about A,

$$20\times 13\times\frac{13}{2} - 169.9\times 5 - R_C\times 13 = 0$$

or, $$R_C = 64.65 \text{ kN}$$

$$\therefore \quad R_A = 20\times 13 - R_B - R_C = 25.45 \text{ kN}$$

Let us draw detailed bending moment diagram, and locate the points of maximum positive, maximum negative and zero bending moments.

Maximum Negative Bending Moment

The maximum negative bending moment accurs at the intermediate support, that is, at $x = 5$m from A.

At $x = 5m$ from A,

B.M. due to U.D.L, $$M_1 = \left(\frac{20\times 13}{2}\right)\times 5 - 20\times\frac{5^2}{2} = 400 \text{ kNm}$$

B.M. due to R_B alone, $$M_2 = 169.9\times\frac{5\times 8}{13} = -522.77 \text{ kNm}$$

$\therefore$ Net bending moment $$M = M_2 - M_1 = -122.77 \text{ kNm}$$

Maximum Positive Bending Moment

The maximum positive bending moments occur at the point of zero shear force. The shear force diagram can be drawn as shown in Fig. 3.4d knowing the support reactions. In the span AB, zero shear force occurs at a distance of 1.27m from A, while in span BC, zero shear force occurs at a distance of 4.77m from C.

At $x = 1.27$ m from A,

Net bending moment $$M = 25.45\times 1.27 - 20\times\frac{1.27^2}{2} = 16.19 \text{ kNm}$$

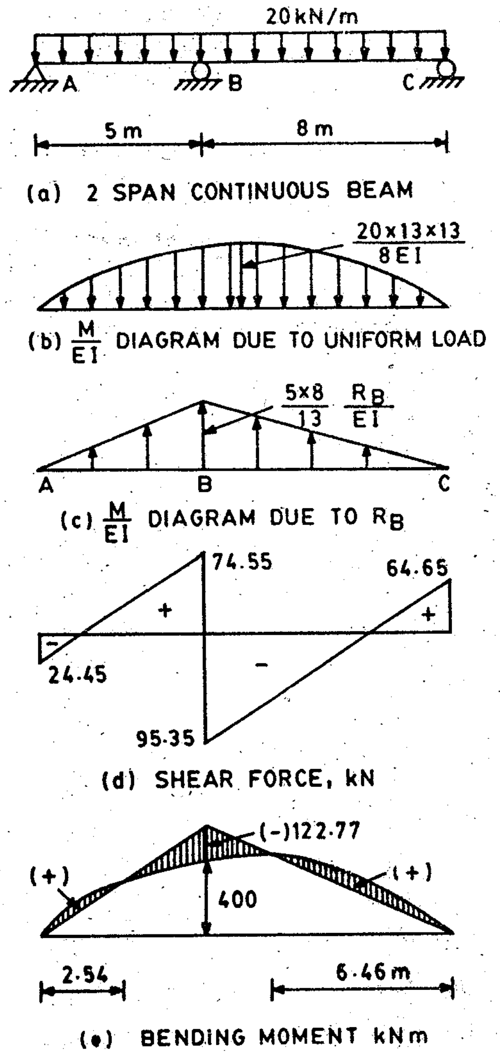


Fig. 3.4

At $x = 4.77$ *m from C,*

$$\text{Net bending moment } M = 64.65 \times 4.77 - 20 \times \frac{4.77^2}{2} = 80.85 \text{ kNm}$$

Points of inflection

Span A – B

Let us assume that point of inflection occurs at x from A.

$$\text{B.M. at x,} \quad \frac{20 \times 13}{2}x - \frac{20x^2}{2} = 169.9 \times \frac{8}{13}x$$

$$\text{or,} \quad x = 2.54 \text{ m}$$

Span B - C

Let us assume that point of inflection occurs at x from C.

$$\text{B. M. at x,} \quad \frac{20 \times 13}{2}x - \frac{20x^2}{2} = 169.9 \times \frac{5}{13}x$$

$$\text{or,} \quad x = 6.46 \text{ m}$$

The bending moment diagram is plotted as follows:

(i) First, the bending moment due to the redundant R_B is plotted. It is hogging in nature and , hence, negative. It is triangular in shape.

(ii) Next, the bending moment due to uniform load in the released structure is plotted. It is sagging in nature and, hence, positive. It is parabolic in shape. It is drawn so as to superimpose on the hogging moment diagram.

(iii) The area common to both the hogging and sagging moments consists of zero bending moment, hence ignored.

(iv) The locations of points of inflection, and points of maximum positive and negative bending moments and their values are indicated on the diagram.

The shear force and bending moment diagrams are shown in Figs. 3.4 d and e.

Example 3.3

Analyze the beam shown in Fig. 3.5a using the method of consistent deformations and draw shear and moment diagrams. Treat R_A as redundant and make use of the law of reciprocal deflections.

Solution

The released beam is shown in Fig. 3.5b. The condition of compatibility requires that the deflection at the free end A due to the combined action of the applied loads and the redundant R_A is zero, that is,

$y_{A1} = -R_A y_{A2}$ (Figs. 3.5b and d)

y_{A1} = Moment of the M/EI loading due to the applied loads about point A in the conjugate beam of Fig. 3.5c

$\Sigma M_B^R = 0$, Taking moment of the loading to the right of hinge B about B, (Fig. 3.5c)

$$\frac{1}{2} \times \frac{36}{EI} \times 1.15 \times \frac{1.15}{3} - \frac{1}{2} \times (2.5 + 1.35) \times \frac{42}{EI}\left(\frac{1.35 + 3.85}{3} + 1.15\right) + R_C \times 5 = 0$$

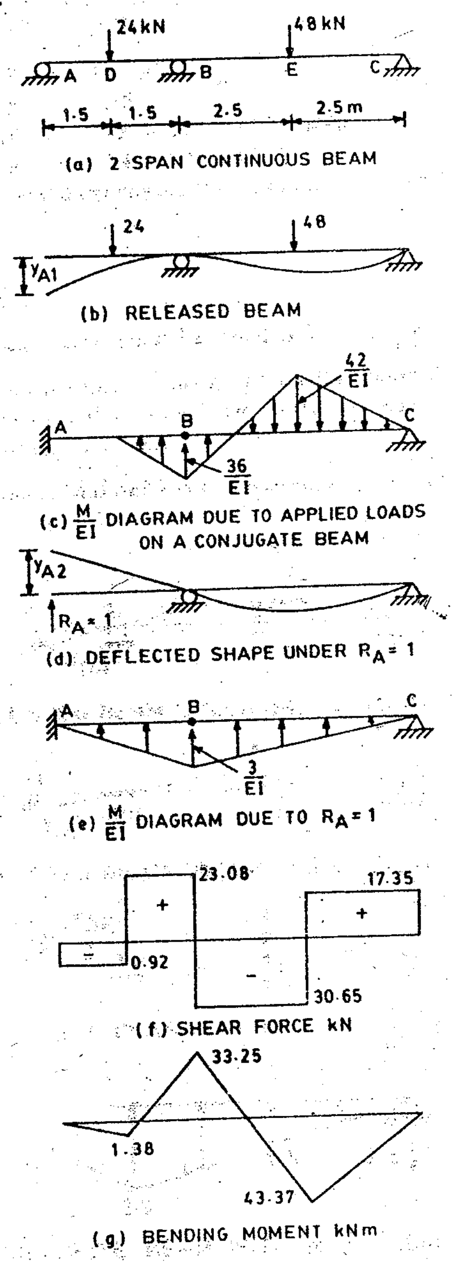


Fig. 3.5

or, $$R_C = \frac{44.98}{EI}$$

$\Sigma F_y = 0$, or, $$R_A + R_C + \frac{1}{2}\times\frac{36}{EI}\times 2.65 - \frac{1}{2}\times\frac{42}{EI}\times 3.85 = 0$$

or, $$R_A = -\frac{11.83}{EI}$$

$\Sigma M_B^L = 0$, Taking moment of the loading to the left of hinge B about B, (Fig. 3.5c)

$$\frac{11.83}{EI}\times 3 - M_A - \frac{1}{2}\times\frac{36}{EI}\times 1.5\times\frac{1.5}{3} = 0$$

or, $$M_A = \frac{22}{EI} = y_{A1} \downarrow$$

y_{A2} = moment of the M/EI loading due to the unit redundant R_A about point A in the conjugate beam of Fig. 3.5e.

$\Sigma F_y = 0$, or, $$R_A + R_C = \frac{1}{2}\times\frac{3}{EI}\times 8 = \frac{12}{EI}$$

$\Sigma M_B^R = 0$, $$\frac{1}{2}\times\frac{3}{EI}\times 5\times\frac{5}{3} = R_C\times 5$$

or, $$R_C = \frac{2.5}{EI}$$

$\therefore$ $$R_A = \frac{9.5}{EI}$$

$\Sigma M_B^L = 0$, $$R_A\times 3 - \frac{1}{2}\times\frac{3}{EI}\times 3\times\frac{3}{3} - M_A = 0$$

or, $$M_A = \frac{24}{EI} = y_{A2} \uparrow$$

For compatibility, $y_{A1} = -R_A\ y_{A2}$ (here R_A = reaction at A in the original beam of Fig. 3.5a)

or, $$\frac{22}{EI} = R_A\times\frac{24}{EI}$$

or, $$R_A = 0.917\text{ kN} \approx 0.92\text{ kN}$$

Reactions

$\Sigma M_C = 0$, (Fig. 3.5a)

$$48 \times 2.5 + 24 \times 6.5 - R_A \times 8 - R_B \times 5 = 0$$

or, $R_B = 53.73$ kN

$\Sigma F_y = 0$, $R_A + R_B + R_C = 24 + 48$

or, $R_C = 17.35$ kN

Moment at B $= R_C \times 5 - 48 \times 2.5 = -33.25$ kNm

$= R_A \times 3 - 24 \times 1.5 = -33.25$ kNm O. K.

The resulting shear and moment diagrams are shown in Figs. 3.5 f and g.

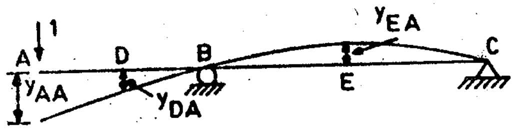
Alternatively

Law of Reciprocal Deflections,

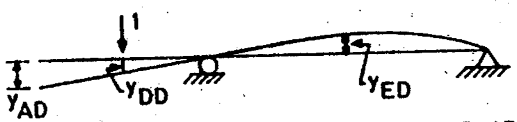
The Maxwell's law of reciprocal theorem may be stated as follows (see 9.14, volume 1):

The deflection of point 1 on a structure due to a load P at point 2 is equal to the deflection of point 2 due to the load P at point 1. Of course, the deflections referred to are in the same directions as the applied loads.

With reference to Figs. 3.6 a and b, Maxwell's law of reciprocal deflections gives,

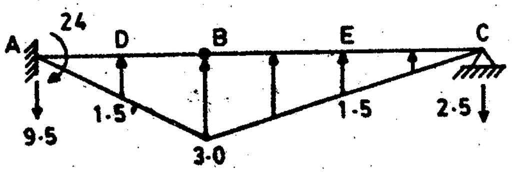


(a) RELEASED BEAM UNDER UNIT LOAD AT A



(b) RELEASED BEAM UNDER UNIT LOAD AT D

(c) RELEASED BEAM UNDER UNIT LOAD AT E



(d) CONJUGATE BEAM WITH M/EI LOADS

Fig. 3.6

$$y_{AD} = y_{DA} \qquad \text{(i)}$$

With reference to Figs. 3.6a and c, Maxwell's law of reciprocal deflections gives,

$$y_{AE} = y_{EA} \qquad \text{(ii)}$$

Using the conjugate beam method with reference to Fig. 3.6d,

$$y_{EA} = \frac{2.5}{EI} \times 2.5 - \frac{1}{2} \times \frac{1.5}{EI} \times 2.5 \times \frac{2.5}{3} = \frac{4.69}{EI} = y_{AE}$$

$$y_{DA} = \frac{9.5}{EI} \times 1.5 - 24 - \frac{1}{2} \times \frac{1.5}{EI} \times 1.5 \times \frac{1.5}{3} = \frac{-10.30}{EI} = y_{AD}$$

$$y_{AD} \text{ due to 24 kN load at D} = \frac{24 \times 10.30}{EI} = \frac{247.2}{EI} \downarrow$$

$$y_{AE} \text{ due to 48 kN load at E} = \frac{48 \times 4.69}{EI} = \frac{225.12}{EI} \uparrow$$

$$\text{Net deflection at A} = \frac{22.08}{EI} \downarrow \approx \frac{22}{EI} \text{ obtained earlier.} \qquad \text{O. K.}$$

Thus, it is easier to calculate the deflections under the applied loads using the law of reciprocal deflections rather than calculating them using conjugate beam method alone.

3.4 BEAMS WITH TWO OR MORE REDUNDANTS

Example 3.4

Calculate the end reactions in the fixed ended beam of Fig. 3.7a. Draw shear force and bending moment diagrams.

Solution

The axial deformation in the beam is ignored. The degree of indeterminacy is 2. Choose M_A and M_B as two redundants and a simply supported beam AB as shown in Fig. 3.7 b is obtained. The values of M_A and M_B will be such so as to make the net slopes θ_A and θ_B equal to zero.

θ_{A1} = slope at A due to external loading.

= shear at A in the conjugate beam of Fig. 3.7c.

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{wL^2}{8EI} \times L$$

θ_{A2} = slope at A due to redundant M_A in the conjugate beam of Fig.3.7d

$$= \frac{1}{2} \times \frac{M_A}{EI} \times L \times \frac{2}{3}$$

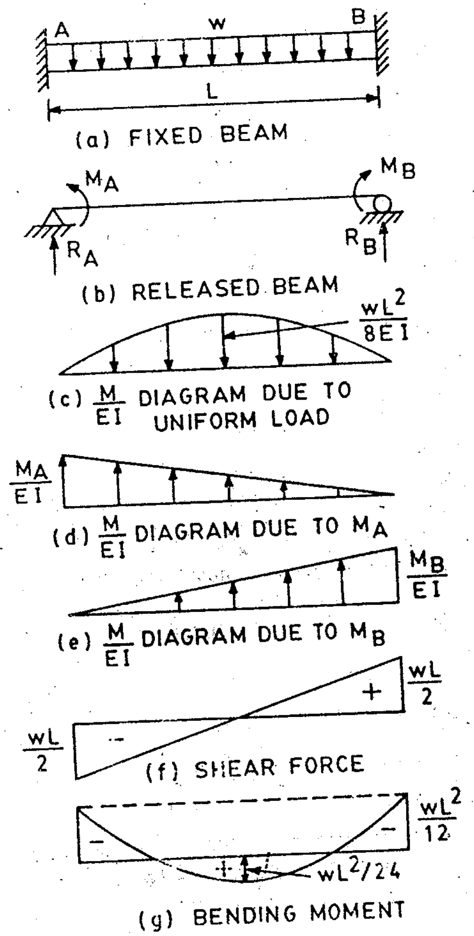


Fig. 3.7

θ_{A3} = slope at A due to redundant M_B in the conjugate beam of Fig.3.7e

$$= \frac{1}{2} \times \frac{M_B}{EI} \times L \times \frac{1}{3}$$

Due to symmetry, $M_A = M_B$

For compatibility,

$$\theta_{A1} - \theta_{A2} - \theta_{A3} = 0$$

or, $$\frac{wL^3}{24EI} - \frac{M_A L}{3EI} - \frac{M_A L}{6EI} = 0$$

$$\therefore \quad M_A = \frac{wL^2}{12} = M_B \quad \text{(hogging)}$$

Reaction $R_A = \dfrac{wL}{2} = R_B$

Maximum moment at mid span $= \dfrac{wL^2}{8} - \dfrac{wL^2}{12} = \dfrac{wL^2}{24}$ (sagging)

The resulting shear and moment diagrams are shown in Figs. 3.7 f and g. Bending moment diagram is plotted as follows:

(i) First, the end moment ordinates M_A and M_B are plotted and joined by a straight line. This is called as *end moment diagram*. In the present case, it is hogging in nature, hence negative.
(ii) Next, free span moment is determined assuming the beam is simply supported and carrying the uniform load. It is sagging in nature, hence positive.
(iii) The free span sagging moment is plotted taking the dash line of the end moment diagram as the base line. The sagging moment ordinates are measured vertically and not perpendicular to the base line.
(iv) The areas common to the negative and positive moments are ignored.
(v) A cantilever span is always statically determinate. Hence, end moment diagram is never plotted in the cantilever span. Only the statically determinate bending moment is plotted.

Example 3.5

Analyze the beam shown in Fig. 3.8a using the consistent deformation method.

Solution

The beam is indeterminate to a degree equal to 3. If, its axial deformation is ignored, its degree of indeterminacy reduces to 2. Choose R_C and M_C as redundants and remove the support C. The compatibility conditions require that the net vertical deflection at C and slope at C under the combined action of the applied load and redundants must be zero, that is, (Figs. 3.8 b, d and f):

$$y_C + y'_C + y''_C = 0 \qquad \text{(i)}$$

$$\theta_C + \theta'_C + \theta''_C = 0 \qquad \text{(ii)}$$

θ_C = slope at C due to external load on the beam [using the moment-area method, Fig. 3.8b]

$$= \frac{Ma}{EI} \quad \text{clockwise}$$

y_C = deflection at C due to external load

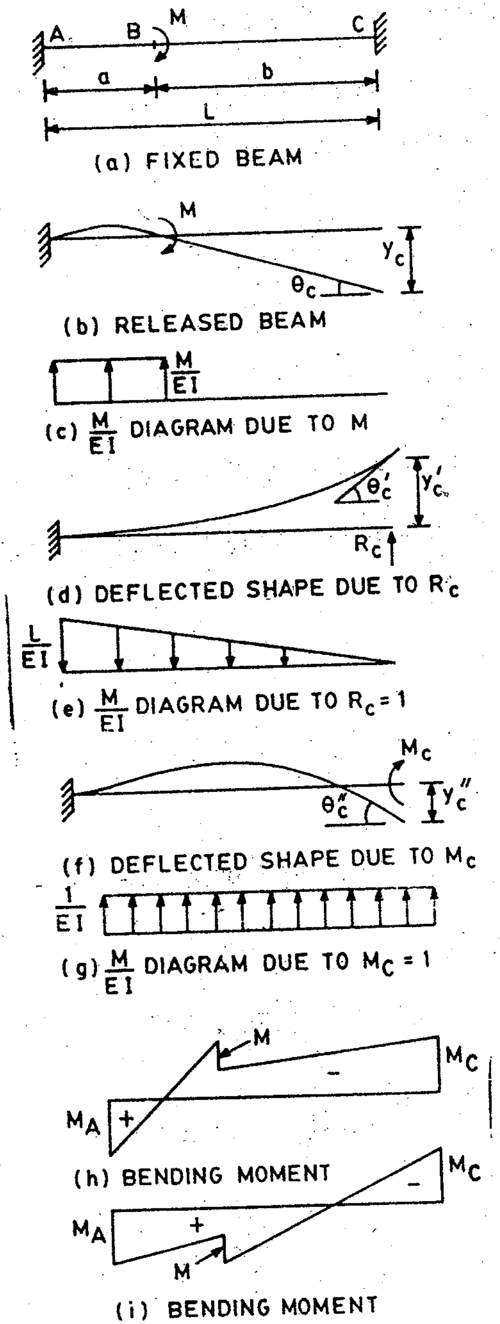


Fig. 3.8

$$y_C = \frac{Ma\left(b+\frac{a}{2}\right)}{EI} = \frac{Ma(a+2b)}{2EI}\downarrow$$

θ'_C = slope at C due to unit upward load at C (Figs. 3.8 d and e)

$$= -\frac{1}{2}\frac{L\times L}{EI} = -\frac{L^2}{2EI} \quad \text{anti clockwise}$$

y'_C = deflection at C due to unit upward load at C

$$= \frac{L^2}{2EI}\times\frac{2}{3}L = \frac{L^3}{3EI}\uparrow$$

θ''_C = slope at C due to unit clockwise rotation at C (Figs. 3.8 f and g)

$$= \frac{L}{EI} \quad \text{clockwise}$$

y''_C = deflection at C due to unit clockwise rotation at C

$$= \frac{L^2}{2EI}\downarrow$$

Substituting the values of θ'_s and y'_s in the compatibility equations gives,

$$\frac{Ma}{EI} - \frac{L^2}{2EI}R_C + \frac{L}{EI}M_C = 0 \qquad \text{(iii)}$$

$$\frac{Ma(a+2b)}{2EI} - \frac{L^3}{3EI}R_C + \frac{L^2}{2EI}M_C = 0 \qquad \text{(iv)}$$

Solution of Eqs. (iii) and (iv) gives,

$$M_C = \frac{Ma}{L^2}(2b-a)$$

and

$$R_C = \frac{6Mab}{L^3}\uparrow = -R_A$$

and

$$M_A = \frac{Mb}{L^2}(2a-b)$$

Two possible bending moment diagrams depending upon the relative values of a, b and M are shown in Figs. 3.8 h and i.

Example 3.6

Analyze the beam shown in Fig 3.9a using the consistent deformation method.

Solution

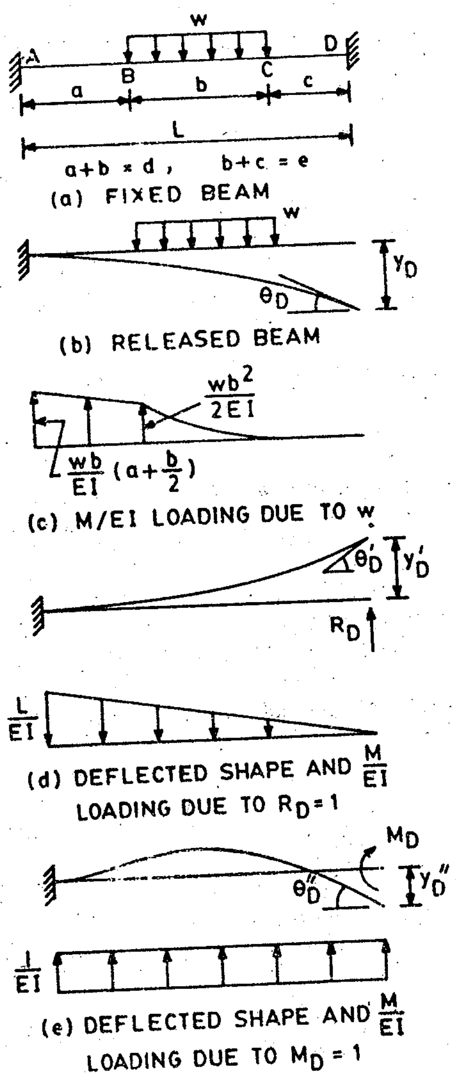


Fig. 3.9

The beam is statically indeterminate to a degree equal to 2. Choose R_D and M_D as redundants and remove the support D. The compatibility conditions require that the net

slope and deflection at D under the combined action of the applied load and redundants must be zero, (Figs. 3.9 b, d and e):

that is,

$$\theta_D - \theta'_D R_D + \theta''_D M_D = 0 \qquad \text{(i)}$$

and

$$y_D - y'_D R_D + y''_D M_D = 0 \qquad \text{(ii)}$$

θ_D = slope at D due to external load on the beam (Fig. 3.9b and c) (using the moment - area method)

$$= \frac{wb^3}{6EI} + \frac{w}{2}\left(ab + \frac{b^2}{2} + \frac{b^2}{2}\right)\frac{a}{EI}$$

$$= \frac{w}{6EI}\left[b^3 + 3ab(a+b)\right] \text{ clockwise}$$

y_D = deflection at D due to external load on the beam

$$= \frac{wb^3}{6EI}\left(c + \frac{3b}{4}\right) + \frac{w}{2EI}ab(a+b)\left\{c + b + \frac{a(3b+4a)}{6(a+b)}\right\}$$

θ'_D = slope at D due to unit upward load at D (Fig. 3.9 d)

$$= -\frac{L^2}{2EI} \text{ anti clockwise}$$

$$y'_D = -\frac{L^3}{3EI}\uparrow$$

θ''_D = slope at D due to unit clockwise moment at D (Fig. 3.9e)

$$\theta''_D = \frac{L}{EI} \text{ clockwise}$$

$$y''_D = \frac{L^2}{2EI}\downarrow$$

Substituting the values of θ'_S and y'_S in the compatibility equations:

$$\frac{w}{6EI}\left[b^3 + 3ab(a+b)\right] - \frac{L^2}{2EI}R_D + \frac{L}{EI}M_D = 0 \qquad \text{(iii)}$$

and

$$\frac{wb^3}{6EI}\left(c + \frac{3b}{4}\right) + \frac{w}{2EI}\left[ab(a+b)(c+b) + \frac{a^2b}{6}(3b+4a)\right]$$

$$-\frac{L^3}{3EI}R_D + \frac{L^2}{2EI}M_D = 0 \qquad \text{(iv}$$

Eq. (iii) gives,

$$R_D = \left\{\frac{w}{6EI}\left[b^3 + 3ab(a+b)\right] + \frac{L}{EI}M_D\right\}\frac{2EI}{L^2}$$

Substituting the value of R_D in Eq. (iv) gives,

$$M_D = \frac{w}{12L^2}\left\{3b^3(4c+3b) + 6ab\left[6(a+b)(b+c) + a(4a+3b)\right] - 8L\left[b^3 + 3ab(a+b)\right]\right\}$$

Substituting $c = L - d$, $b = d - a$ and simplifying leads to:

$$M_D = \frac{w}{12L^2}\left[d^3(4L-3d) - a^3(4L-3a)\right] \quad \textit{clockwise and hogging}$$

By considering the equilibrium of the beam,

$$M_A = \frac{w}{12L^2}\left[e^3(4L-3e) - c^3(4L-3c)\right] \quad \textit{anti clockwise and hogging}$$

Example 3.7

Determine the end moments at A and D in the non-prismatic beam of Fig. 3.10a using the consistent deformation method.

Solution

The beam is statically indeterminate to a degree 2. Choose M_A and M_D as redundants and introduce moment release at A and D. The released beam is shown in Fig. 3.10 b. The compatibility condition requires that net slope at the supports under the combined action of the applied load and redundants must be zero, that is (Fig. 3.10 b and d)

$$\theta_A - \theta'_A - \theta''_A = 0 \qquad \text{(i)}$$

and $$\theta_D - \theta'_D - \theta''_D = 0 \qquad \text{(ii)}$$

θ''_A and θ''_D are end slopes due to a redundant clockwise moment M_D applied at D (not shown). The slopes can be evaluated using the conjugate beam method. Due to symmetry, only the slopes at end A need be evaluated.

$$\text{Moment at B in Fig. 3.10 b} = \frac{wL}{2}\times\frac{L}{4} - w\times\left(\frac{L}{4}\right)^2\times\frac{1}{2} = \frac{3}{32}wL^2$$

$$\therefore \quad \text{M/ EI ordinate } B-2 = \frac{3}{32}\frac{wL^2}{EI} \quad \text{(Fig.3.10c)}$$

$$\text{M/ EI ordinate } B-1 = \frac{3}{64}\frac{wL^2}{EI}$$

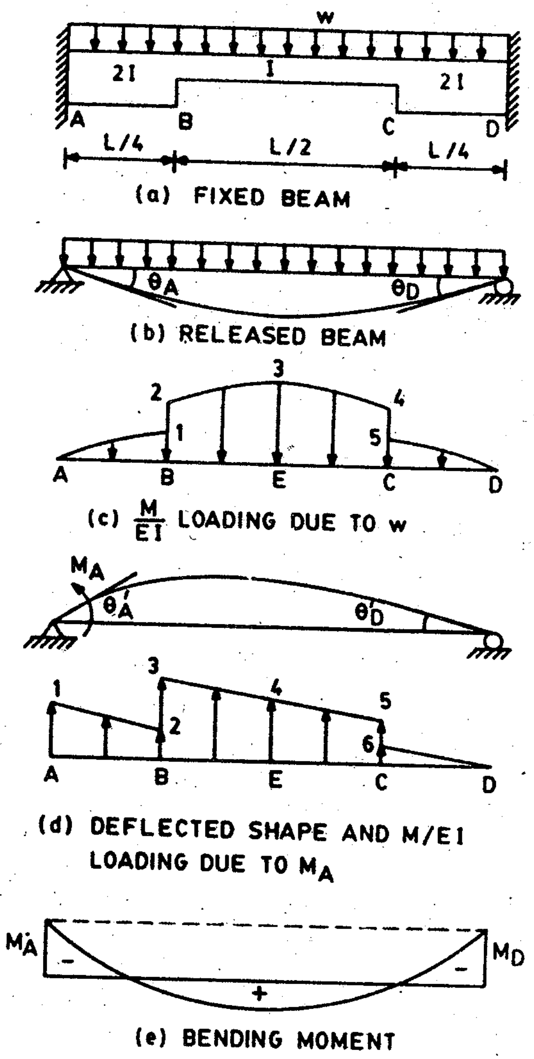


Fig. 3.10

$$\text{M/ EI ordinate } E-3 = \frac{wL^2}{8EI}$$

Let us first calculate areas of various M/EI loading segments and their center of gravity.
Area of segment A − B − 1 (Fig. 3.10 c) :

$$A_1 = \frac{w\left(\frac{L}{4}\right)^2}{12}\frac{\left(3L - 2\times\frac{L}{4}\right)}{2EI} = \frac{5}{768}\frac{wL^3}{EI}$$

Area of segment B – 2 – 3 – E (Fig. 3.10 c)

$$A_2 = \frac{w}{12}\left[3L\left(\frac{L^2}{4}-\frac{L^2}{16}\right)-2\left(\frac{L^3}{8}-\frac{L^3}{64}\right)\right]\frac{1}{EI} = \frac{11}{384}\frac{wL^3}{EI}$$

$\theta_A = A_1 + A_2$, using the conjugate beam method

$$\text{Area of segment } A-B-2-1 \text{ (Fig. 3.10 d)} = \frac{L}{4}\left(\frac{\frac{1}{2}+\frac{3}{8}}{2}\right)\frac{M_A}{EI} = \frac{7}{64}\frac{M_AL}{EI}$$

$$\text{c.g. of segment } A-B-2-1 \text{ from D} = \frac{3L}{4}+\frac{1}{3}\times\frac{L}{4}\left[\frac{\frac{3}{8}+2\times\frac{1}{2}}{\frac{3}{8}+\frac{1}{2}}\right] = \frac{37}{42}L$$

$$\text{c.g. of segment } A-B-2-1 \text{ from A} = L-\frac{37}{42}L = \frac{5}{42}L$$

$$\text{Area of segment } B-C-5-3 \text{ (Fig. 3.10 d)} = \frac{L}{2}\left[\frac{\frac{3}{4}+\frac{1}{4}}{2}\right]\frac{M_A}{EI} = \frac{M_AL}{4EI}$$

$$\text{c.g. of segment } B-C-5-3 \text{ from D} = \frac{L}{4}+\frac{1}{3}\times\frac{L}{2}\left[\frac{\frac{1}{4}+2\times\frac{3}{4}}{1}\right] = \frac{13}{24}L$$

$$\text{c.g. of segment } B-C-5-3 \text{ from A} = L-\frac{13}{24}L = \frac{11}{24}L$$

$$\text{Area of segment } C-D-6 \text{ (Fig. 3.10 d)} = \frac{1}{2}\times\frac{L}{4}\times\frac{M_A}{8EI} = \frac{M_AL}{64EI}$$

$$\text{c.g. of segment } C-D-6 \text{ from D} = \frac{2}{3}\times\frac{L}{4} = \frac{L}{6}$$

$$\text{c.g. of segment } C-D-6 \text{ from A} = L-\frac{L}{6} = \frac{5L}{6}$$

Using the conjugate beam method,

$$\theta'_A = R_A = \left[\frac{7}{64}\frac{M_AL}{EI}\times\frac{37}{42}+\frac{M_AL}{4EI}\times\frac{13}{24}+\frac{M_AL}{64EI}\times\frac{1}{6}\right]$$

$$= \frac{90}{384}\frac{M_AL}{EI}$$

$$\theta'_D = R_D = \left[\frac{7}{64}\times\frac{M_AL}{EI}\times\frac{5}{42}+\frac{M_AL}{4EI}\times\frac{11}{24}+\frac{M_AL}{64EI}\times\frac{5}{6}\right]$$

$$= \frac{54}{384}\frac{M_AL}{EI} = \theta''_A$$

Compatibility condition : Substituting values of the slopes at A in Eq.(i) gives

$$\left(\frac{5}{768}+\frac{11}{384}\right)\frac{wL^3}{EI}-\frac{90}{384}\frac{M_AL}{EI}-\frac{54}{384}\frac{M_AL}{EI} = 0$$

or, $M_A = \dfrac{3}{32}wL^2$ *hogging*

Maximum positive bending moment occurs at the midspan where shear force is zero.

$$\text{Moment at } L/2 = \frac{wL}{2}\times\frac{L}{2}-M_A-w\left(\frac{L}{2}\right)^2\frac{1}{2}$$

$$= \frac{wL^2}{8}-\frac{3wL^2}{32} = \frac{wL^2}{32} \quad \textit{sagging}$$

Net bending moment diagram is shown in Fig. 3.10e.. It is obtained by superimposing the end moment diagram and free span moment diagram.

Example 3.8

Analyze the continuous beam shown in Fig. 3.11a using the consistent deformation method and draw shear and moment diagrams.

Solution

The beam is statically indeterminate to a degree 2. Select R_C and R_D acting upward as redundants. By removing the supports at C and D, simply supported beam AE is chosen as the released beam as shown in Fig. 3.11b. The compatibility conditions

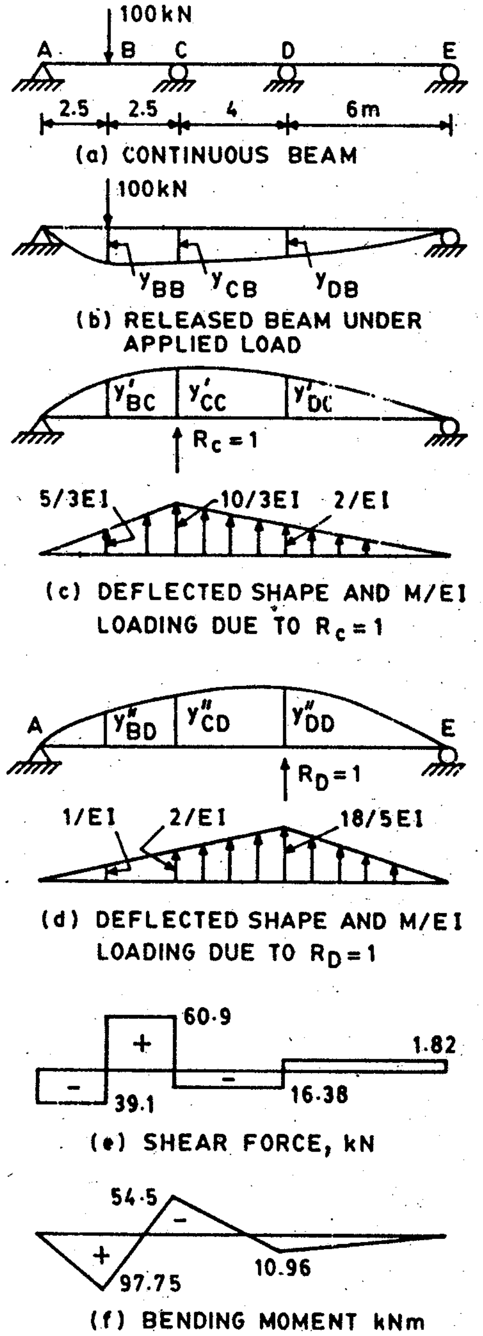


Fig. 3.11

require that the deflections at both C and D of the released beam under the combined action of the applied loads and redundants R_C and R_D must be zero, that is : (Fig. 3.11 b, c and d)

$$y_{CB} = y'_{CC} R_C + y''_{CD} R_D \qquad \text{(i)}$$

and $$y_{DB} = y'_{DC} R_C + y''_{DD} R_D \qquad \text{(ii)}$$

where, y_{ij} = deflection at i due to certain load applied at j.

y'_{BC} = moment of the M/EI loading diagram due to unit load at C about B in the conjugate beam of Fig. 3.11 c

$$= \frac{1}{2} \times 15 \times \frac{10}{3EI} \times \left(\frac{10+15}{3}\right)\left(\frac{2.5}{15}\right) - \frac{1}{2} \times 2.5 \times \frac{5}{3EI} \times \frac{2.5}{3} = \frac{33}{EI}$$

$$y'_{CC} = \frac{1}{2} \times 15 \times \frac{10}{3EI} \times \left(\frac{10+15}{3}\right) \times \frac{5}{15} - \frac{1}{2} \times 5 \times \frac{10}{3EI} \times \frac{5}{3} = \frac{55.56}{EI}$$

y'_{DC} = moment of the M/EI loading diagram about D in the conjugate beam of Fig. 3.11 c

$$= \frac{1}{2} \times 15 \times \frac{10}{3EI} \times \left(\frac{5+15}{3}\right)\frac{6}{15} - \frac{1}{2} \times 6 \times \frac{2}{EI} \times \frac{6}{3}$$

$$= \frac{54.67}{EI}$$

y''_{BD} = moment of the M/EI loading diagram (due to the unit load at D) about B in the conjugate beam of Fig. 3.11 d

$$= \frac{1}{2} \times 15 \times \frac{18}{5EI} \times \left(\frac{6+15}{3}\right)\frac{2.5}{15} - \frac{1}{2} \times 2.5 \times \frac{1}{EI} \times \frac{2.5}{3} = \frac{30.46}{EI}$$

y''_{CD} = y'_{DC} using the reciprocal theorem (Figs. 3.11 c and d)

y''_{DD} = moment of the M/EI loading diagram about D in the conjugate beam of Fig. 3.11 d

$$= \frac{1}{2} \times 15 \times \frac{18}{5EI} \times \left(\frac{9+15}{3}\right)\frac{6}{15} - \frac{1}{2} \times 6 \times \frac{18}{5EI} \times \frac{6}{3} = \frac{64.8}{EI}$$

Using the Maxwell's reciprocal theorem:

$$y_{CB} = 100 \times y'_{BC} = \frac{3300}{EI}$$

$$y'_{DC} = 100 \times y''_{BD} = \frac{3046}{EI}$$

Using the compatibility Eqs. (i) and (ii)

$$3300 = 55.56 R_C + 54.67 R_D$$
$$3046 = 54.67 R_C + 64.80 R_D$$

Matrix solution,

$$\begin{bmatrix} 55.56 & 54.67 \\ 54.67 & 64.80 \end{bmatrix} \begin{Bmatrix} R_C \\ R_D \end{Bmatrix} = \begin{Bmatrix} 3300 \\ 3046 \end{Bmatrix} \quad \text{or,} \quad \begin{Bmatrix} R_C \\ R_D \end{Bmatrix} = \begin{Bmatrix} 77.28 \\ -18.20 \end{Bmatrix}$$

Using the equations of static equilibrium,

$$R_A = 39.10 \text{ kN} \quad \text{and} \quad R_E = 1.82 \text{ kN}$$

The shear and moment diagrams are shown in Figs. 3.11 e and f.

3.5 REACTIONS DUE TO YIELDING OF SUPPORTS

Example 3.9

Determine reactions in a fixed ended beam due to a vertical settlement of Δ at support B as shown in Fig. 3.12 a. Draw shear force and bending moment diagrams.

Solution

The degree of indeterminacy is 2. Select R_B and M_B as redundants. By removing the support B, a cantilever beam AB is obtained as the released beam. The compatibility conditions require that under the combined action of the applied load and redundants, net vertical deflection at B should be Δ and net slope must be zero(Figs. 3.12b, d and e) :

that is,

$$y_B + y'_B - y''_B = \Delta \qquad \text{(i)}$$

$$\theta_B + \theta'_B - \theta''_B = 0 \qquad \text{(ii)}$$

R_B is applied upward while M_B is applied clockwise. The slopes and deflections can be obtained using the moment area theorems.

y_B = moment of the M/EI loading diagram about B in the beam of Fig. 3.12 c

$$= \frac{wL^3}{6EI} \times \frac{3L}{4} = \frac{wL^4}{8EI} \downarrow$$

θ_B = area of the M/EI diagram (Fig. 3.12 c)

$$= \frac{wL^3}{6EI} \quad \text{clockwise}$$

$$y'_B = \frac{M_B}{EI} L \times \frac{L}{2} = \frac{M_B L^2}{2EI} \downarrow \quad \text{(Fig. 3.12 d)}$$

$$\theta'_B = \frac{M_B L}{EI} \quad \text{clockwise}$$

$$y''_B = \frac{1}{2}\frac{R_B L^2}{EI} \times \frac{2}{3} L = \frac{R_B L^3}{3EI} \uparrow \text{(Fig. 3.12 e)}$$

$$\theta''_B = \frac{R_B L^2}{2EI} \quad \text{anti clockwise}$$

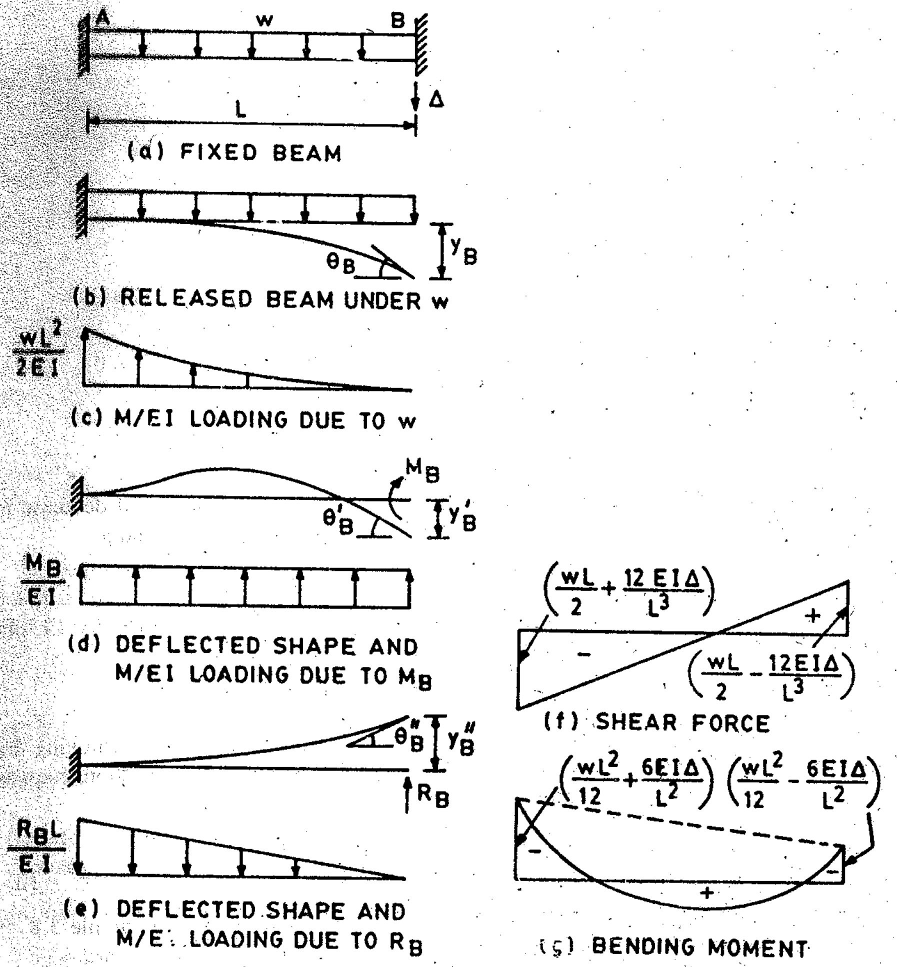


Fig. 3.12

Substituting the values in Eqs. (i) and (ii) give,

$$\frac{wL^4}{8EI}+\frac{M_BL^2}{2EI}-\frac{R_BL^3}{3EI}=\Delta \qquad \text{(iii)}$$

and
$$\frac{wL^3}{6EI}+\frac{M_BL}{EI}-\frac{R_BL^2}{2EI}=0 \qquad \text{(iv)}$$

Solution of Eqs.(iii) and (iv) gives,

$$R_B = \frac{wL}{2}-\frac{12EI}{L^3}\Delta$$

and
$$M_B = \frac{wL^2}{12}-\frac{6EI}{L^2}\Delta$$

$$\Sigma F_y = 0 \text{ gives, } R_A = \frac{wL}{2}+\frac{12EI}{L^3}\Delta$$

$$\Sigma M_A = 0 \text{ gives, } M_A+\left(\frac{wL}{2}-\frac{12EI}{L^3}\Delta\right)L-\frac{wL^2}{12}+\frac{6EI}{L^2}\Delta-\frac{wL^2}{2}=0$$

or,
$$M_A = \frac{wL^2}{12}+\frac{6EI}{L^2}\Delta$$

Thus, yielding of support B results in an increase in moment at A and decrease in moment at B. The shear and moment diagrams are shown in Figs. 3.12 f and g.

Example 3.10

Analyze the beam shown in Fig. 3.13 a due to settlement of support C by 10 mm using the method of consistent deformation. Take EI = constant.

Solution

The degree of indeterminacy is 2. Select R_C and R_D as redundants. By removing the supports C and D, the released structure is obtained. The compatibility conditions require (Refer Figs. 3.11 c and d of Example 3.8) :

$$y'_{CC}\,R_C + y''_{CD}\,R_D - 0.01 = 0 \qquad \text{(i)}$$

and
$$y'_{DC}\,R_C + y''_{DD}\,R_D = 0 \qquad \text{(ii)}$$

The values of y'_{CC}, y'_{DC}, y''_{CD} and y''_{DD} can be obtained from Example 3.8. It should be noted that in Example 3.8, R_C and R_D are assumed to be acting upward.

$$55.56\ R_C + 54.67\ R_D - 0.01\ EI = 0$$
$$54.67\ R_C + 64.80\ R_D = 0$$

or,
$$\begin{bmatrix}55.56 & 54.67\\ 54.67 & 64.80\end{bmatrix}\begin{Bmatrix}R_C\\ R_D\end{Bmatrix}=\begin{Bmatrix}0.01EI\\ 0\end{Bmatrix}=\begin{Bmatrix}400\\ 0\end{Bmatrix}$$

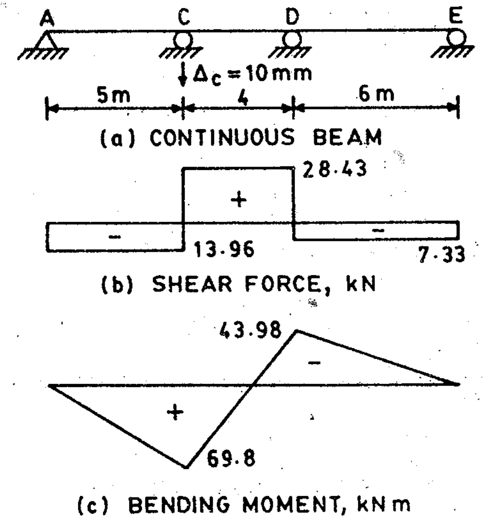


Fig. 3.13

or
$$R_C = 42.39 \text{ kN} \uparrow$$
$$R_D = -35.76 \text{ kN} \downarrow$$

In the present problem, since support C sinks, the correct direction of R_C is downward and that of R_D is upward. Therefore, the reactions R_A and R_E are 13.96 kN ↑ and − 7.33 kN ↓, respectively.

The shear force and bending moment diagrams are shown in Figs. 3.13 b and c.

3.6 FRAMES

Example 3.11

Figure 3.14 a shows a portal frame with hinged supports A and F. Draw shear force and bending moment diagrams and elastic curve of the frame.

Solution

The statical indeterminacy of the frame is 1. Choose R_{FX} as redundant. Replace the hinge support by a roller support at F to obtain the released structure as shown in Fig. 3.14 b. The compatibility condition requires that the net horizontal deflection at F under the combined action of the applied loads and redundant R_{FX} should be zero. The deflections can be calculated using the unit load method as discussed in sec 9.11 and 10.3 of volume 1 of this book. The free body diagram due to applied loads on the frame is shown in Fig. 3.14c, and that due to unit redundant $R_{Fx} = 1$ in Fig. 3.14f.

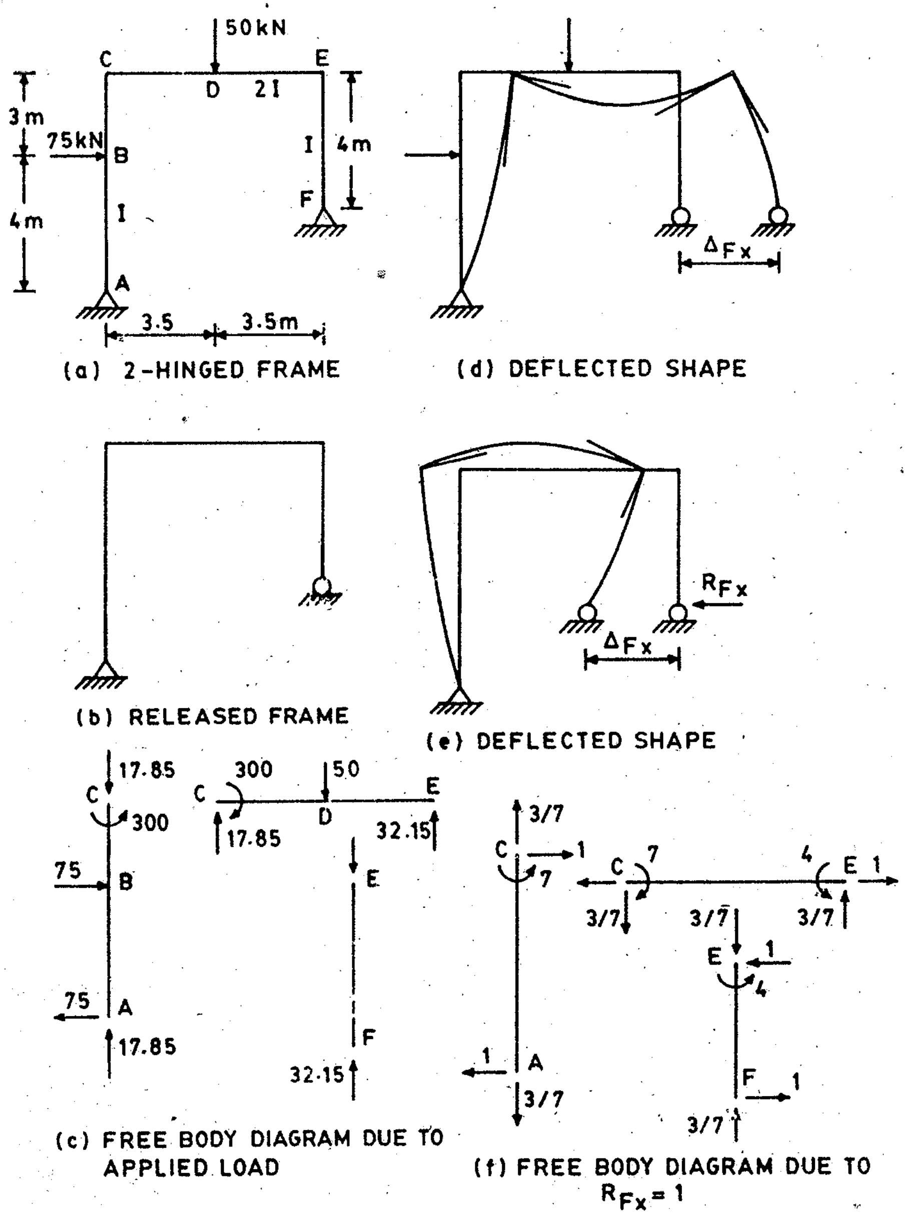


Fig. 3.14 Portal frame with hinged supports (continue)

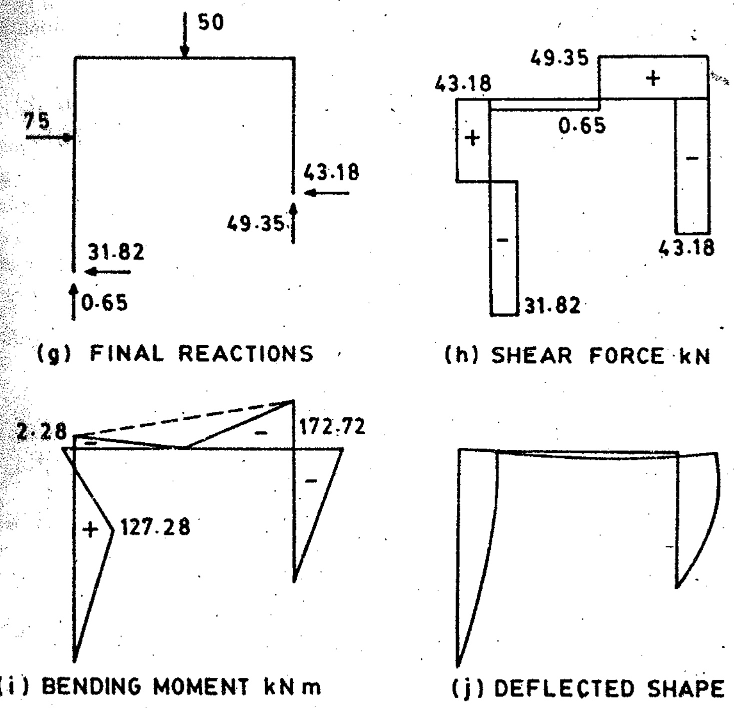


Fig. 3.14 Portal frame with hinged supports

Horizontal Deflection at F due to Applied Loads

Segment	FE	ED	DC	CB	BA
Origin	F	E	D	C	B
Limits	0 – 4	0 – 3.5	0 – 3.5 m	0 – 3	0 – 4 m
I	I	2I	2I	I	I
M (Fig. 3.14 c)	0	67.85x	67.85(x + 3.5) – 50x = (17.85x + 237.48)	300	(300 – 75x)
m (Fig. 3.14 f)	x	$4+\frac{3x}{7}$	$4+\frac{3}{7}(x+3.5)$ $=5.5+\frac{3x}{7}$	(7 – x)	7 – (x+3) = 4 – x

$$\Delta_{Fx} = \int_0^L \frac{Mm\,dx}{EI}$$

$$= 0 + \int_0^{3.5} 67.85x\left(4+\frac{3x}{7}\right)\frac{dx}{2EI} + \int_0^{3.5}(17.85x+237.48)\times\left(5.5+\frac{3x}{7}\right)\frac{dx}{2EI} +$$

$$\int_0^{3} 300(7-x)\frac{dx}{EI} + \int_0^{4}(300-75x)(4-x)\frac{dx}{EI}$$

$$= \frac{33.925}{EI}\left|4\frac{x^2}{2}+\frac{3}{7}\frac{x^3}{3}\right|_0^{3.5} + \frac{0.5}{EI}\left|98.175\frac{x^2}{2}+7.65\frac{x^3}{3}+1306.14x+\right.$$

$$101.78\frac{x^2}{2}\Big|_0^{3.5} + \frac{300}{EI}\left|7x-\frac{x^2}{2}\right|_0^{3} + \frac{1}{EI}\left|1200x-600\frac{x^2}{2}+75\frac{x^3}{3}\right|_0^{4}$$

$$\Delta_{Fx} = \frac{10541.86}{EI}$$

Horizontal Deflection at F due to a Unit Load at F

$$\Delta'_{Fx} = \int_0^L \frac{mm\,dx}{EI}$$

$$= \int_0^4 \frac{x^2dx}{EI} + \int_0^{3.5}\left(4+\frac{3x}{7}\right)^2\frac{dx}{2EI} + \int_0^{3.5}\left(5.5+\frac{3x}{7}\right)^2\frac{dx}{2EI} + \int_0^3 (7-x)^2\frac{dx}{EI} + \int_0^4 (4-x)^2\frac{dx}{EI}$$

$$= \frac{1}{EI}\left|\frac{x^3}{3}\right|_0^4 + \frac{1}{2EI}\left|16x+3.43\frac{x^2}{2}+0.1836\frac{x^3}{3}\right|_0^{3.5}$$

$$+\frac{1}{2EI}\left|30.25x+4.71\frac{x^2}{2}+0.1836\frac{x^3}{3}\right|_0^{3.5} + \frac{1}{EI}\left|49x-14\frac{x^2}{2}+\frac{x^3}{3}\right|_0^{3}$$

$$+\frac{1}{EI}\left|16x-8\frac{x^2}{2}+\frac{x^3}{3}\right|_0^{4}$$

$$\Delta'_{Fx} = \frac{244.16}{EI}$$

For compatibility,

$$\Delta_{Fx} + \Delta'_{Fx}\,R_{Fx} = 0$$

or, $R_{Fx} = (-)\ 10541.86/244.16 = (-)\ 43.18$ kN

The reaction R_{Fx} is acting towards left. Final reactions are shown in Fig. 3.14 g. The resulting shear force and bending moment diagrams are shown in Figs. 3.14 h and i. The deflected shape of the frame is shown in Fig. 3.14 j.

Example 3.12

Figure 3.15 a shows a portal frame with fixed supports A and F. Draw shear force and bending moment diagrams.

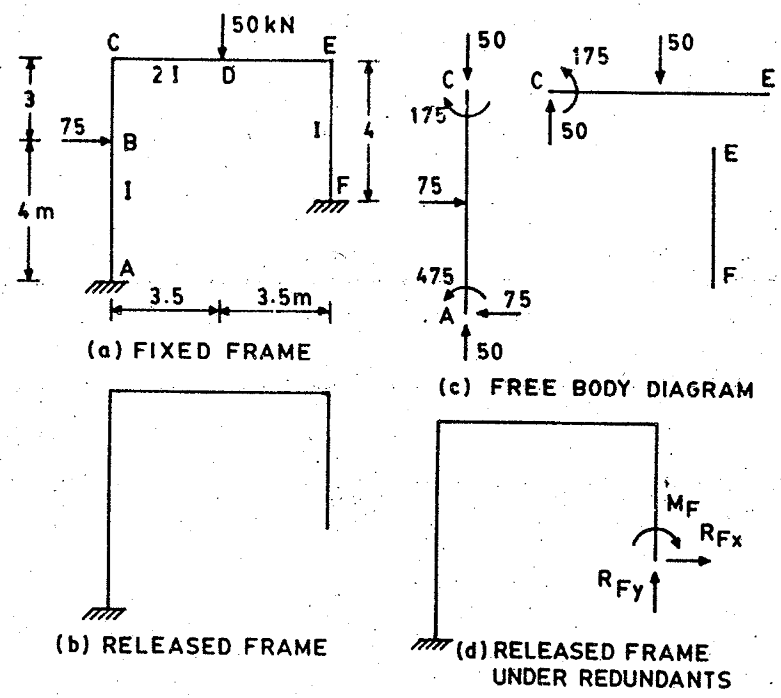


Fig. 3.15 Portal frame with fixed supports (continue)

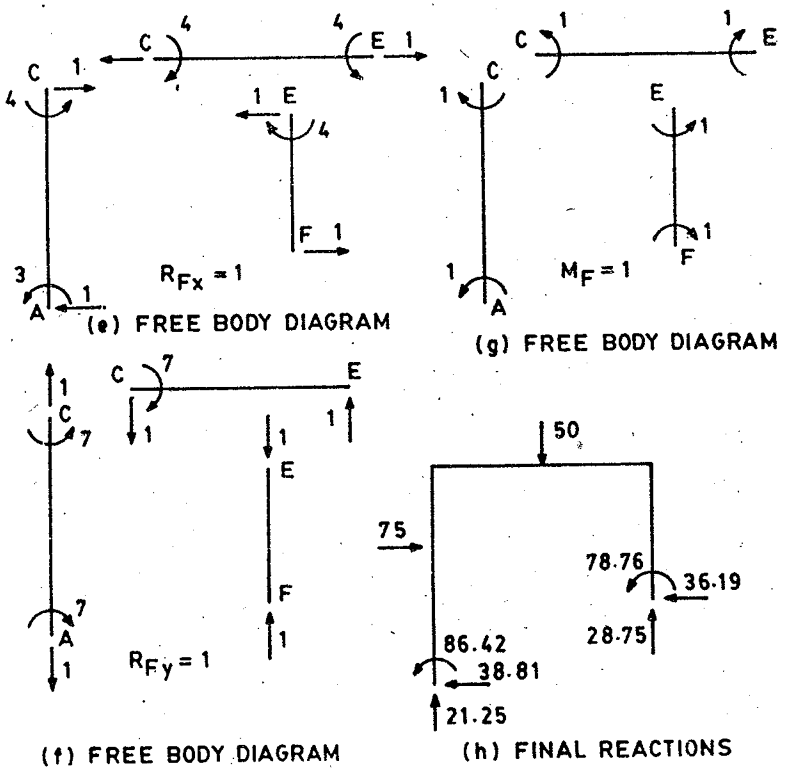


Fig. 3.15 Portal frame with fixed supports (continue)

Solution

The frame is statically indeterminate to a degree 3. Choose R_{Fx}, R_{Fy} and M_F as redundants and remove the fixed support F. The released structure is a cantilever as shown in Fig. 3.15 b. Free body diagram due to applied loads is shown in Fig. 3.15c.

The compatibility conditions may be written as follows:

(a) The sum of horizontal displacements at F due to the applied loads in Fig. 3.15 b and redundants in Fig. 3.15 d must be zero.

$$\delta_{Fx1} + \delta_{Fx2} + \delta_{Fx3} + \delta_{Fx4} = 0 \qquad \text{(i)}$$

(b) The sum of vertical displacements at F due to the applied loads in Fig. 3.15 b and redundants in Fig. 3.15 d must be zero.

$$\delta_{Fy1} + \delta_{Fy2} + \delta_{Fy3} + \delta_{Fy4} = 0 \qquad \text{(ii)}$$

(c) The sum of clockwise rotations at F due to the applied loads in Fig. 3.15 b and redundants in Fig. 3.15 d must be zero.

$$\theta_{F1} + \theta_{F2} + \theta_{F3} + \theta_{F4} = 0 \qquad \text{(iii)}$$

Each of the three redundants will produce displacements δ_{Fx}, δ_{Fy} and θ_F. These displacements can be computed using the unit load method due to each loading separately. Free body diagrams due to each unit redundant are shown in Figs. 3.15e, f and g.

Displacement at F Due to Applied Loads

Segment	DC	CB	BA
Origin	D	C	B
I	2 I	I	I
Limits	0 – 3.5 m	0 – 3	0 – 4 m
M(Fig. 3.15 c)	– 50 x	– 175	– 175 – 75 x
m (Fig. 3.15 e)	4	4 – x	4 – (x + 3) = 1 – x
m (Fig. 3.15 f)	1(x + 3.5)	7	7
m (Fig. 3.15 g)	– 1	– 1	– 1

$$\delta_{Fx1} = \int_0^{3.5}(-50x)\frac{4dx}{2EI} + \int_0^{3}(-175)(4-x)\frac{dx}{EI} + \int_0^{4}(-175-75x)(1-x)\frac{dx}{EI}$$

$$= -100\left|\frac{x^2}{2}\right|_0^{3.5} + 175\left|\frac{x^2}{2} - 4x\right|_0^{3} + \left|-175x + 100\frac{x^2}{2} + 75\frac{x^3}{3}\right|_0^{4}$$

$$\delta_{Fx1} = -\frac{225}{EI}$$

$$\delta_{Fy1} = \int_0^{3.5}(-50x)(x+3.5)\frac{dx}{2EI} + \int_0^{3}(-175)(7)\frac{dx}{EI} + \int_0^{4}(-175-75x)7\frac{dx}{EI}$$

$$= -\frac{25}{EI}\left|\frac{x^3}{3} + 3.5\frac{x^2}{2}\right|_0^{3.5} - 1225\left|x\right|_0^{3} - \left|1225x + 525\frac{x^2}{2}\right|_0^{4}$$

$$\delta_{Fy1} = -\frac{13668.23}{EI}$$

$$\theta_{F1} = \int_0^{3.5}(-50x)(-1)\frac{dx}{2EI} + \int_0^{3}(-175)(-1)\frac{dx}{EI} + \int_0^{4}(-175-75x)(-1)\frac{dx}{EI}$$

$$= \frac{50}{2}\left|\frac{x^2}{2}\right|_0^{3.5} + 175\,|x|_0^3 + \left|175x + 75\frac{x^2}{2}\right|_0^4$$

$$\theta_{F1} = \frac{1980}{EI}$$

Displacements at F Due to Unit Horizontal Load at F

Segment	FE	EC	CA
Origin	F	E	C
Limits	0 – 4m	0 – 7m	0 – 7m
I	I	2I	I
m (Fig. 3.15 e)	x	4	4 – x
m (Fig. 3.15 f)	0	x	7
m (Fig. 3.15 g)	– 1	– 1	– 1

$$\delta_{Fx2} = \int_0^4 \frac{x^2dx}{EI} + \int_0^7 \frac{4^2dx}{2EI} + \int_0^7 (4-x)^2\frac{dx}{EI} = \left|\frac{x^3}{3}\right|_0^4 + 8|x|_0^7 + \left|16x - 8\frac{x^2}{2} + \frac{x^3}{3}\right|_0^7$$

$$\delta_{Fx2} = \frac{107.67}{EI}$$

$$\delta_{Fy2} = 0 + \int_0^7 \frac{4x\,dx}{2EI} + \int_0^7 (4-x)7\frac{dx}{EI} = 0 + |x^2|_0^7 + \left|28x - 7\frac{x^2}{2}\right|_0^7$$

$$\delta_{Fy2} = \frac{73.5}{EI} = \delta_{Fx3}$$

$$\theta_{F2} = \int_0^4 -x\frac{dx}{EI} + \int_0^7 (-4)\frac{dx}{2EI} + \int_0^7 (-4+x)\frac{dx}{EI} = \left|-\frac{x^2}{2}\right|_0^4 - 2|x|_0^7 + \left|-4x + \frac{x^2}{2}\right|_0^7$$

$$\theta_{F2} = -\frac{25.5}{EI} = \delta_{Fx4}$$

Displacements at F Due to Unit Vertical Load at F

$$\delta_{Fx3} = \frac{73.5}{EI}$$

$$\delta_{Fy3} = 0 + \int_0^7 \frac{x^2dx}{2EI} + \int_0^7 \frac{7^2dx}{EI} = 0 + \left|\frac{x^3}{6}\right|_0^7 + |49x|_0^7$$

$$\delta_{Fy3} = \frac{400.17}{EI}$$

$$\theta_{F3} = 0 + \int_0^7 \frac{(-x)dx}{2EI} + \int_0^7 (-7)\frac{dx}{EI} = -\left|\frac{x^2}{4}\right|_0^7 + |-7x|_0^7$$

$$\theta_{F3} = -\frac{61.25}{EI} = \delta_{Fy4}$$

Displacements at F Due to Unit Moment at F

$$\delta_{Fx4} = -\frac{25.5}{EI}$$

$$\delta_{Fy4} = -\frac{61.25}{EI}$$

$$\theta_{F4} = \int_0^4 \frac{dx}{EI} + \int_0^7 \frac{dx}{2EI} + \int_0^7 \frac{dx}{EI}$$

$$\theta_{F4} = \frac{14.5}{EI}$$

Substituting the values of δ_{Fx}, δ_{Fy} and θ_F in the compatibility Eqs. (i), (ii) and (iii)

$$-\frac{225}{EI} + \frac{107.67}{EI}R_{Fx} + \frac{73.5}{EI}R_{Fy} - \frac{25.5}{EI}M_F = 0$$

$$-\frac{13668.23}{EI} + \frac{73.5}{EI}R_{Fx} + \frac{400.17}{EI}R_{Fy} - \frac{61.25}{EI}M_F = 0$$

$$\frac{1980}{EI} - \frac{25.5}{EI}R_{Fx} - \frac{61.25}{EI}R_{Fy} + \frac{14.5}{EI}M_F = 0$$

Rearranging the above equations in the matrix form:

$$\begin{bmatrix} 107.67 & 73.5 & -25.5 \\ & 400.17 & -61.25 \\ \text{Symmetric} & & 14.5 \end{bmatrix} \begin{Bmatrix} R_{Fx} \\ R_{Fy} \\ M_F \end{Bmatrix} = \begin{Bmatrix} 225 \\ 13668.23 \\ -1980 \end{Bmatrix}$$

$$\text{or,}\qquad \begin{Bmatrix} R_{Fx} \\ R_{Fy} \\ M_F \end{Bmatrix} = \begin{Bmatrix} -36.19 \\ 28.75 \\ -78.76 \end{Bmatrix}$$

The final reactions in the frame are shown in Fig. 3.15h. The bending moment diagram can be drawn in three parts:

(i) First end moment diagram due to the redundants alone is drawn as shown in Fig. 3.15i.

(ii) Next, free span moment diagram is drawn due to the applied loads assuming each member to be simply supported as shown in Fig. 3.15j.

(iii) Now, the bending moment diagrams so obtained are superimposed to get the net bending moment diagram as shown in Fig. 3.15k. The location of the point of inflection, maximum positive and negative bending moment and their magnitudes can be determined by considering the free body diagram of each member (beam and column) separately.

The shear force diagram is shown in Fig. 3.15 *l*. It is drawn with the help of free body diagram due to the applied loads and final reactions.

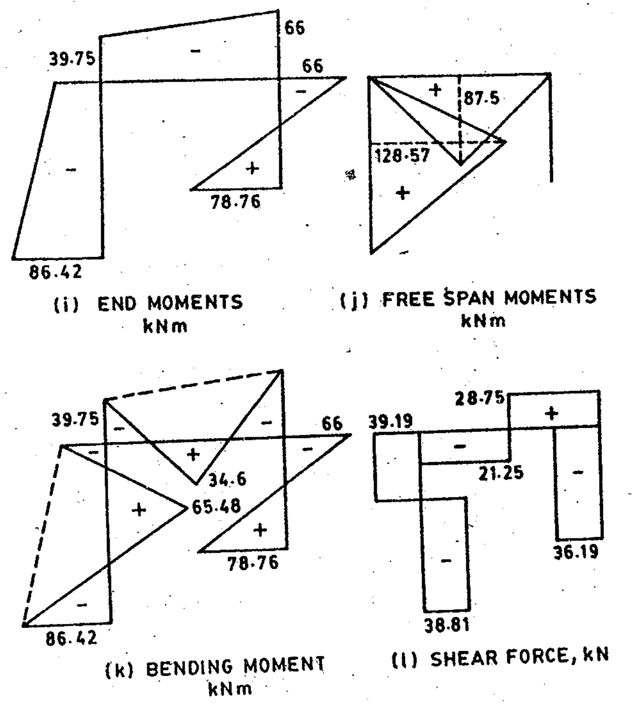


Fig. 3.15 Portal frame with fixed supports

3.7 TRUSSES

A truss may be externally indeterminate or internally indeterminate or both. If a truss is externally indeterminate, the redundants may include excess reactions over three so that the truss remains stable. However, it is possible to obtain a released structure by choosing only member forces as redundants. When a reaction is chosen as a redundant, the constraint in the direction of the reaction is removed and its action is replaced by unknown force R as done in the case of a statically indeterminate beam or a frame. The compatibility condition is that the deflection in the direction of the constraint must be zero.

If a truss is internally indeterminate or externally as well as internally indeterminate, member forces may be chosen as redundants. The redundant members are removed from the given truss such that the truss remains stable. A pair of unknown member forces are applied pulling on the joints. This change will not alter forces in other members of the truss which has now become perfect and statically determinate. The compatibility condition requires that the relative movement of the joints away from each other is equal to the elongation of the member removed from the joints.

Consider a truss shown in Fig. 3.16a. Remove member 3 – 5 and apply a pair of unknown member forces. Mathematically it may be written as:

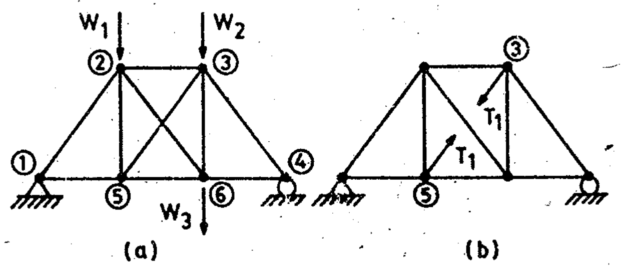


Fig. 3.16 Statically indeterminate truss

at joints 3 – 5 (Fig. 3.16 b)

$$\Delta - T\,\delta = \frac{T_1 L_1}{A_1 E_1} \tag{3.1}$$

By unit load method,

$$\Delta = \sum \frac{F'uL}{AE} \tag{3.2a}$$

$$\delta = \sum \frac{u^2 L}{AE} \tag{3.2b}$$

where
F' = member forces due to the applied loads
u = member forces due to a unit load pulling on the joints
T_1 = redundant force
Δ = axial displacements due to the applied loads where redundant members have been removed.
δ = axial displacements due to a pair of unit loads pulling on the joints

The unit load method was discussed in section 9.11 of volume 1 of this book. Alternatively, the compatibility condition may be expressed as:

Joints 3 and 5 will come closer by an amount = $\sum \frac{FLu}{AE}$

But member 3 - 5 will elongate by an amount = $\frac{T_1 L_1}{A_1 E_1}$

For compatibility,

$$\sum \frac{FLu}{AE} + \frac{T_1 L_1}{A_1 E_1} = 0 \tag{3.3}$$

The total force in any member $F = F' + uT_1$ (3.4)

There is only one unknown quantity T_1 in Eq. 3.3 which can be easily evaluated. Eq. 3.3 can be expanded as:

$$\sum \frac{F'Lu}{AE} + \sum \frac{u^2 L T_1}{AE} + \frac{T_1 L_1}{A_1 E_1} = 0$$

or,

$$\sum \frac{F'Lu}{AE} + T_1 \sum \frac{u^2 L}{AE} + \frac{T_1 L_1}{A_1 E_1} = 0$$

The last term may be rewritten = $\frac{T_1 L_1 u^2}{A_1 E_1}$, since the force in this member is unity (u=1). It can now be included in the second term.

Thus,

$$\sum_{i=1}^{n-1} \frac{F'Lu}{AE} + T_1 \sum_{i=1}^{n} \frac{u^2 L}{AE} = 0 \tag{3.5}$$

or,

$$T_1 = (-)\frac{\sum_{i=1}^{n-1} \frac{F'Lu}{AE}}{\sum_{i=1}^{n} \frac{u^2 L}{AE}} \tag{3.6}$$

where n = total number of members in the given truss

If there are k redundant members, there will be k equations similar to Eq. 3.5. These can be simultaneously solved to get the redundants. If there are more than one redundants, it is more convenient to use Eq. 3.5 rather than Eq. 3.1.

Example 3.13

Find forces in various members of statically indeterminate truss shown in Fig. 3.17 a.

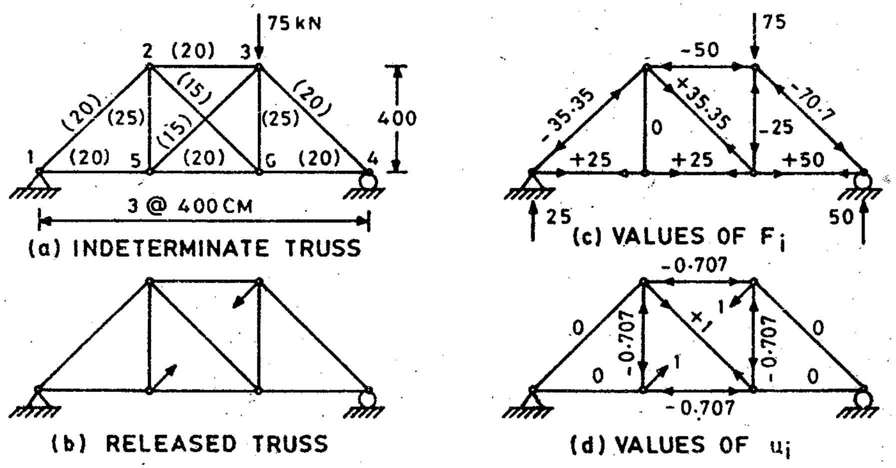


Fig. 3.17 Statically indeterminate truss

Solution

The truss is statically internally indeterminate by degree 1. Let us remove the member 3-5 and the released truss is shown in Fig. 3.17b. Analyze the truss due to the applied loads as shown in Fig. 3.17c. Now apply a pair of unit loads at joints 3 and 5. Due to this unit force, there will be no support reactions and member forces are shown in Fig. 3.17d. The physical properties of the members and stresses due to external loads and due to unit load are shown in Table 3.1.

Force in member 3 – 5, $F_{3-5} = -\frac{1969.23}{111.4} = -17.68$ kN compression

Total force in a member = $F' + u \times (-17.68)$, that is, as shown in last column of Table 3.1.

Table 3.1 : Computations for example 3.13

Member	Length L cm	Area A cm²	Force F ' kN	Force u kN	$\frac{F'uL}{A}$	$\frac{u^2 L}{A}$	Total force kN
2 - 3	400.0	20	– 50	– 0.707	707.00	10.0	– 37.50
2 - 6	565.6	15	35.35	1.000	1332.93	37.7	17.67
2 - 5	400.0	25	0	– 0.707	0	8.0	12.50
5 - 6	400.0	20	25	– 0.707	– 353.50	10.0	37.50
3 - 5	565.6	15	0	1.000	0	37.7	–17.68
3 - 6	400.0	25	– 25	– 0.707	282.80	8.0	–12.50
				Total	Σ1969.23	Σ111.4	

+ve forces : tension
–ve forces : compression

Example 3.14

Analyze the statically externally indeterminate truss shown in Fig. 3.18 a. Area of chord members = 100 cm², and area of web members = 60 cm².

Solution

The truss can be made statically determinate by removing the roller support at joint 8. The truss remains stable. The released truss is shown in Fig. 3.18b. Let us determine forces F'_i in the members due to the external loads, and forces u_i due to a unit load applied at joint 8. The member forces are shown in Figs. 3.18c and d, and also in Table 3.2.

$$\sum \frac{F'uL}{AE} = -\frac{4150.42}{E}$$

$$\sum \frac{u^2L}{AE} = \frac{60.51}{E}$$

For compatibility, net deflection at support 8 must be zero,

$$\sum \frac{F'uL}{AE} + \sum \frac{u^2L}{AE} R = 0$$

$$\text{or,} \qquad R = -\frac{\sum \frac{F'uL}{AE}}{\sum \frac{u^2L}{AE}}$$

$$\text{that is,} \quad R = (-)\frac{(-)4150.42}{60.51} = +68.6\,\text{kN} \quad \text{upward}$$

The net force in each member can be determined by superposition, that is, $(F'_i + u_i R)$. The final member forces are shown in Table 3.3.

Table 3.2 : Computations for Example 3.14

	Member	Length L cm	Area A cm²	Force F' kN	Force u kN	$\frac{F'uL}{A}$	$\frac{u^2 L}{A}$
Top Chord	1 - 2	600	100	– 76.67	1.000	– 460.00	6.000
	2 - 3	600	100	– 76.67	1.000	– 460.00	6.000
	3 - 4	600	100	– 93.33	0.500	– 280.00	1.500
	4 - 5	600	100	– 93.33	0.500	– 280.00	1.500
Bottom Chord	6 - 7	600	100	– 11.67	– 0.500	35.00	1.500
	7 - 8	600	100	– 11.67	– 0.500	35.00	1.500
	8 - 9	600	100	65.00	– 0.750	– 292.50	3.370
	9 - 10	600	100	65.00	– 0.750	– 292.50	3.370
	10 - 11	600	100	21.66	– 0.250	– 32.50	0.375
	11 - 12	600	100	21.66	– 0.250	– 32.50	0.375
Web Member	1 - 6	1000	60	– 63.89	0.837	– 891.27	11.670
	1 - 7	800	60	0	0	0	0
	1 - 8	1000	60	63.89	– 0.837	– 891.27	11.670
	2 - 8	800	60	0	0	0	0
	3 - 8	1000	60	– 63.89	– 0.416	443.00	2.920
	3 - 9	800	60	0	0	0	0
	3 - 10	1000	60	– 36.10	0.416	– 250.30	2.920
	4 - 10	800	60	0	0	0	0
	5 - 10	1000	60	36.10	– 0.416	– 250.30	2.920
	5 - 11	800	60	0	0	0	0
	5 - 12	1000	60	– 36.10	0.416	– 250.30	2.920
				Total		Σ– 4150.42	Σ60.51

Table 3.3 Final member forces

Member	Force kN	Member	Force kN
1 - 2	– 7.866	6 - 1	– 6.555
2 - 3	– 7.866	7 - 1	0
3 - 4	– 58.933	1 - 8	6.555
4 - 5	– 58.933	8 - 2	0
6 - 7	– 46.067	8 - 3	–92.556
7 - 8	– 46.067	9 - 3	0
8 - 9	13.399	10 - 3	– 7.444
9 - 10	13.399	10 - 4	0
10 - 11	4.466	10 - 5	7.444
11 - 12	4.466	11 - 5	0
		5 - 12	– 7.444

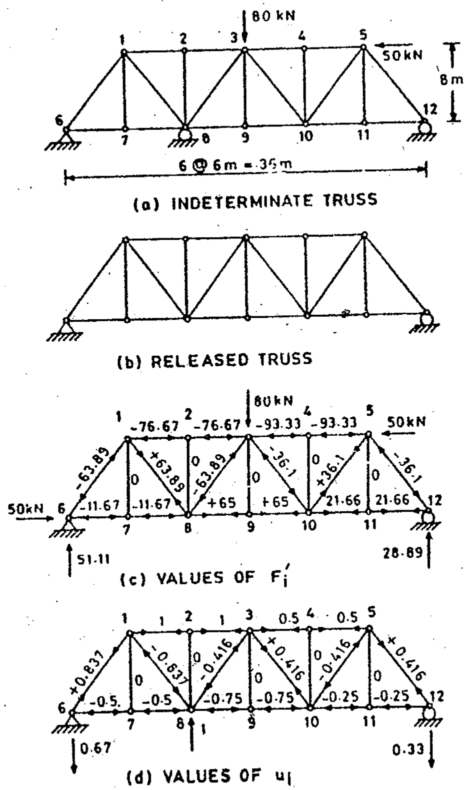


Fig. 3.18

Example 3.15

A truss shown in Fig. 3.19a represents an industrial frame. Determine the member forces.

Solution

The truss is statically indeterminate to degree two. It can be made statically determinate in several ways selecting any two redundants, for example :

(a) select members 1 - 4 and 3 - 6

(b) select members 2 - 3 and 4 - 5

(c) select reaction R_{5x} and member 1 - 4

(d) select reaction R_{5x} and member 3 - 6

The last choice is bad as there is no member at joint 6 to balance the horizontal force R_{6x}. Let us select the first option.

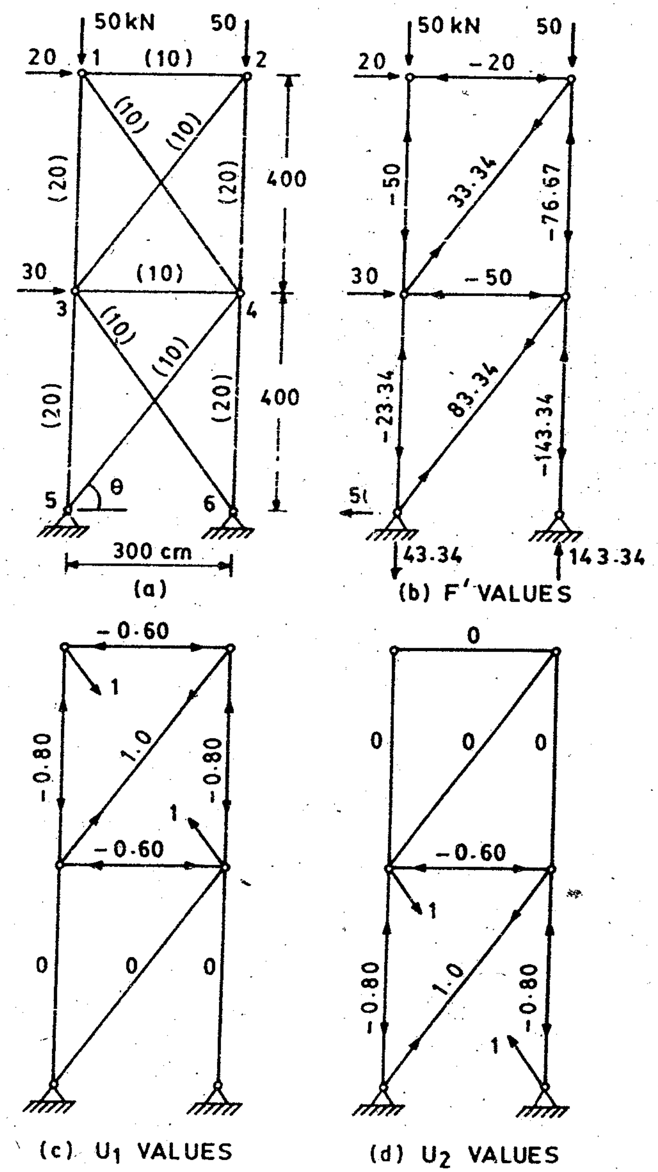


Fig. 3.19

The compatibility conditions are

$$\sum_{1}^{10} \frac{F' u_1 L}{AE} = 0 \quad \text{and} \quad \sum_{1}^{10} \frac{F u_2 L}{AE} = 0 \qquad \text{(i)}$$

but $$F = F' + u_1 F_{1\text{-}4} + u_2 F_{3\text{-}6} \qquad \text{(ii)}$$

where, u_1 = forces in members due to unit load in location 1- 4

u_2 = forces in members due to unit load in location 3- 6

Substituting for F in the compatibility conditions give,

$$\sum \frac{F' u_1 L}{AE} + \sum \frac{u_1^2 L}{AE} F_{1-4} + \sum \frac{u_1 u_2 L}{AE} F_{3-6} = 0 \qquad \text{(iii)}$$

and $$\sum \frac{F' u_2 L}{AE} + \sum \frac{u_1 u_2 L}{AE} F_{1-4} + \sum \frac{u_2^2 L}{AE} F_{3-6} = 0 \qquad \text{(iv)}$$

The values of F ', u_1 and u_2 are shown in Figs. 3.19b, c and d and also in Table 3.4.
Eqs. (iii) and (iv) become,

$$4953.72 + 147.2\, F_{1-4} + 10.8\, F_{3-6} = 0$$

and $$7733.88 + 10.8\, F_{1-4} + 136.4\, F_{3-6} = 0$$

or, $$\begin{bmatrix} 147.2 & 10.8 \\ 10.8 & 136.4 \end{bmatrix} \begin{Bmatrix} F_{1-4} \\ F_{3-6} \end{Bmatrix} = \begin{Bmatrix} -4953.72 \\ -7733.88 \end{Bmatrix}$$

or, $$\begin{Bmatrix} F_{1-4} \\ F_{3-6} \end{Bmatrix} = \begin{Bmatrix} -29.66 \\ -54.35 \end{Bmatrix}$$

The net force in each member can be computed by Eq.(ii), and the values are shown in Table 3.4

Table 3.4 : Computations for example 3.15

Member	$\frac{L}{A}$	F ' kN	u_1 kN	u_2 kN	$\frac{F' u_1 L}{A}$	$\frac{F' u_2 L}{A}$	$u_1^2 \frac{L}{A}$	$u_2^2 \frac{L}{A}$	$u_1 u_2 \frac{L}{A}$	F kN
1-2	30	–20	–0.6	0	360	0	10.8	0	0	–2.20
3-4	30	–50	–0.6	–0.6	900	900	10.8	10.8	0	0.41
1-3	20	–50	–0.8	0	800	0	12.8	0	0	–26.27
3-5	20	–23.34	0	–0.8	0	373.44	0	12.8	0	20.14
2-4	20	–76.67	–0.8	0	1226.72	0	12.8	0	0	–52.94
4-6	20	–143.34	0	–0.8	0	2293.44	0	12.8	0	–99.86
1-4	50	0	1	0	0	0	50	0	0	–29.66
3-6	50	0	0	1	0	0	0	50	0	–54.35
2-3	50	33.34	1	0	1667.0	0	50	0	0	3.67
4-5	50	83.34	0	1	0	4167	0	50	0	28.99
					Σ 4953.72	7733.88	147.2	136.4	10.8	

PROBLEMS

Note: Clockwise moments are shown as positive while anticlockwise moments are shown as negative in all problems. M_{ba} represents moment at B in the span BA. Similarly, M_{ab} represents moment at A in the span AB.

3.1 Reduce the following structures in Fig. P3.1 a-g into all possible released structures and indicate the redundants.

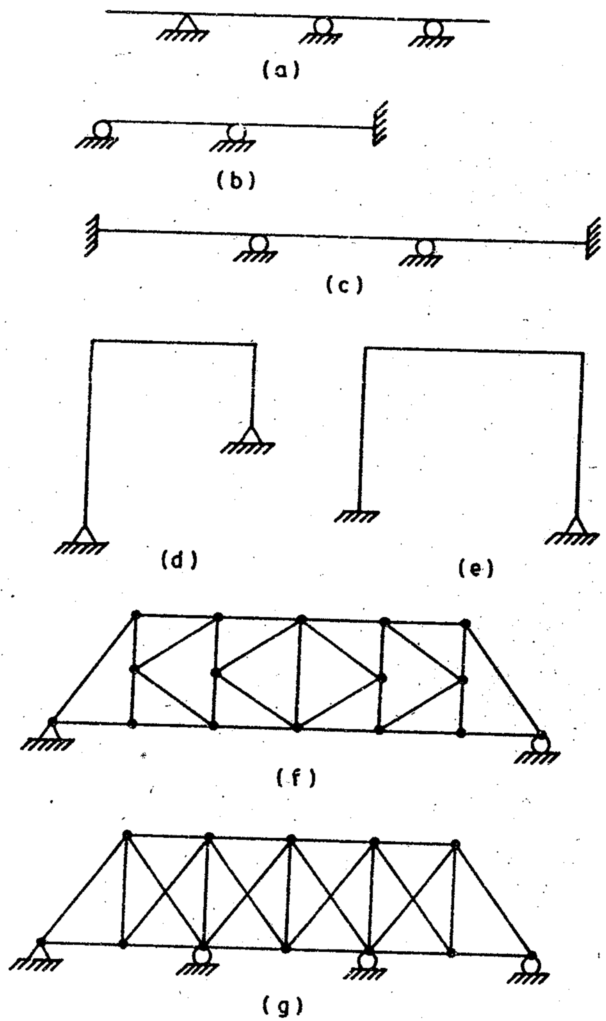
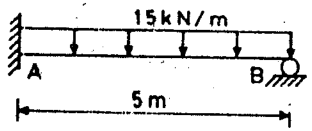


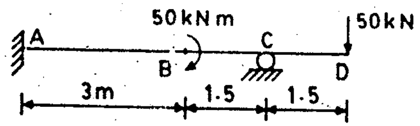
Fig. P 3.1

3.2 Determine reactions in the beams shown in Figs. P3.2 - P3.7 using the method of consistent deformations and draw shear force and bending moment diagrams. Also draw deflected shape of the beams. Take EI = constant.

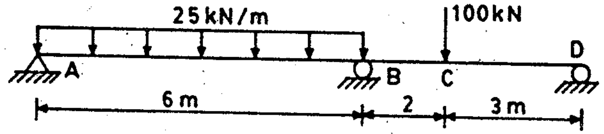
(P3.2)	R_{by}	=	*28 kN↑,*	M_a	=	*– 47.55 kNm,*		
(P3.3)	R_{cy}	=	*89.81 kN↑,*	M_a	=	*54.14 kNm,*		
(P3.4)	R_{ay}	=	*57.5 kN↑,*	R_{by}	=	*173.5 kN↑,*	R_{dy}	= *19 kN↑,*
	M_{ba}	=	*105 kNm,*	M_{bc}	=	*– 105 kNm*		
(P3.5)	R_{by}	=	*65.83 kN↑,*	R_{cy}	=	*114.17 kN↑,*	R_{dy}	= *10 kN↑,*
	M_{cb}	=	*85.0 kNm*					
(P3.6)	R_{ay}	=	*29.18 kN↑,*	R_{by}	=	*95.74 kN↑,*	M_{ba}	= *43.27 kNm,*
	M_{cb}	=	*43.67 kNm,*	R_{cy}	=	*30.08 kN↑,*		
(P3.7)	R_{by}	=	*84.36 kN↑,*	R_{cy}	=	*33.90 kN↑,*	M_{ba}	= *79.4 kNm,*
	M_{cb}	=	*33.3 kNm,*					



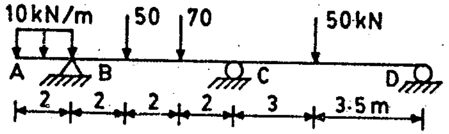
P 3.2



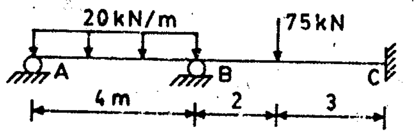
P 3.3



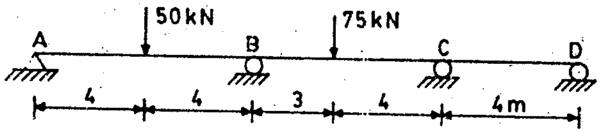
P 3.4



P 3.5



P 3.6



P 3.7

Fig. P 3.2 - 3.7

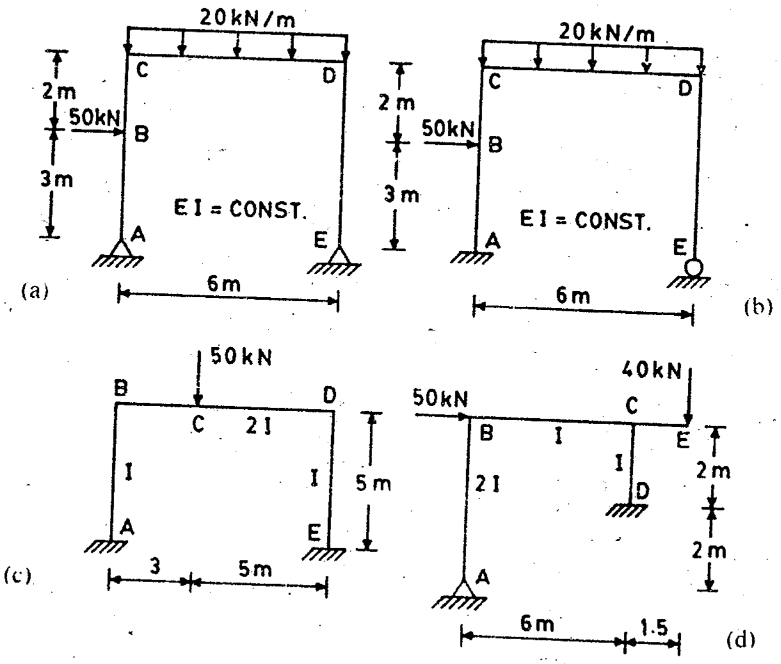
3.3 Determine all reactions due to a vertical settlement of 7.5 mm at support B of the beam in Fig. P3.4. Determine slopes at sections A, B, and D of the final elastic curve and draw shear and bending moment diagrams. Take E = 200×10^6 kN/m^2 and I = 150 × 10^{-6} m^4.

(P3.4) R_{ay} = *63.75 kN↑,* R_{by} = *159.75 kN↑,* R_{dy} = *26.5 kN↑,* M_{ba} = *67.5 kNm*

3.4 Determine all reactions due to a vertical settlement of 8 mm at support B of the beam in Fig. P3.7. Determine slopes at A, B, C and D of the final elastic curve and draw shear and bending moment diagrams. Take E = 200 × 10^6 kN/m^2 and I = 200 × 10^{-6} m^4.

(P3.7) R_{by} = *67.11 kN↑,* R_{cy} = *52.45 kN↑,* M_{ba} = *40.46 kNm,* M_{cb} = *58.04 kNm,*

3.5(a) Reduce the portal frames shown in Figs. P3.8 a - d into all possible released structures and indicate the redundants.



Figs. P 3.8

3.5(b) Find the reactions in portal frames shown in Figs. P3.8 a - d using the most appropriate released structure. Draw thrust, shear force and bending moment diagrams.

(P3.8a) $R_{ax} = 25.57\ kN\leftarrow$, $R_{ex} = 24.43\ kN\leftarrow$, $M_{cb} = -27.84\ kNm$,
$M_{dc} = 122.14\ kNm$

(P3.8b) $R_{ey} = 61.07\ kN\uparrow$, $M_{ab} = -143.58\ kNm$

(P3.8c) $R_{ax} = 8.65\ kN\rightarrow$, $R_{ay} = 31.60\ kN\uparrow$, $M_{ab} = 13.05\ kNm$,
$M_{ba} = 30.22\ kNm$, $R_{ex} = 8.65\ kN\leftarrow$, $R_{ey} = 18.40\ kN\uparrow$,
$M_{ed} = -15.80\ kNm$, $M_{de} = -27.46\ kNm$

(P3.8d) $R_{ax} = 8.14\ kN\leftarrow$, $R_{ay} = 5.30\ kN\downarrow$, $M_{ba} = -32.53\ kNm$
$R_{dx} = 41.86\ kN\leftarrow$, $R_{dy} = 45.30\ kN\uparrow$, $M_{dc} = -84.44\ kNm$

3.6 Determine all reactions due to a 15mm vertical settlement at the hinged support E of the frame shown in Fig. P 3.8 a. Draw free-body, shear force and bending moment diagrams. Take E = 200 GPa and I = 300 × 10^{-6} m^4.

(P3.8a) $R_{ax} = 25.57\ kN\leftarrow$, $R_{ex} = 24.43\ kN\leftarrow$, $M_{ca} = -27.84\ kNm$,
$M_{dc} = 122.14\ kNm$

3.7 Determine all reactions due to a vertical settlement of 10 mm at support A and also a rotational slip of 0.003 radian clockwise of support A of the portal frame shown in Fig. P3.8c. Draw free-body, shear force and bending moment diagrams. Take E = 200 GPa and I = 300 × 10^{-6} m^4.

(P3.8c) $R_{ax} = 23.61\ kN\rightarrow$, $R_{ay} = 24.32\ kN\uparrow$, $M_{ab} = 103.08\ kNm$,
$M_{ba} = 14.95\ kNm$, $R_{ex} = 23.61\ kN\leftarrow$, $R_{ey} = 25.68\ kN\uparrow$,
$M_{ed} = -47.60\ kNm$, $M_{de} = -70.43\ kNm$

3.8 Determine force in member 2 - 4 of the truss shown in Fig. P 3.9. Area of each member = 15 cm^2.

(P3.9) $F_{1-4} = 8.08\ kN$, $F_{2-4} = 4.62\ kN$, $F_{3-4} = -2.31\ kN$

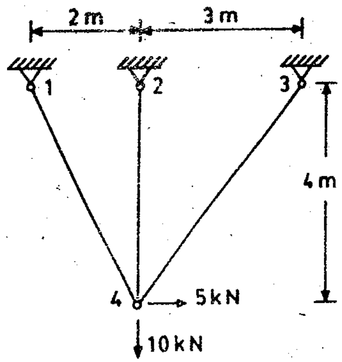


Fig. P 3.9

3.9 Analyze the truss shown in Fig. P3.10. Take area of top and bottom chords = 100 cm^2 and area of web members = 70 cm^2. Treat R_{6y} as redundant.

(P3.10) $R_{4y} = 82.3\ kN\uparrow$, $R_{6y} = 285.5\ kN\uparrow$, $R_{8y} = 132.2\ kN\uparrow$, $F_{1-2} = 35.51\ kN$,

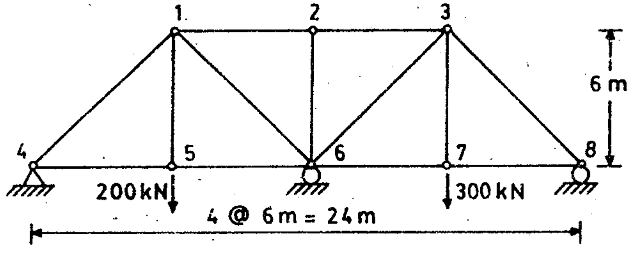


Fig. P3.10

3.10 Analyze the truss shown in Fig. P3.11; Numbers on the members indicate the areas in cm^2.

(a) Treat members 5 - 2 and 2 - 7 as redundants.

(b) Treat members 1 - 6 and 6 - 3 as redundants.

(P3.11) $R_{4y} = 256.30\ kN\uparrow$, $R_{8y} = 268.70\ kN\uparrow$, $F_{5-2} = -23.96\ kN$,
$F_{1-6} = 126.30\ kN$, $F_{6-3} = 121.40\ kN$, $F_{2-} = -11.19\ kN$,

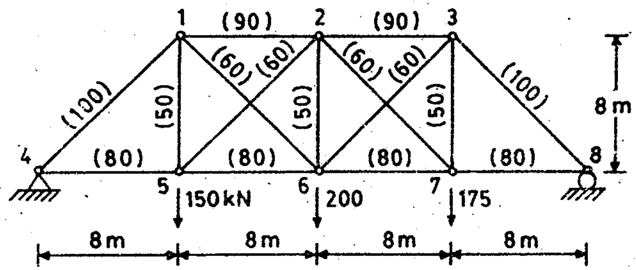


Fig. P3.11

3.11 Analyze the truss shown in Fig. P 3.12. Treat member 2- 3 as internal redundant and R_{5y} as external redundant.

(P3.12) $R_{1x} = 59.43\ kN\leftarrow$, $R_{1y} = 62.34\ kN\uparrow$, $R_{3x} = 59.43\ kN\rightarrow$,
$R_{3y} = 42.38\ kN\uparrow$, $R_{5y} = 45.28\ kN\uparrow$, $F_{1-4} = 88.15\ kN$,
$F_{2-3} = -59.94\ kN$,

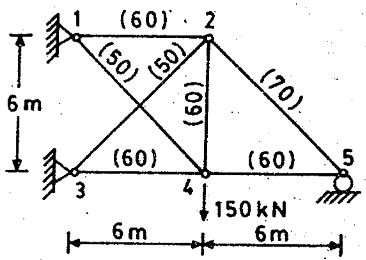


Fig. P3.12

CHAPTER
four

THREE - MOMENT EQUATION

4.1 INTRODUCTION

– Continuous beams are commonly used in buildings and bridges. The spans may be different and cross sections of the beams in different spans may also be different. Moreover, there may be unequal settlement of supports. Such beams can always be analyzed by the method of consistent deformations. However, this will involve computations of a large number of deflections or slopes. The three moment equation is a more convenient approach. It is based on the continuity of the elastic curve of a beam over any intermediate support. The unknown moments at the supports are treated as redundants. The compatibility condition requires that slopes of the elastic curve at either side of an intermediate support must be equal. In this manner each span may be treated individually acted upon by the loads on it and the moments at both ends, if any. The computation of slopes at either side of a support requires a knowledge of loading on the two adjacent spans as well as the bending moment at three successive supports including the one before and one after the support under consideration. Since this approach requires three bending moments at supports, it is called as *three - moment equation.* It is also called *Clapeyron equation* who proposed it in 1857. It was later modified by Mohr in 1860 to account for the effect of settlement of supports.

4.2 DERIVATION OF THREE-MOMENT EQUATION

Let us consider two adjacent spans AB and BC of a continuous beam as shown in Fig. 4.1a. Due to uneven settlements, the supports A' and C ' are at higher elevations relative to the support B by amounts h_A and h_C. The elastic curve passes through A' , B and C ' shown in dotted lines. The free span bending moment diagrams due to the applied loads on these spans AB and BC are shown in Fig. 4.1b. A free span moment diagram is drawn assuming a span to be simply supported. If the support moments are M_A , M_B and M_C , the end moment diagrams are shown in Fig. 4.1c. This aspect may be examined in detail as follows.

The total loading on any span can be split in two parts assuming it to be simply supported and loaded with

(a) applied loads on the span, and

(b) support moments M_A and M_B.

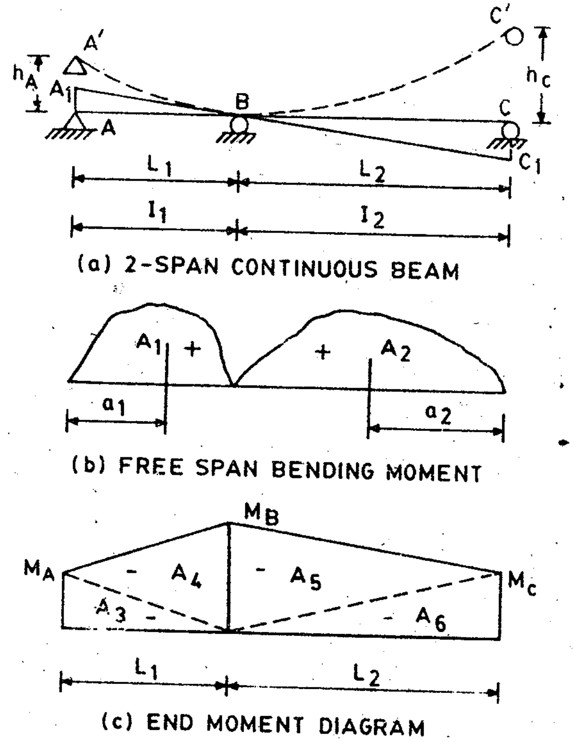


Fig. 4.1

The moment diagram can also be split accordingly. The net moment is the sum of free span moment and the end moments as shown in Figs. 4.2b and 4.2c. By superposition, the net moment diagram is obtained as shown in Fig. 4.2a. The free span moments are usually sagging while the end moments are hogging. Therefore due care should be taken while superposing the moments. The sagging and hogging moments are treated as positive and negative moments, respectively and are drawn on opposite sides of a base line as shown in Fig.4.2 a. Alternatively, they may be drawn on the same side as shown in Fig. 4.2 d, but it is difficult to get net ordinate in this manner. Therefore, the most desirable way is to plot the two diagrams as shown in Fig. 4.2e. The net moment

diagram is shown by the shaded portion. The superposition of loads or moment diagrams is permissible within the elastic range only.

In Fig. 4.1b, A_1 and A_2 represent area of the free span moment diagrams on spans AB and BC. A_3 and A_4 represent area of the end moment diagram on span AB while

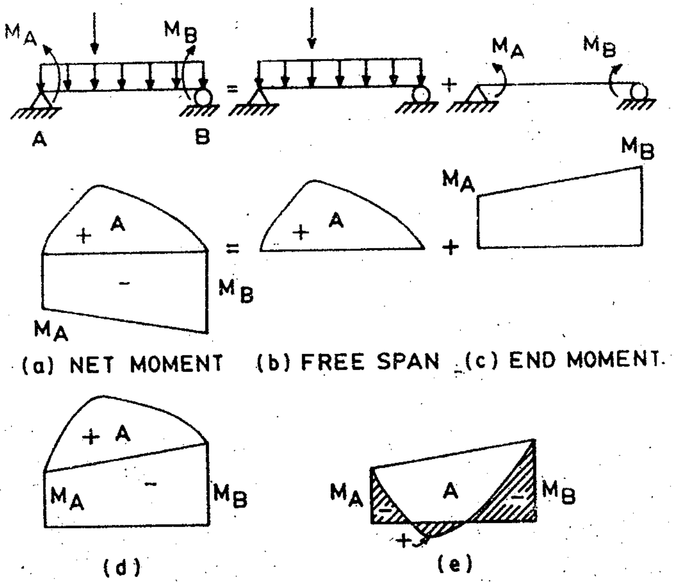


Fig. 4.2

A_5 and A_6 represent area of the end moment diagram on span BC. The free span moments are known while the support moments M_A, M_B and M_C are to be determined.

In Fig. 4.1a, lines A_1 B and BC_1 are tangents to A'B and BC', at B respectively. Since the elastic curve A'BC' is continuous at B, the two tangents at B must be in the same straight line.

$$\therefore \quad \frac{AA_1}{L_1} = \frac{CC_1}{L_2} \tag{4.1}$$

where, $AA_1 = h_A - A'A_1$, and $CC_1 = C'C_1 - h_C$ (4.2)

$A'A_1$ = vertical deflection of A with respect to the tangent at B

$C'C_1$ = vertical deflection of C with respect to the tangent at B.

Both $A'A_1$ and $C'C_1$ can be obtained with the help of moment area theorem. Let a_1 and a_2 be the distances of the center of gravity of the M/EI loading on spans AB and BC from A and C, respectively.

$$\therefore \quad A'A_1 = \int_0^{L_1} \frac{M\,dx\,x}{EI_1} = \frac{1}{EI_1}\int_0^{L_1} M\,dx\,x$$

$$= \frac{1}{EI_1}\left[A_1a_1 - A_3\frac{L_1}{3} - A_4\frac{2L_1}{3}\right]$$

$$= \frac{1}{EI_1}\left[A_1a_1 - \frac{1}{6}M_AL_1^2 - \frac{1}{3}M_BL_1^2\right] \tag{4.3}$$

$$C'C_1 = \int_0^{L_2} \frac{M\,dx\,x}{EI_2} = \frac{1}{EI_2}\int_0^{L_2} M\,dx\,x$$

$$= \frac{1}{EI_2}\left[A_2a_2 - A_5\frac{2L_2}{3} - A_6\frac{L_2}{3}\right]$$

$$= \frac{1}{EI_2}\left[A_2a_2 - \frac{1}{3}M_BL_2^2 - \frac{1}{6}M_CL_2^2\right] \tag{4.4}$$

Substituting Eqs. 4.2, 4.3 and 4.4 in Eq. 4.1,

$$\frac{hA}{L_1} - \frac{1}{EI_1L_1}\left[A_1a_1 - \frac{1}{6}M_AL_1^2 - \frac{1}{3}M_BL_1^2\right]$$

$$= \frac{1}{EI_2L_2}\left[A_2a_2 - \frac{1}{3}M_BL_2^2 - \frac{1}{6}M_CL_2^2\right] - \frac{h_C}{L_2}$$

Multiplying every term in the above expression by 6E and upon simplifying,

$$-M_A\left(\frac{L_1}{I_1}\right) - 2M_B\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) - M_C\left(\frac{L_2}{I_2}\right)$$

$$= -\frac{6A_1a_1}{I_1L_1} - \frac{6A_2a_2}{I_2L_2} + \frac{6Eh_A}{L_1} + \frac{6Eh_C}{L_2} \tag{4.5}$$

This is the three - moment equation. h_A and h_C are taken as positive if the supports A and C are higher than support B. A sagging moment is considered as positive while a hogging moment is considered as negative. If solution of three - moment equations gives a positive value of M_A, M_B or M_C, it means hogging, while a negative value means sagging.

The three - moment equations can be written for all intermediate supports of a continuous beam. Hence number of three - moment equations is the same as the

$M_{BA} = -\ 50$ kNm, $\qquad M_{DE} = -\ 20 \times 1.5 = -\ 30$ kNm

Due to moment-joint equilibrium

$M_{BC} = -\ 50$ kNm, and $M_{DC} = -\ 30$ kNm

There will be only one three - moment equation for one unknown, that is, M_C. Let us consider the free span moment in span CD as shown in Fig. 4.5b.

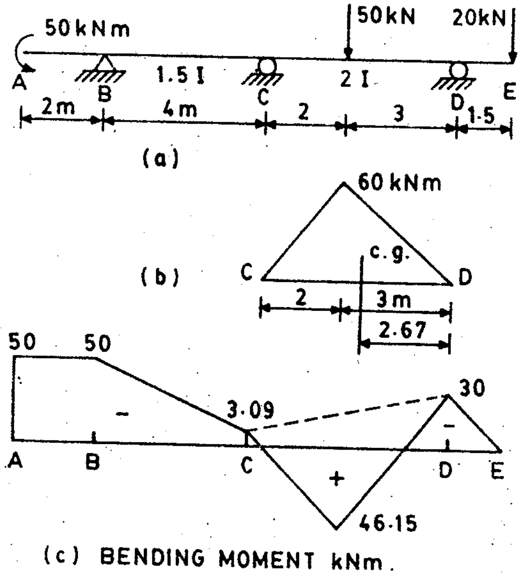


Fig. 4.5

Area $\quad A = 1/2 \times 5 \times 60 = 150$

and $\quad a = 2.67$ m from D

Three-moment equation at support C becomes,

$$-M_B\left(\frac{4}{1.5}\right) - 2M_C\left(\frac{4}{1.5}+\frac{5}{2}\right) - M_D\left(\frac{5}{2}\right) = -\frac{6\times 0}{4\times 1.5} - \frac{6\times 150\times 2.67}{5\times 2}$$

or, $\quad -50\times\frac{4}{1.5} - 10.34M_C - 30\times 2.5 = -240.3$

or, $\quad M_C = 3.09$ kNm

The resulting bending moment diagram is shown in Fig. 4.5c.

Example 4.3

Analyze the beam shown in Fig. 4.6a using the three - moment equation. Draw bending moment diagram.

Solution

The free span moment diagrams, considering each span as a simple beam subjected to the applied loads, are shown in Fig. 4.6b. By inspection $M_D = 25 \times 3 = 75$ kNm hogging. Since support A is fixed, an imaginary span A'A of length L_O and with $I = \infty$ is added.

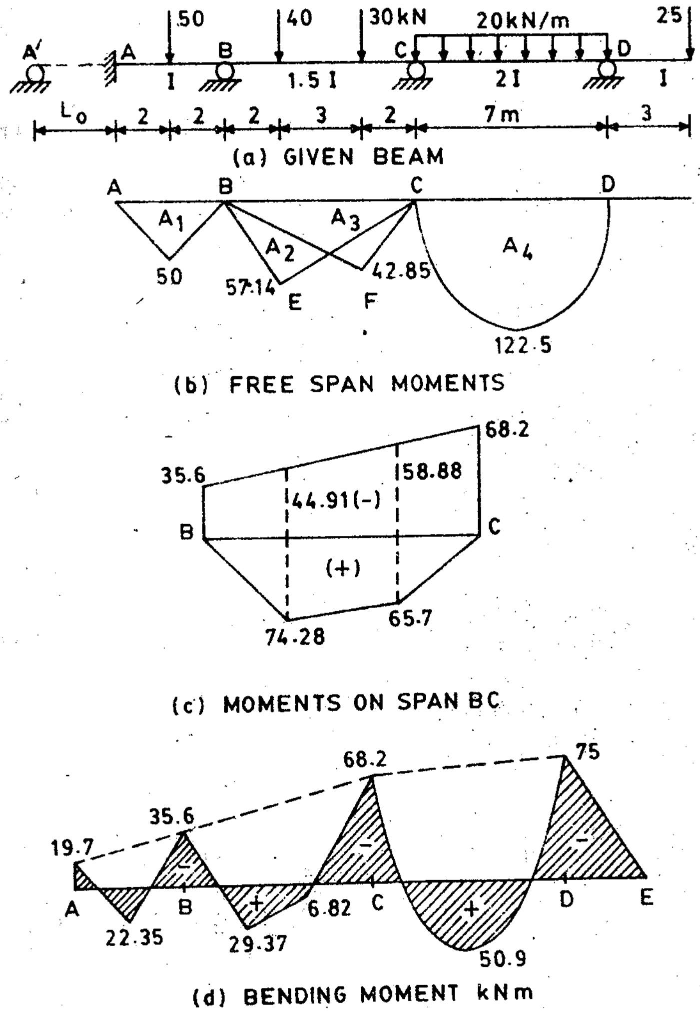


Fig. 4.6

Support C

$$-M_B\left(\frac{4}{I}\right)-2M_C\left(\frac{4}{I}+\frac{6}{I}\right)-M_D\left(\frac{6}{I}\right)=\frac{6E(-0.01)}{4}+0$$

or, $\quad -4\,M_B - 20\,M_C = -0.015\,E\,I = -600$

Matrix Solution

$$\begin{bmatrix}-18 & -4\\ -4 & -20\end{bmatrix}\begin{Bmatrix}M_B\\ M_C\end{Bmatrix}=\begin{Bmatrix}1080\\ -600\end{Bmatrix}$$

or,

$$\begin{Bmatrix}M_B\\ M_C\end{Bmatrix}=\begin{Bmatrix}-69.80\\ 43.98\end{Bmatrix}$$

It gives M_B as sagging and M_C as hogging moments. These are the same values as obtained by the method of consistent deformation in Example 3.10.

4.5 FRAMES

Example 4.5

Analyze the portal frame shown in Fig. 4.8a using the three - moment equation.

Solution

Since joints B and C are rigid, the portal frame may be considered as a continuous beam having three spans. For the sake of convenience, the vertical members may be opened on to form a straight continuous beam as shown in Fig. 4.8b. Since support A is fixed, an imaginary span A'A of length L_1 with $I = \infty$ is added. Similarly, an imaginary span DD' is added.

Free span moments are shown in Fig. 4.8c. Let us first calculate areas and centers of gravity of the free span moments.

Span AB $\quad$ Area AE $= -\dfrac{1}{2}\times 3\times 9 = -13.5$ hogging moment

c.g. from B $= 2 + 1 = 3$ m, c.g. from A $= 2$ m

Area EB $= \dfrac{1}{2}\times 2\times 6 = 6$

c.g. from B $= 1.33$ m, $\quad$ c.g. from A $= 3 + \dfrac{2}{3} = 3.67$ m

Span BC $\quad$ Area $= \dfrac{2}{3}\times 320\times 8 = 1706.67$

c.g. $= 4$ m

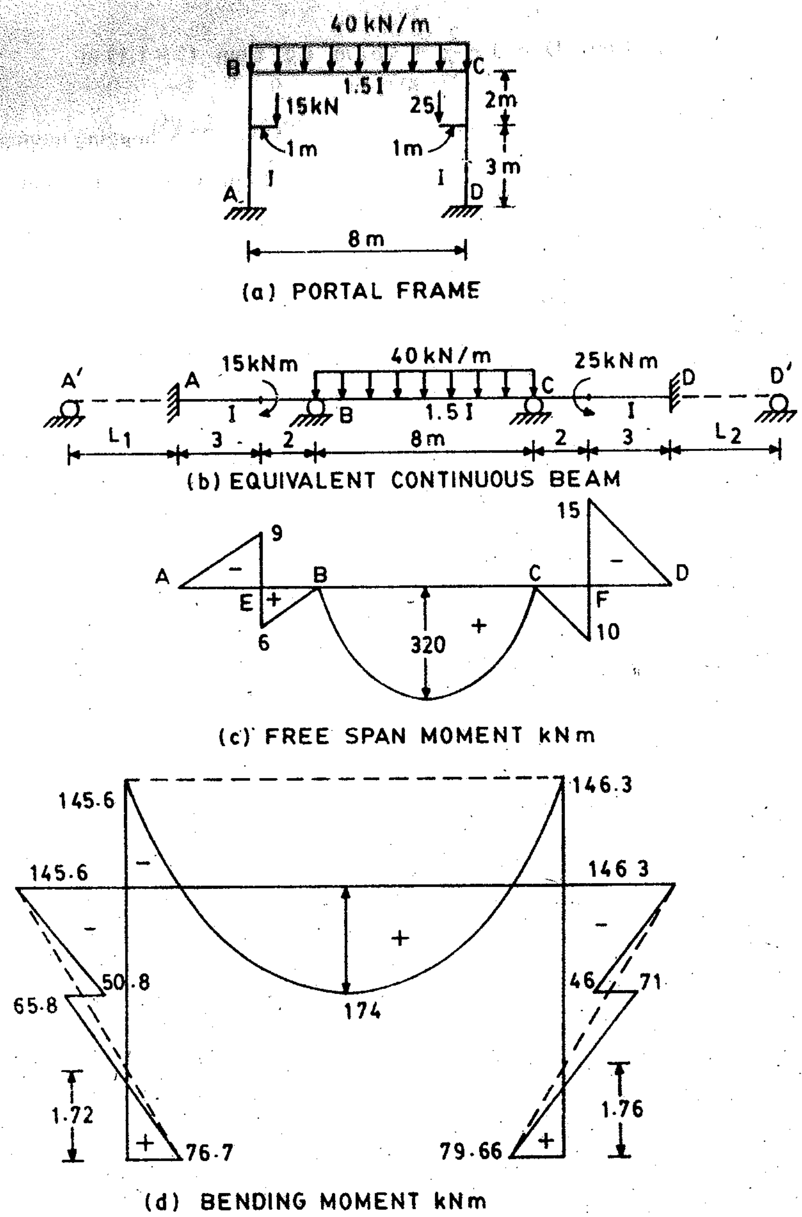


Fig. 4.8

Span CD

Area CF $= \frac{1}{2} \times 2 \times 10 = 10$

c.g. from D $= 3 + \frac{2}{3} = 3.67$ m, c.g. from C $= 1.33$ m

Area FD $= -\frac{1}{2} \times 3 \times 15 = -22.5$ hogging moment

c.g. from D $= 2$ m, c.g. from C $= 2 + 1 = 3$ m

Let us write the three - moment equations for each interior support.

$$-M_A\left(\frac{L_1}{I_1}\right) - 2M_B\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) - M_C\left(\frac{L_2}{I_2}\right) = -\frac{6A_1a_1}{L_1I_1} - \frac{6A_2a_2}{L_2I_2}$$

Support A

$$-M'_A\left(\frac{L_1}{\infty}\right) - 2M_A\left(\frac{L_1}{\infty} + \frac{5}{I}\right) - M_B\left(\frac{5}{I}\right) = -\frac{6}{5I}(-13.5 \times 3 + 6 \times 1.33)$$

or, $$-10\,M_A - 5\,M_B = +39 \qquad \text{(i)}$$

Support B

$$-M_A\left(\frac{5}{I}\right) - 2M_B\left(\frac{5}{I} + \frac{8}{1.5I}\right) - M_C\left(\frac{8}{1.5I}\right) = -\frac{6}{5I}(-13.5 \times 2 + 6 \times 3.67) - \frac{6}{1.5I \times 8}(1706.67 \times 4)$$

or, $$-5\,M_A - 20.67\,M_B - 5.34\,M_C = -3407.36 \qquad \text{(ii)}$$

Support C

$$-M_B\left(\frac{8}{1.5I}\right) - 2M_C\left(\frac{8}{1.5I} + \frac{5}{I}\right) - M_D\left(\frac{5}{I}\right) = -\frac{6}{1.5I \times 8}(1706.67 \times 4) - \frac{6}{5I}(10 \times 3.67 - 22.5 \times 2)$$

or, $$-5.34\,M_B - 20.67\,M_C - 5\,M_D = -3403.38 \qquad \text{(iii)}$$

Support D

$$-M_C\left(\frac{5}{I}\right) - 2M_D\left(\frac{5}{I} + \frac{L_2}{\infty}\right) - M'_D\left(\frac{L_2}{\infty}\right) = -\frac{6}{5I}(10 \times 1.33 - 22.5 \times 3)$$

or, $$-5\,M_C - 10\,M_D = 65.04 \qquad \text{(iv)}$$

Matrix Solution

$$\begin{bmatrix} -10 & -5 & 0 & 0 \\ -5 & -20.67 & -5.34 & 0 \\ 0 & -5.34 & -20.67 & -5 \\ 0 & 0 & -5 & -10 \end{bmatrix} \begin{Bmatrix} M_A \\ M_B \\ M_C \\ M_D \end{Bmatrix} = \begin{Bmatrix} 39 \\ -3407.36 \\ -3403.38 \\ 65.04 \end{Bmatrix} \qquad \text{(v)}$$

or, $$\begin{Bmatrix} M_A \\ M_B \\ M_C \\ M_D \end{Bmatrix} = \begin{Bmatrix} -76.70 \\ 145.60 \\ 146.30 \\ -79.66 \end{Bmatrix} \qquad \text{(vi)}$$

The resulting bending moment diagram is shown in Fig. 4.8d.

PROBLEMS

Note: Clockwise moments are shown as positive while anticlockwise moments are shown as negative in all problems. M_{ba} represents moment at B in the span BA. Similarly, M_{ab} represents moment at A in the span AB.

4.1 Analyze the continuous beams shown in Fig. P4.1 – P4.4 using the three moment equation. Draw shear force and bending moment diagrams. Compute slopes of the elastic curve at all supports.

(P4.1) $M_{ba} = 160$ kNm, $M_{cb} = 10$ kNm
(P4.2) $M_{ba} = 34.92$ kNm, $M_{cb} = 24.70$ kNm, $M_{ec} = 40$ kNm
(P4.3) $M_{ba} = 117.0$ kNm, $M_{cb} = 134.30$ kNm, $M_{dc} = -67.15$ kNm
(P4.4) $M_{ab} = -101.2$ kNm, $M_{ba} = 37.60$ kNm, $M_{cb} = 18.8$ kNm

4.2 Analyze the continuous beam of Fig. P4.2 for a 5 mm settlement of support B without the applied loads. Draw shear force and bending moment diagrams. Take E = 200 GPa and $I = 150 \times 10^{-6}$ m^4.

(P4.2) $R_b = 12.31$ kN $\downarrow$, $R_c = 9.6$ kN $\uparrow$, $M_{ba} = -32.15$ kNm, $M_{ce} = -14.87$ kNm

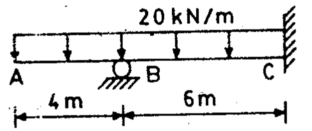
4.3 Analyze the continuous beam of Fig. P4.4 for a 5 mm settlement of support B. Draw shear force and bending moment diagrams. Take $E = 200 \times 10^6$ kN/m^2, $I = 300 \times 10^{-6}$ m^4.

(P4.4) $M_{ab} = -189.81$ kNm, $M_{ba} = -55.25$ kNm, $M_{cb} = 77.87$ kNm

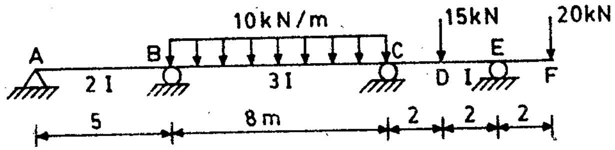
4.4 Analyze the portal frame shown in Fig. P4.5 using the three-moment equation and draw shear force and bending moment diagrams. Also draw its deflected shape.

(P4.5) $M_{ab} = 13.05$ kNm, $M_{ba} = 30.22$ kNm, $M_{ed} = -15.8$ kNm, $M_{de} = -27.46$ kNm

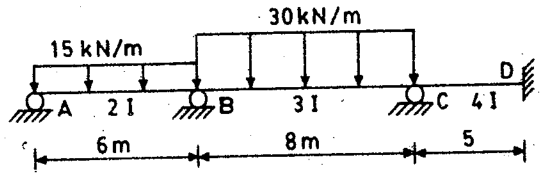
P 4.1



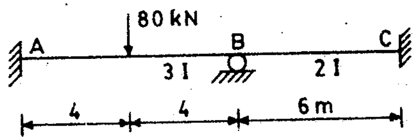
P 4.2



P 4.3



P 4.4



P 4.5

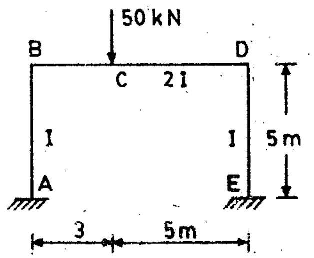


Fig. P 4.1 – P 4.5

CHAPTER
five

STRAIN ENERGY METHOD

5.1 INTRODUCTION

A structure undergoes elastic deformations under the application of external forces or other causes. The applied loads suffer displacement and work is done. This work is stored as strain energy in the structure. The strain energy is proportional to the strains produced in the structure. The work done by external forces or actions on an elastic system is equal to the strain energy stored internally on the system. The external forces are assumed to act gradually on the system from zero to their final values so that there is no dynamic effect. The external and internal actions or forces are in static equilibrium at all times.

The slopes and deflections produced in a structure depend upon the strains developed as a result of external actions. The strains may be axial, shear, flexural or torsional. Apparently, there is a relationship between strain energy and deformations in a structure. This relationship can be used to determine the slopes and deflections in a structure both statically determinate as well as statically indeterminate. Thus, strain energy method is one of the most powerful tools for the analysis of any structure.

The energy principles discussed in this chapter are applicable to linear elastic or non-linear elastic deformable systems. These principles are applied generally for the computation of deflections in statically determinate structures. Nevertheless it is desirable to understand scope of their application to the analysis of statically indeterminate structures. Strain energy and complementary energy methods are used for the analysis of structures. Both these methods may be considered as special cases of the principle of virtual work. The principle of virtual work may be associated with the principle of virtual forces. Thus, energy theorems may be grouped as follows:

1. Strain energy theorems:
   (a) Castigliano's first theorem
   (b) Principle of virtual displacement or unit displacement method
   (c) Principle of stationary potential energy.

2. Complementary energy theorems:
   (a) Castigliano's second theorem
   (b) Principle of virtual forces or unit load method
   (c) Principle of stationary complementary energy.

The theorems in the first group are used for the stiffness method or displacement method of analysis, while the theorems in the latter group are used for the flexibility method or force method of analysis.

5.2 WORK AND COMPLEMENTARY WORK

A force is defined as an action that tends to change the state of motion of a body to which it is applied. Force is equal to mass times acceleration according to the Newton's laws of motion. In structural mechanics, it is defined as the stress times area. When the point of application of a force moves, then a work is said to be done. The work is equal to force times the displacement in the direction of the applied force. Consider a structure acted upon by a load P_0 producing a displacement Δ_0. A typical load- displacement characteristics is shown in Fig. 5.1a for a linear structure when the load is applied gradually. The work done by a load P in causing a small displacement $d\Delta$ is shown by the shaded strip ,

$$dW = P\, d\Delta \qquad (5.1a)$$

The work done by P_0 is represented by the area OAB which is equal to the strain energy stored in the structure. The total work is found by summing all the increments of work represented by the similar strips as the displacement increases gradually from zero to the final value Δ_0. This is equal to the area under the force-deformation curve :

$$\therefore \quad W = \int_0^{\Delta_0} P\, d\Delta = U \qquad (5.1b)$$

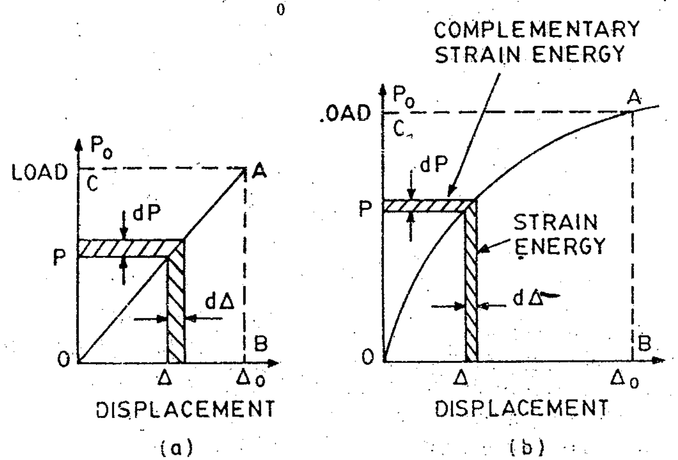


Fig. 5.1 Load displacement curves in elastic (a) linear structures and (b) non-linear structures

The work done may also be obtained by considering the horizontal strip shown in Fig. 5.1a :

$$dW^* = \Delta\, dP \qquad (5.2a)$$

The term W* represents complementary work. The load is increased by a amount dP. The total work done as the load increases from zero to its final value P_0 is obtained by summing all the increments of work represented by the similar strips. This is equal to the area above the force-deformation curve.

$$\therefore \quad W^* = \int_0^{P_0} \Delta\, dP = U^* \qquad (5.2b)$$

For a linear elastic system as shown in Fig. 5.1a,

$$W = W^*$$

and

$$U = U^*$$

and it is therefore not necessary to distinguish between work and complementary work, or strain energy and complementary strain energy whenever the Hook's law holds. Then, the integrals representing the areas under load-deformation curves are reduced to area of the triangles. Because of the relative simplicity in the evaluation of these expressions, linear elasticity is generally assumed in structural analysis if the material characteristic can be represented conveniently as linear.

For a nonlinear elastic system as shown in Fig. 5.1b, the work and complementary work are not equal. Therefore, it is necessary to clearly distinguish them. The area OAC represents the complementary work or complementary strain energy. The complementary strain energy is useful in computing deflections in a nonlinear elastic structure using Engesser's theorem.

5.3 STRAIN ENERGY

Let us first derive the expressions for various types of strain energy.

Axial Strain Energy

Consider a member of length L subjected to an axial load P. Let the area of cross-section of the member be A at a distance x from one end as shown in Fig. 5.2.

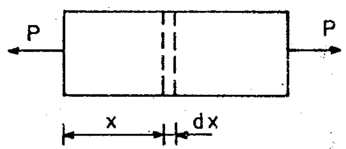


Fig. 5.2 Axial deformation

Stress in member $\quad \sigma = \dfrac{P}{A}$

Strain in member $\quad \varepsilon = \dfrac{\sigma}{E} = \dfrac{P}{AE}$

Elongation in a length $\quad dx = \dfrac{P\,dx}{AE}$

The total elongation in the member will be

$$\Delta L = \int_0^L \frac{P\,dx}{AE}$$

If the force P was increased gradually from zero to P, average force causing the elongation ΔL

$$= \frac{0+P}{2} = \frac{P}{2}$$

Therefore, work done U = Force × displacement

$$dU = \frac{P}{2} \times \Delta L$$

or,
$$U = \frac{P}{2}\int_0^L \frac{P\,dx}{AE} = \int_0^L \frac{1}{2}\cdot\frac{P^2dx}{AE} \qquad (5.3)$$

If area of cross-section of the member is uniform,

Total strain energy
$$U = \frac{P^2L}{2AE} \qquad (5.4)$$

The strain may be tensile or compressive.

Shear Strain Energy

Consider a small element of a structure dx, dy and dz subjected to a shear force V as shown in Fig.5.3. It causes a shear strain equal to θ and a lateral displacement equal to 'a' at the top fiber. If the shear force is applied gradually,

average shear force $= \dfrac{0+V}{2} = \dfrac{V}{2}$

or, $\quad V = \tau\ dy\ dz$

$\quad a = dx\,\theta$

$$\frac{\text{shear stress}}{\text{shear strain}} = \text{modulus of rigidity}$$

or, $\quad \dfrac{\tau}{\theta} = G \quad$ or $\quad \theta = \dfrac{\tau}{G}$

$\therefore \quad a = \dfrac{\tau}{G}\,dx$

$\therefore$ Work done $\quad U = \text{force} \times \text{displacement}$

$$dU = \frac{V}{2} \times a = \frac{V}{2}\frac{\tau}{G}\,dx \qquad (5.4)$$

$$= \frac{1}{2}\frac{\tau^2}{G}\,dx\,dy\,dz$$

Total strain energy
$$U = \iiint_V \frac{1}{2}\frac{\tau^2}{G}\,dx\,dy\,dz \qquad (5.5)$$

Alternatively, $\quad \tau = \dfrac{V}{A_e}$

where, A_e = effective area of cross-section in shear

$\therefore$ Eq. 5.4 becomes

$$dU = \frac{1}{2}\cdot\frac{V^2}{A_eG}\,dx$$

$\therefore$
$$U = \int_0^L \frac{1}{2}\frac{V^2}{A_eG}\,dx \qquad (5.6)$$

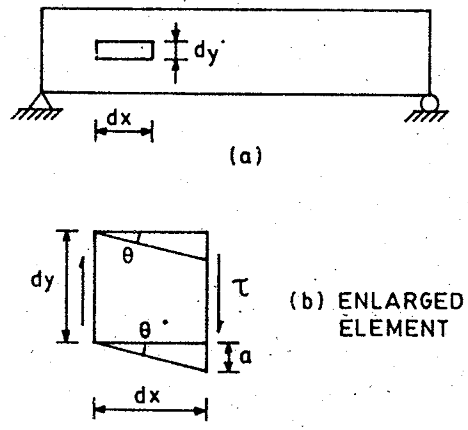


Fig. 5.3 Shear deformation

Flexural Strain Energy

Consider a small element of length ds subjected to a uniform bending moment M which produces an angle dθ in this length d as shown in Fig. 5.4. If moment is applied gradually,

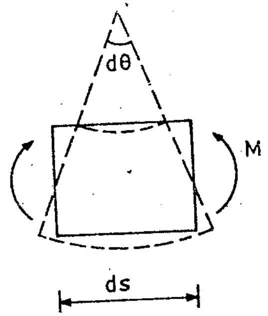


Fig. 5.4 Flexural deformation

$$\text{average bending moment} \quad = \frac{0+M}{2} = \frac{M}{2}$$

$$\text{Work done} \qquad dU = \frac{M}{2}\,d\theta$$

$$\text{but} \qquad d\theta = \frac{Mds}{EI} \qquad \text{by moment - area theorem}$$

$$\therefore \qquad dU = \frac{M^2 ds}{2EI}$$

$$\text{Total strain energy} \qquad U = \int \frac{1}{2}\frac{M^2 ds}{2EI} \tag{5.7}$$

Torsional Strain Energy

Consider a small element of length dx subjected to a torsional moment T which produces an angle of twist dθ as shown in Fig. 5.5.

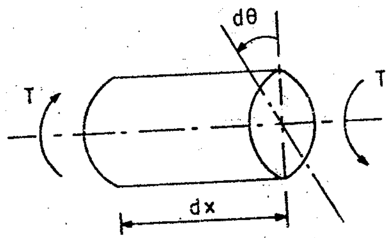


Fig. 5.5 Torsional deformation

$$\text{Work done} \qquad dU = \frac{T}{2} \times d\theta$$

$$\text{but} \qquad d\theta = \frac{Tdx}{GJ}$$

where $\quad G$ = shear modulus = $\dfrac{E}{2(1+\nu)}$

$\quad J$ = torsional inertia of the member

$$\text{Hence} \qquad dU = \frac{T^2 dx}{2GJ}$$

$$\text{Total strain energy} \qquad U = \int \frac{1}{2}\frac{T^2 dx}{GJ} \tag{5.8}$$

Combined Strain Energy

In a general plane beam element, there can be three force components: axial force P, shear force V and bending moment M. Total strain energy under the combined effect of these forces can be written as

$$U = \int \frac{P^2 dx}{2AE} + \int \frac{V^2 dx}{2A_e G} + \int \frac{M^2 dx}{2EI} \tag{5.9a}$$

If torsion is also included,

$$U = \int \frac{P^2 dx}{2AE} + \int \frac{V^2 dx}{2A_e G} + \int \frac{M^2 dx}{2EI} + \int \frac{T^2 dx}{2GJ} \tag{5.9b}$$

In general, the effect of axial deformation and shear deformation is very small (< 5%), therefore, only flexural strain energy given by Eq.5.7 is used extensively.

5.4 ENERGY THEOREMS

A. Castigliano published two theorems in 1879 to determine deflections in structures and redundants in statically indeterminate structures. These theorems may be stated as follows:

First Theorem

If a linearly elastic structure is subjected to a set of loads, the partial derivative of the total strain energy with respect to the deflection at any point is equal to the load applied at that point.

$$\frac{\partial U}{\partial \delta_i} = P_i \tag{5.10}$$

Second Theorem

If a linearly elastic structure is subjected to a set of loads, the partial derivative of the total strain energy with respect to a load applied at any point is equal to the deflection at that point.

The load may be a force, moment, shear or torsion. The corresponding displacement will be a deflection, slope, shear deformation or angle of twist. This theorem states that:

$$\frac{\partial U}{\partial P_1} = \delta_1, \qquad \frac{\partial U}{\partial M_1} = \theta_1 \tag{5.11}$$

where,
U = total strain energy
P_1, M_1 = loads on the structure
δ_1, θ_1 = displacements in the direction of the loads

If $$U = \int \frac{M^2 dx}{2EI}$$

$$\frac{\partial U}{\partial P} = \frac{\partial U}{\partial M}\frac{\partial M}{\partial P} = \int \frac{M dx}{EI}\frac{\partial M}{\partial P}$$

$$\delta = \int \frac{M}{EI}\frac{\partial M}{\partial P} dx \tag{5.12}$$

The integral is evaluated as it stands to give deflection under a load P. The value of the bending moment M at any other section is determined in terms of the load P. If no general expression for M in terms of P can be derived to cover the entire beam, and hence the limits of the integral, the beam can be divided into several segments as may be required, and the results added. Quite often, deflection (or slope) is required at a point or in a direction in which there is no load applied. In such cases, an imaginary load P is introduced in the required direction, the integral obtained in terms of P and then evaluated setting P equal to zero.

The first theorem is not very convenient to use since it requires expressing the strain energy in terms of deflections. The second theorem is very effective in the analysis of statically indeterminate structures. In fact, the application of the second theorem leads to the same expression as obtained by the method of virtual work.

Theorem of Least Work

It is applicable to any statically indeterminate structure and may be stated as follows:

For any statically indeterminate structure, the redundants should be such so as to make the total strain energy within a structure a minimum.

Thus, $$\frac{\partial U}{\partial P} = 0 \tag{5.13}$$

This means that the strain energy is either maximum or minimum. The maximum value of strain energy corresponds to unstable equilibrium. For a stable equilibrium, the strain energy should be minimum. The validity of the theorem of least work comes directly for the Castigliano's second theorem. Both these theorems are applicable to statically indeterminate beams, trusses as well as frames. By differentiating the total strain energy of the structure with respect to each unknown redundant separately and equating to zero, there will be as many equations as the number of unknown reactions. The values of these redundants can be determined by the solution of linear simultaneous equations. The following examples illustrate the application of these theorems in the solution of statically indeterminate structures.

Principle of Virtual Work

It may be stated as follows:

If a body is in equilibrium under a virtual force system and remains in equilibrium while it is subjected to a small deformation, the virtual work done by the external forces is equal to the virtual work done by the internal stresses due to those forces.

This is very useful in computing deflections in any structure using:

$$\Delta = \left(\frac{\partial U}{\partial P}\right)_{P=0} \tag{5.14}$$

or, $$\Delta = \int \frac{Mmds}{EI} \tag{5.15}$$

where
M = moment due to external load
m = moment due to unit load

This is the same expression as derived by the unit load method.

The main difference between the virtual work method and the Castigliano's method is in the order of differentiation and integration. In the virtual work method, the partial differential term (M m ds/ EI) is placed inside the integral and then summed up. In the Castigliano's method, the strain energy is integrated first and then differentiated.

Betti's Reciprocal Theorem

This theorem is very useful and is applicable to any type of structure. It was proved by Betti in 1872 and may be stated as follows:

If a structure is acted upon by two force systems P_A and P_B, in equilibrium separately, the external virtual work done by a system of forces P_B during the deformations caused by another systems of forces P_A is equal to the external virtual work done by the P_A system during the deformations caused by the P_B system.

Betti's law is a generalization of the Maxwell's law. The latter can be stated as follows:

> *The deflection of point 1 on a structure due to a load P at point 2 is equal to the deflection of point 2 due to the load P acting at point 1. Of course, the deflections referred to are in the same direction as the applied loads.*

This law is very useful in the analysis of statically indeterminate structures as well as in constructing influence lines for structures.

These theorems were derived in chapter 9 of volume 1 of this book.

Engesser's Complimentary Strain Energy Theorem

> *If a linearly on nonlinearly elastic structure is subjected to a set of loads, the partial derivative of the complementary strain energy with respect to any load P_i is equal to the displacement δ_i of the point of application of that force in the direction of its line of action.*

$$\frac{\partial U^*}{\partial P_i} = \Delta_i \tag{5.16}$$

This theorem is very similar to the Castigliano's second theorem which is applicable to a linear elastic structure.

The applications of various theorems is illustrated in the following examples. The strain energy method is very similar to the consistent deformation method discussed in chapter 3.

5.5 BEAMS - ILLUSTRATIVE EXAMPLES

Example 5.1

Analyze the propped cantilever beam shown in Fig. 5.6 using the strain energy method. Take EI = constant.

Solution

The beam is statically indeterminate to degree 1. Let us consider reaction R_C as redundant and remove the support C.

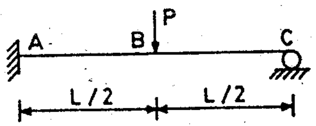


Fig. 5.6

$$\text{Strain energy } U = \int_0^L \frac{1}{2}\frac{M^2 dx}{EI}$$

Using Castigliano's second theorem

$$\Delta_C = \frac{\partial U}{\partial R_C} = \int \frac{M}{EI}\frac{\partial M}{\partial R_C}dx$$

$$\text{For U to be minimum,} \quad \frac{\partial U}{\partial R_C} = 0$$

Let us evaluate the integrals as follows:

Segment	CB	BA
Origin	C	B
Limits	0 – L/2	0 – L/2
M	$R_C\,x$	$R_C\,(L/2+x) - Px$
$\frac{\partial M}{\partial R_C}$	x	L/2 + x

$$\frac{\partial U}{\partial R_C} = \frac{1}{EI}\int_0^{L/2} R_C x^2 dx + \frac{1}{EI}\int_0^{L/2}\left[R_C\left(\frac{L}{2}+x\right) - Px\right]\left(\frac{L}{2}+x\right)dx$$

$$= \frac{R_C}{EI}\left|\frac{x^3}{3}\right|_0^{L/2} + \frac{1}{EI}\left|\frac{R_C}{3}\left(\frac{L}{2}+x\right)^3 - \frac{Px^2L}{4} - \frac{Px^3}{3}\right|_0^{L/2}$$

$$= \frac{R_C L^3}{24EI} + \frac{1}{EI}\left[\frac{R_C L^3}{3} - \frac{PL^3}{16} - \frac{PL^3}{24} - \frac{R_C L^3}{24}\right]$$

$$= \frac{R_C L^3}{3EI} - \frac{5PL^3}{48EI} = \Delta_C$$

$$\frac{\partial U}{\partial R_C} = 0 \text{ for minimum potential energy}$$

or,

$$R_C = \frac{5P}{16}$$

Moment at A $\quad M_A = \dfrac{5PL}{16} - \dfrac{PL}{2} = -\dfrac{3PL}{16}$ hogging

Net moment at B $\quad M_B = \dfrac{5PL}{32}$ sagging

Example 5.2

Determine the contributions in deflection due to bending and shear deformations at the midspan of beam AB shown in Fig. 5.7a.

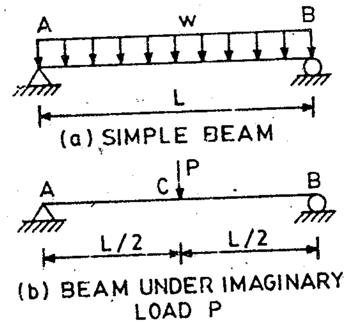


Fig 5.7

Solution

In order to determine deflection at midspan of the simple beam AB, an imaginary vertical load P is applied at C as shown in Fig. 5.7b. The strain energy minimized at P = 0 will give the desired deflection.

Segment AC

Deflection due to Bending Deformations

$$M = \frac{wL}{2}x - \frac{wx^2}{2} + \frac{P}{2}x, \qquad \frac{\partial M}{\partial P} = \frac{x}{2}$$

$$\Delta_b = \left(\frac{\partial U}{\partial P}\right)_{P=0} = \int M \frac{\partial M}{\partial P}\frac{dx}{EI}$$

$$\frac{\partial U}{\partial P} = 2\int_0^{L/2}\left(\frac{wLx}{2} - \frac{wx^2}{2} + \frac{Px}{2}\right)\left(\frac{x}{2}\right)\frac{dx}{EI}$$

$$= \frac{1}{EI}\left|\frac{wL}{2}\frac{x^3}{3} - \frac{w}{2}\frac{x^4}{4} + \frac{P}{2}\frac{x^3}{3}\right|_0^{L/2}$$

$$\frac{\partial U}{\partial P} = \frac{5}{384}\frac{wL^4}{EI} + \frac{PL^3}{48EI}$$

$$\therefore \quad \Delta_b = \left(\frac{\partial U}{\partial P}\right)_{P=0} = \frac{5}{384}\frac{wL^4}{EI}$$

Deflection Due to Shear Deformations

$$V = \frac{wL}{2} - wx + \frac{P}{2}, \qquad \frac{\partial V}{\partial P} = \frac{1}{2}$$

$$\Delta_v = \left(\frac{\partial U}{\partial P}\right)_{P=0}$$

$$\frac{\partial U}{\partial P} = \int V \frac{\partial V}{\partial P}\frac{dx}{A_eG}$$

$$= 2\int_0^{L/2}\left[\frac{wL}{2} - wx + \frac{P}{2}\right]\frac{1}{2}\frac{dx}{A_eG} = \frac{1}{A_eG}\left|\frac{wL}{2}x - \frac{wx^2}{2} + \frac{Px}{2}\right|_0^{L/2}$$

$$= \frac{wL^2}{8A_eG} + \frac{PL}{4A_eG}$$

$$\therefore \quad \Delta_V = \left(\frac{\partial U}{\partial P}\right)_{P=0} = \frac{wL^2}{8A_eG}$$

Hence,
$$\frac{\text{Deflection due to shear}}{\text{Deflection due to bending}} = \frac{wL^2/8A_eG}{5wL^4/384EI}$$

$$\frac{\Delta_V}{\Delta_b} = 9.6\left(\frac{E}{G}\right)\left(\frac{I}{L^2A_e}\right) \qquad \text{(i)}$$

This equation is valid for simply supported beam of any cross-section subjected to a uniform load.

$$G = \frac{E}{2(1+\nu)} \qquad \text{if } \nu = 0.25, \quad G = 0.4\,E$$

For a steel beam of I section

A_e = Area of web where shear stress may be considered to be constant.
$= D\,t_w$

$$\therefore \quad \frac{\Delta_V}{\Delta_b} = \frac{24\,I}{L^2 A_e} = \frac{24\,I}{A_e D^2}\left(\frac{D}{L}\right)^2 \qquad \text{(ii)}$$

$$= C\left(\frac{D}{L}\right)^2$$

where D = depth of I-section
t_w = thickness of web of I section

$$C = \frac{24\,I}{A_e\,D^2} \qquad \text{(iii)}$$

For rolled steel sections generally used in beams, C varies as follows:

$$\text{For ISMB 100, } C = \frac{24 \times 257.5}{(10 \times 0.4) \times 10^2} = 15.45$$

$$\text{For ISMB 600, } C = \frac{24 \times 91813}{(60 \times 1.2) \times 60^2} = 8.5$$

For a rectangular section

$$\text{Eq. (ii) gives} \quad \frac{\Delta_V}{\Delta_b} = \frac{24 \times \frac{bD^3}{12}}{L^2 \times bD} = 2\left(\frac{D}{L}\right)^2$$

If,	D/L	=	1	0.8	0.5	0.2	0.1
Deflection ratio	(%)	=	200%	128%	50%	8%	2%

In the majority of practical cases, D/L lies around 0.1, hence contribution of shear deformation is very small. In deep beams such as pier caps and well caps in bridge structures, the effect of shear deformations is quite appreciable.

Example 5.3

Analyze a two span continuous beam shown in Fig. 5.8 using the strain energy method and draw bending moment diagram.

Solution

The beam is indeterminate to a degree 1. Let us treat R_B as redundant and remove the support B.

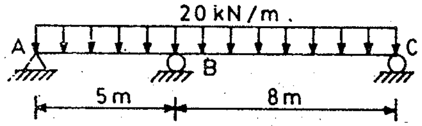


Fig. 5.8

$$\text{Strain energy } U = \int \frac{1}{2}\frac{M^2 dx}{EI}$$

Using Castigliano's second theorem

$$\frac{\partial U}{\partial R_C} = \int \frac{M}{EI}\frac{\partial M}{\partial R_C} dx$$

The integrals can be evaluated as follows:

Segment	AB	CB
Origin	A	C
Limits	0 – 5 m	0 – 8 m
M due to UDL	$130x - 20\frac{x^2}{2}$	$130x - 20\frac{x^2}{2}$
M due to R_B	$-\frac{8}{13}R_B x$	$-\frac{5}{13}R_B x$

$$\frac{\partial U}{\partial R_C} = \int_0^5 \left(130x - 10x^2 - \frac{8}{13}R_B x\right)\left(-\frac{8}{13}x\right)\frac{dx}{EI} + \int_0^8 \left(130x - 10x^2 - \frac{5}{13}R_B x\right)\left(-\frac{5}{13}x\right)\frac{dx}{EI}$$

$$= \frac{-8}{13EI}\left|130\frac{x^3}{3} - 10\frac{x^4}{4} - \frac{8}{13}R_B\frac{x^3}{3}\right|_0^5 + \frac{-5}{13EI}\left|130\frac{x^3}{3} - 10\frac{x^4}{4} - \frac{5}{13}R_B\frac{x^3}{3}\right|_0^8$$

$$= \frac{(-1)}{13EI}\left[30833.36 - 205.12\,R_B + 59733.35 - 328.2\,R_B\right]$$

$$\text{or,} \quad \frac{\partial U}{\partial R_C} = 0 \quad \text{for minimum strain energy}$$

$$\text{or,} \quad R_B = 169.81 \text{ kN}$$

This is the same value as obtained in Ex. 3.2 using the method of consistent deformation. O. K.

5.6 FRAMES - ILLUSTRATIVE EXAMPLES

Example 5.4

Determine contributions in the total vertical deflection due to bending, axial and shear deformations at the free end of a portal frame shown in Fig. 5.9a. Take EI = Constant, AE = 50 EI, $A_eG = 25$ EI

Solution

This is a statically determinate frame. Let us draw bending moment, shear and thrust diagrams due to applied loads as shown in Figs. 5.9 b, c and d. In order to determine vertical deflection at F, the external loads are removed, a vertical load W is applied at F and moment, shear and thrust diagrams are plotted as shown in Figs. 5.9 f, g and h. The various integrals can now be evaluated.

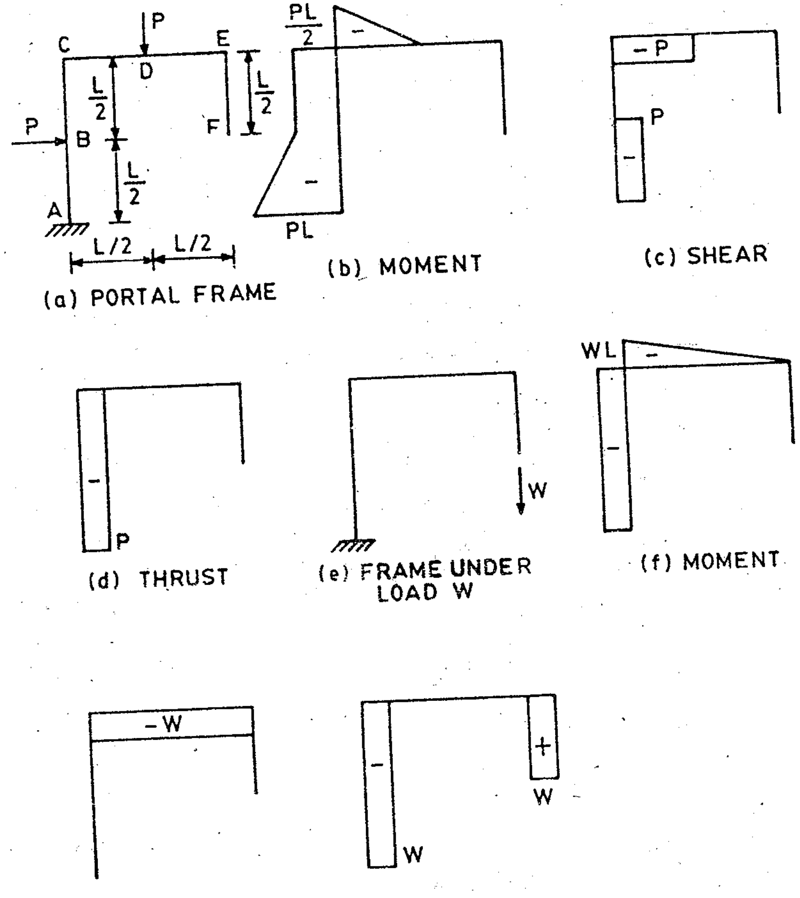


Fig. 5.9 Portal frame and force diagrams due to different loadings

Total strain energy $U = \int \frac{1}{2}\frac{M^2}{EI}dx + \int \frac{1}{2}\frac{V^2}{A_eG}dx + \int \frac{1}{2}\frac{T^2dx}{AE}$

Castigliano's second theorem gives

$$\Delta = \frac{\partial U}{\partial W}$$

Member EF

Bending moment M = 0,

$$\therefore \int \frac{M}{EI}\frac{\partial M}{\partial W}dx = 0 \qquad (1)$$

Shear V = 0

$$\therefore \int \frac{V}{A_eG}\frac{\partial V}{\partial W}dx = 0 \qquad (2)$$

Thrust T = W

$$\frac{\partial T}{\partial W} = 1$$

$$\int \frac{T}{AE}\frac{\partial T}{\partial W}dx = \int_0^{L/2} \frac{W}{AE}dx = \frac{WL}{2AE} \qquad (3)$$

Member DE

$$M = 0 - Wx, \qquad \frac{\partial M}{\partial W} = -x$$

$$\int \frac{M}{EI}\frac{\partial M}{\partial W}dx = \int_0^{L/2} (-Wx)(-x)\frac{dx}{EI} = \left|\frac{WL^3}{3EI}\right|_0^{L/2} = \frac{WL^3}{24EI} \qquad (4)$$

$$V = -W, \qquad \frac{\partial V}{\partial W} = -1$$

$$\int \frac{V}{A_eG}\frac{\partial V}{\partial W}dx = \int_0^{L/2}(-W)(-1)\frac{dx}{A_eG} = \frac{WL}{2A_eG} \qquad (5)$$

$$T = 0$$

$$\int \frac{T}{AE}\frac{\partial T}{\partial W}dx = 0 \tag{6}$$

Member CD

$$M = -Px - W\left(\frac{L}{2}+x\right)$$

$$\frac{\partial M}{\partial W} = -\left(\frac{L}{2}+x\right)$$

$$\int M\frac{\partial M}{\partial W}\frac{dx}{EI} = \int_0^{L/2}\left[-Px - W\left(\frac{L}{2}+x\right)\right](-)\left(\frac{L}{2}+x\right)\frac{dx}{EI}$$

$$= \int_0^{L/2}\left[\frac{PLx}{2}+Px^2+W\left(\frac{L}{2}+x\right)^2\right]\frac{dx}{EI}$$

$$= \left|\frac{PLx^2}{4}+\frac{Px^3}{3}+\frac{W}{3}\left(\frac{L}{2}+x\right)^3\right|_0^{L/2}$$

$$= \frac{5}{48}\frac{PL^3}{EI}+\frac{7}{24}\frac{WL^3}{EI} \tag{7}$$

$$V = -P - W, \qquad \frac{\partial V}{\partial W} = -1$$

$$\int V\frac{\partial V}{\partial W}\frac{dx}{A_eG} = \int_0^{L/2}(-P-W)(-1)\frac{dx}{A_eG} = \left|\frac{(P+W)x}{A_eG}\right|_0^{L/2} = \frac{(P+W)L}{2A_eG} \tag{8}$$

$$\int T\frac{\partial T}{\partial W}\frac{dx}{AE} = 0 \tag{9}$$

Member BC

$$M = -\frac{PL}{2} - WL, \qquad \frac{\partial M}{\partial W} = -L$$

$$\int M\frac{\partial M}{\partial W}\frac{dx}{EI} = \int_0^{L/2}\left(\frac{PL^2}{2}+WL^2\right)\frac{dx}{EI} = \frac{PL^3}{4EI}+\frac{WL^3}{2EI} \tag{10}$$

$$V = 0$$

$$\int V\frac{\partial V}{\partial W}\frac{dx}{A_eG} = 0 \tag{11}$$

$$T = -P - W, \qquad \frac{\partial T}{\partial W} = -1$$

$$\int T\frac{\partial T}{\partial W}\frac{dx}{AE} = \int_0^{L/2}(P+W)\frac{dx}{AE} = (P+W)\frac{L}{2AE} \tag{12}$$

Member AB

$$M = -\frac{PL}{2} - Px - WL, \qquad \frac{\partial M}{\partial W} = -L$$

$$\int M\frac{\partial M}{\partial W}\frac{dx}{EI} = \int_0^{L/2}\left(\frac{PL^2}{2}+PLx+WL^2\right)\frac{dx}{EI} = \frac{3}{8}\frac{PL^3}{EI}+\frac{WL^3}{2EI} \tag{13}$$

$$V = -P, \qquad \frac{\partial V}{\partial W} = 0$$

$$\int V\frac{\partial V}{\partial W}\frac{dx}{A_eG} = 0 \tag{14}$$

$$T = -P - W, \qquad \frac{\partial T}{\partial W} = -1$$

$$\int T\frac{\partial T}{\partial W}\frac{dx}{AE} = (P+W)\frac{L}{2AE} \tag{15}$$

Net vertical deflection at F = $\left(\frac{\partial U}{\partial W}\right)_{W=0}$

The various strain energy derivative terms obtained through Eqs. 1 to 15 can now be combined to get the net vertical deflection at F.

To evaluate the net vertical deflection at F due to *flexural deformation alone*, let us combine Eqs. 1, 4, 7, 10 and 13, that is,

$$\Delta_F = \left(\frac{\partial U}{\partial W}\right)_{W=0} = \int M \frac{\partial M}{\partial W}\frac{dx}{EI}$$

bending $$\Delta_F = \frac{5}{48}\frac{PL^3}{EI} + \frac{PL^3}{4EI} + \frac{3}{8}\frac{PL^3}{EI} = \frac{35}{48}\frac{PL^3}{EI}$$

To evaluate the net vertical deflection at F due to *shear deformations alone*, let us combine Eqs. 2, 5, 8, 11 and 14, that is,

$$\Delta_F = \left(\frac{\partial U}{\partial W}\right)_{W=0} = \int V \frac{\partial V}{\partial W}\frac{dx}{A_e G}$$

shear $$\Delta_F = \frac{PL}{2A_eG} = \frac{PL}{50EI}$$

To evaluate the net vertical deflection at F due to *axial deformations alone*, let us combine Eqs. 3, 6, 9, 12 and 15, that is,

$$\Delta_F = \left(\frac{\partial U}{\partial W}\right)_{W=0} = \int T \frac{\partial T}{\partial W}\frac{dx}{AE}$$

thrust $$\Delta_F = \frac{PL}{2AE} + \frac{PL}{2AE} = \frac{PL}{AE} = \frac{PL}{50EI}$$

Total deflection, if L = 3 m

$$\Delta_F = \frac{P}{EI}\left\{\frac{35}{48}\times 3^3 + \frac{3}{50} + \frac{3}{50}\right\}$$

$$= (26.98 + 0.06 + 0.06)\frac{P}{EI} = 27.1\frac{P}{EI}$$

The proportion of deflections due to various causes with respect to the total deflection are :

bending 99. 56 %, shear 0. 22 %, and thrust 0. 22 %

Example 5.5

Analyze the portal frame shown in Fig. 5.10 for horizontal thrust at F using the strain energy method.

Solution

The frame is statically indeterminate to a degree 1. Introduce a roller support at F and treat horizontal reaction at F, R_{Fx} as a redundant. The bending moment diagrams in the released structure due to the applied loads and redundant R_{Fx} are shown in Figs. 3.13 c and f. The various integrals can be evaluated as follows.

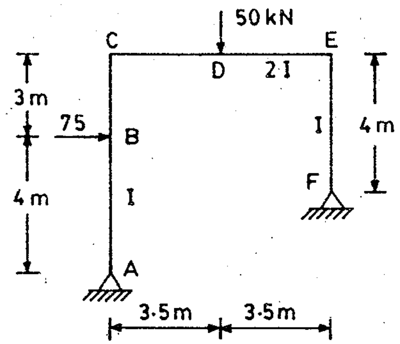


Fig. 5.10

Member FE

$$M = 0 + R_{Fx}\,x, \quad \frac{\partial M}{\partial R_{Fx}} = x$$

$$\int M \frac{\partial M}{\partial R_{Fx}}\frac{dx}{EI} = \int_0^4 R_{Fx}x^2\frac{dx}{EI} = \frac{1}{EI}\left|R_{Fx}\frac{x^3}{3}\right|_0^4 = \frac{21.34}{EI}R_{FX} \qquad (1)$$

Member ED

$$M = 67.85\,x + \left(4 + \frac{3x}{7}\right)R_{Fx}, \quad \frac{\partial M}{\partial R_{Fx}} = 4 + \frac{3x}{7}$$

$$\int M \frac{\partial M}{\partial R_{Fx}}\frac{dx}{EI} = \int_0^{3.5}\left[67.85x + \left(4 + \frac{3x}{7}\right)R_{Fx}\right]\left(4 + \frac{3x}{7}\right)\frac{dx}{EI}$$

$$= \int_0^{3.5}\left[271.4x + 29.08x^2 + \left(4 + \frac{3x}{7}\right)^2 R_{Fx}\right]\frac{dx}{2EI}$$

$$= \frac{1}{2EI}\left|271.4\frac{x^2}{2} + 29.08\frac{x^3}{3} + \frac{1}{3}\left(4 + \frac{3x}{7}\right)^3 R_{Fx}\left(\frac{7}{3}\right)\right|_0^{3.5}$$

$$= (1039 + 39.81\,R_{Fx})\frac{1}{EI} \qquad (2)$$

Member DC

$$M = 17.85\,x + 237.48 + \left(5.5 + \frac{3x}{7}\right)R_{Fx}, \quad \frac{\partial M}{\partial R_{Fx}} = 5.5 + \frac{3x}{7}$$

$$\int M \frac{\partial M}{\partial R_{Fx}} \frac{dx}{EI} = \int_0^{3.5} \left[17.85x + 237.48 + \left(5.5 + \frac{3x}{7}\right) R_{Fx}\right]\left(5.5 + \frac{3x}{7}\right)\frac{dx}{2EI}$$

$$= \int_0^{3.5} \left[7.65x^2 + 199.96x + 1306.14 + \left(5.5 + \frac{3x}{7}\right)^2 R_{Fx}\right] \frac{dx}{2EI}$$

$$= \frac{1}{2EI}\left| 7.65\frac{x^3}{3} + 1306.14x + 199.96\frac{x^2}{2} + \frac{1}{3}\left(5.5 + \frac{3x^3}{3}\right)\frac{7}{3} R_{Fx}\right|_0^{3.5}$$

$$= \frac{1}{EI}\left[2952.79 + 68.69\, R_{Fx}\right] \tag{3}$$

Member C B

$$M = 300 + (7 - x) R_{Fx}, \qquad \frac{\partial M}{\partial R_{Fx}} = (7 - x)$$

$$\int M \frac{\partial M}{\partial R_{Fx}} \frac{dx}{EI} = \int_0^{3} \left[300 + (7-x) R_{Fx}\right](7-x)\frac{dx}{EI} = \int_0^{3} \left[2100 - 300x + (7-x)^2 R_{Fx}\right]\frac{dx}{EI}$$

$$= \left| 2100x - 300\frac{x^2}{2} - \frac{(7-x)^3}{3} \right|_0^3 \frac{1}{EI}$$

$$= (4950 + 93\, R_{Fx})\frac{1}{EI} \tag{4}$$

Member B A

$$M = (300 - 75x) + (4 - x) R_{Fx}, \qquad \frac{\partial M}{\partial R_{Fx}} = (4 - x)$$

$$\int M \frac{\partial M}{\partial R_{Fx}} \frac{dx}{EI} = \int_0^{4} \left[300 - 75x + (4-x) R_{Fx}\right](4-x)\frac{dx}{EI}$$

$$= \int_0^{4} \left[1200 - 600x + 75x^2 + (4-x)^2 R_{Fx}\right]\frac{dx}{EI}$$

$$= \left| 75\frac{x^3}{3} - 600\frac{x^2}{2} + 1200x - \frac{(4-x)^3}{3} R_{Fx}\right|_0^4 \frac{1}{EI}$$

$$= (1600 + 21.34\, R_{Fx})\frac{1}{EI} \tag{5}$$

For minimum strain energy, $\dfrac{\partial U}{\partial R_{Fx}} = 0$

$$\frac{1}{EI}\left[21.34\, R_{Fx} + 1039 + 39.81\, R_{Fx} + 2952.79 + 68.69\, R_{Fx} + 4950 + 93\, R_{Fx} + 1600 + 21.34\, R_{Fx}\right] = 0$$

or, $244.18\, R_{Fx} = -10541.79$

or, $R_{Fx} = -43.17$ kN

This is the same value as obtained using the method of consistent deformations in Ex. 3.11. O. K.

Example 5.6

Analyze the portal frame shown in Fig 5.11a using the strain energy method.

Solution

The frame is statically indeterminate to degree 1. Let us replace the hinge support at D by a roller support. The released structure is shown in Fig 5.11b. The free body diagram due to the applied loads is shown in Fig 5.11c and that due to the redundant R_{Dx} is shown in Fig. 5. 11d. The various integrals can be evaluated as follows :

Member C D

$M = 10.825\,x - R_{Dx}\, x \tan 60°$ (Using the orthogonal system of axes)

$$\frac{\partial M}{\partial R_{Dx}} = -\,x \tan 60° = -\sqrt{3}\,x$$

$$\int M \frac{\partial M}{\partial R_{Dx}} \frac{dx}{EI} = \int_0^3 \left[10.825x - R_{Dx} x\sqrt{3}\right]\left(-x\sqrt{3}\right)\frac{dx}{EI}$$

$$= \frac{1}{EI}\int_0^3 \left[-10.825x^2\sqrt{3} + 3x^2 R_{Dx}\right]dx = \frac{1}{EI}\left| -10.825\sqrt{3}\frac{x^3}{3} + x^3 R_{Dx}\right|_0^3$$

$$= \frac{1}{EI}\left[-168.75 + 27\, R_{Dx}\right] \tag{1}$$

Member B C

$$M = 32.475 + 10.825\,x - 3\sqrt{3}\,R_{Dx}, \qquad \frac{\partial M}{\partial R_{Dx}} = -3\sqrt{3}$$

$$\int M \frac{\partial M}{\partial R_{Dx}} \frac{dx}{EI} = \int_0^6 \left[32.475 + 10.825 - 3\sqrt{3}\,R_{Dx}\right]\left(-3\sqrt{3}\right)\frac{dx}{2EI}$$

$$= \frac{1}{2EI}\left|-168.74x - 56.24\frac{x^2}{2} + 27R_{Dx}x\right|_0^6$$

$$= \frac{1}{EI}\left[-1012.38 + 81\,R_{Dx}\right] \qquad (2)$$

Member A B

$$M = 25\,x\,\tan 60^\circ - 10.825\,x - R_{Dx}\,x\tan 60^\circ, \qquad \frac{\partial M}{\partial R_{Dx}} = -x\sqrt{3}$$

$$\int M \frac{\partial M}{\partial R_{Dx}} \frac{dx}{EI} = \int_0^3 \left[25\sqrt{3}x - 10.825x - R_{Dx}x\sqrt{3}\right]\left(-x\sqrt{3}\right)\frac{dx}{EI}$$

$$= \frac{1}{EI}\left|-75\frac{x^3}{3} + 10.825\sqrt{3}\frac{x^3}{3} + 3R_{Dx}\frac{x^3}{3}\right|_0^3$$

$$= \frac{1}{EI}\left[-506.26 + 27\,R_{Dx}\right] \qquad (3)$$

For minimum strain energy,

$$\frac{\partial U}{\partial R_{Fx}} = 0, \text{ combining Eqs. 1, 2 and 3}$$

or, $$\frac{1}{EI}\left[-168.75 + 27\,R_{Dx} - 1012.38 + 81\,R_{Dx} - 506.26 + 27\,R_{Dx}\right] = 0$$

or, $R_{Dx} = 12.50$ kN

Net moment at B

$$= 97.425 - 3\sqrt{3} \times 12.50 = 32.50 \text{ kNm}$$

The net bending moment diagram is shown in Fig. 5.11e.

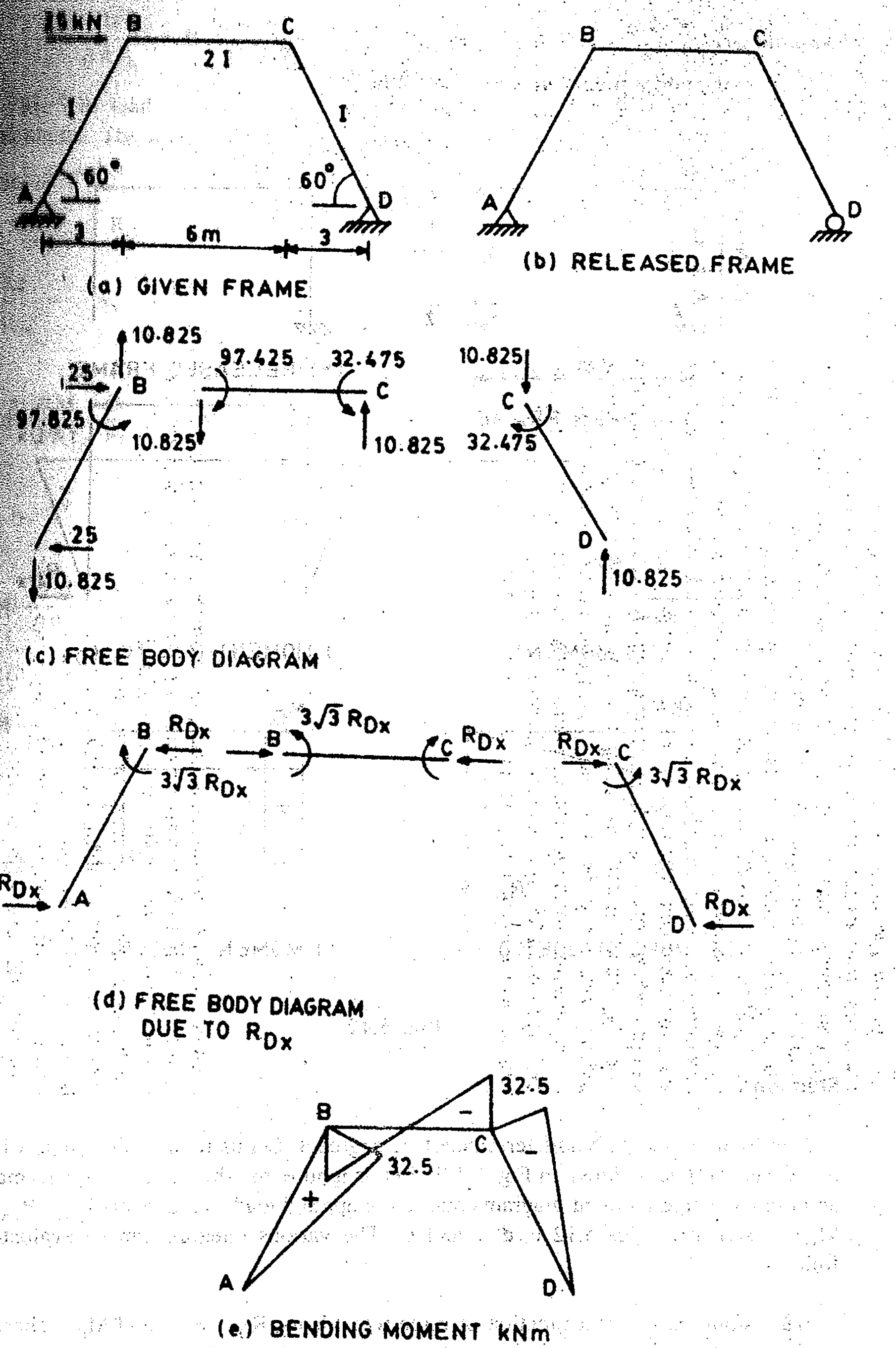


Fig. 5.11 Portal frame with inclined hinged legs

Example 5.7

Analyze the portal frame shown in Fig 5.12a.

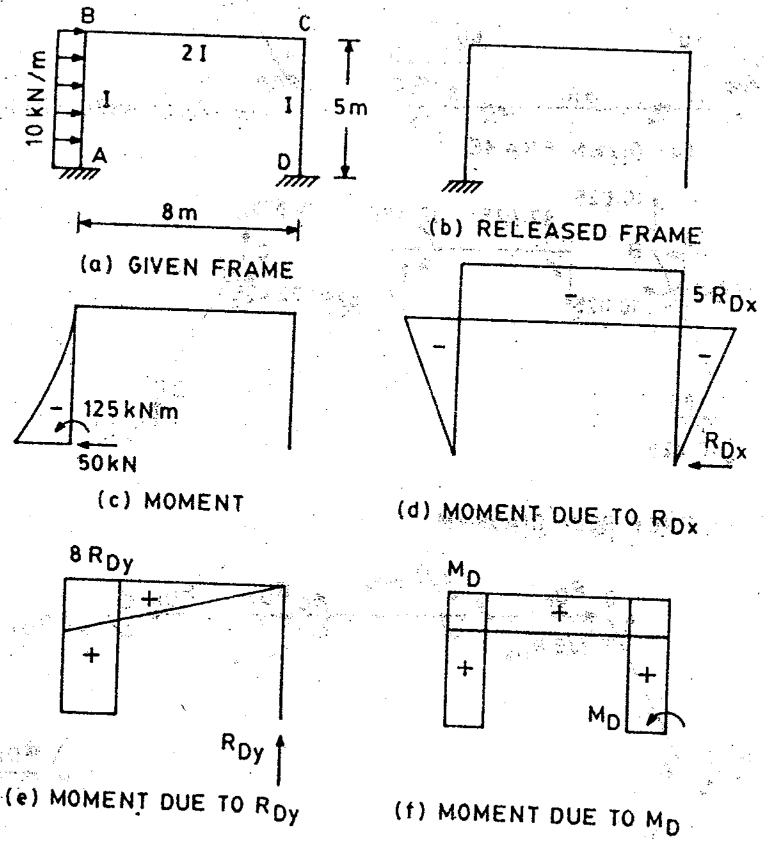


Fig. 5.12

Solution

The frame is statically indeterminate to a degree 3. Let us remove the support D and the released frame is shown in Fig. 5.12b. Let us ignore the shear and axial deformations and draw bending moment diagrams due to the applied loads, redundants R_{Dx}, R_{Dy} and M_D as shown in Figs 5.12 c, d, e and f. The various integrals can be evaluated as follows.

The strain energy is a function of three redundants, R_{Dx}, R_{Dy} and M_D. Therefore, strain energy needs to be minimized with respect to each one of them, that is :

$$\frac{\partial U}{\partial R_{Dx}} = 0, \quad \frac{\partial U}{\partial R_{Dy}} = 0 \text{ and } \frac{\partial U}{\partial M_D} = 0$$

This will lead to three linear simultaneous equations in terms of three unknown redundants. The solution of these linear equations will give the values of the redundants.

Member CD

$$M = -R_{Dx}\,x + M_D$$

$$\frac{\partial M}{\partial R_{Dx}} = -x, \quad \frac{\partial M}{\partial R_{Dy}} = 0, \quad \frac{\partial M}{\partial M_D} = 1$$

$$\int M\frac{\partial M}{\partial R_{Dx}}\frac{dx}{EI} = \int_0^5 (M_D - R_{Dx}x)(-x)\frac{dx}{EI} = \left| -M_D\frac{x^2}{2} + R_{Dx}\frac{x^3}{3}\right|_0^5 \frac{1}{EI}$$

$$= [-12.5\,M_D + 41.67\,R_{Dx}]\frac{1}{EI} \qquad (1)$$

$$\int M\frac{\partial M}{\partial R_{Dy}}\frac{dx}{EI} = 0 \qquad (2)$$

$$\int M\frac{\partial M}{\partial M_D}\frac{dx}{EI} = \int_0^5 (M_D - R_{Dx}x)\frac{dx}{EI} = \left| M_D x - R_{Dx}\frac{x^2}{2}\right|_0^5 \frac{1}{EI}$$

$$= [5\,M_D - 12.5\,R_{Dx}]\frac{1}{EI} \qquad (3)$$

Member BC

$$M = 0 - 5\,R_{Dx} + R_{Dy}\,x + M_D$$

$$\frac{\partial M}{\partial R_{Dx}} = -5, \quad \frac{\partial M}{\partial R_{Dy}} = x, \quad \frac{\partial M}{\partial M_D} = 1$$

$$\int M\frac{\partial M}{\partial R_{Dx}}\frac{dx}{EI} = \int_0^8 \frac{1}{2EI}\left[-5R_{Dx} + R_{Dy}x + M_D\right](-5)dx$$

$$= \frac{1}{2EI}\left|25x\,R_{Dx} - 5\frac{x^2}{2}R_{Dy} - 5x\,M_D\right|_0^8$$

$$= \frac{1}{EI}[100\,R_{Dx} - 80\,R_{Dy} - 20\,M_D] \qquad (4)$$

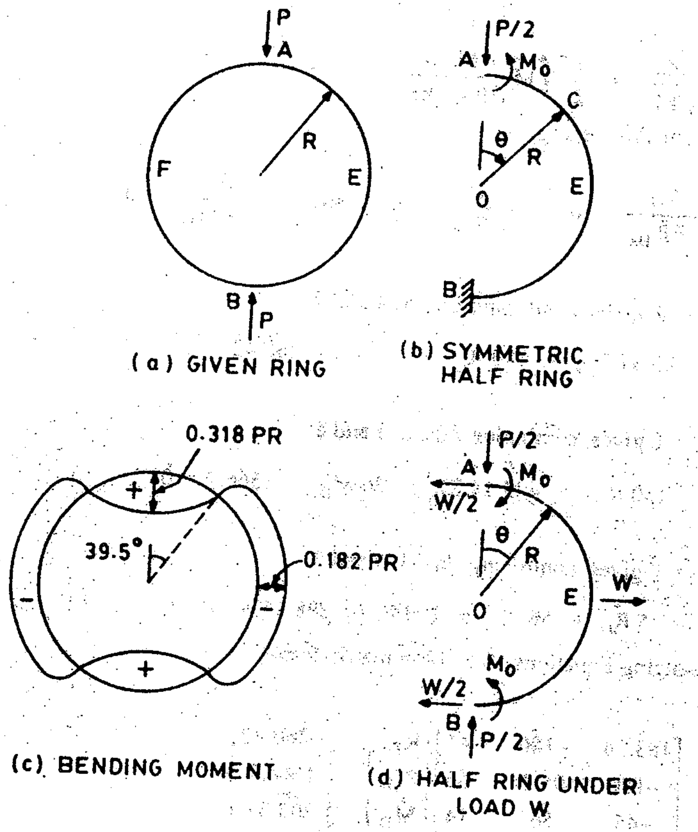


Fig. 5.13 Circular ring frame

Solution

Due to symmetry along the diagonal AB, the free body diagram is shown in Fig. 5.13b. The half ring carries a load equal to P/2 and moment M_o at A. The frame is indeterminate to a degree 3. Due to symmetry, the shear at A is zero. Let us ignore the thrust at A.

Strain energy $U = \int \frac{1}{2}\frac{M^2 dx}{EI}$

$$\frac{\partial U}{\partial M_o} = \int_0^{\pi} M \frac{\partial M}{\partial M_o}\frac{ds}{EI}$$

At any point C, the moment is given by,

$$M = M_o + \frac{P}{2} R \sin\theta, \quad \frac{\partial M}{\partial M_o} = 1, \quad ds = R\, d\theta$$

$$\therefore \quad \frac{\partial U}{\partial M_o} = \int_0^{\pi}\left(M_o + \frac{P}{2} R \sin\theta\right)\frac{R\, d\theta}{EI} = 0 \quad \text{for minimum strain energy}$$

or, $$\left| M_o\theta - \frac{P}{2} R \cos\theta \right|_0^{\pi} = 0$$

or, $$M_o = -\frac{PR}{\pi} \quad \text{sagging}$$

$\therefore$ Bending moment at C, $M_C = -\frac{PR}{\pi} + \frac{P}{2} R \sin\theta$

at, $\theta = 0°$, $M_C = -\frac{PR}{\pi} = -0.318$ PR sagging

at $\theta = 90°$, $M_C = -\frac{PR}{\pi} + \frac{PR}{2} =$ 0. 182 PR hogging

$M_C = 0$ occurs at, $-\frac{PR}{\pi} + \frac{P}{2} R \sin\theta = 0$

or, $\sin\theta = +\frac{2}{\pi} = +0.636$, or, $\theta = 39.5°$

The bending moment diagram is shown in Fig. 5.13c.

Deflection of the Loaded Section

$$\Delta = \frac{\partial U}{\partial P} \quad \text{along the direction of the load P}$$

$$= \frac{2}{EI}\int_0^{\pi} M \frac{\partial M}{\partial P} ds, \qquad \text{where } M = -\frac{PR}{\pi} + \frac{PR \sin\theta}{2}$$

or, $$\Delta = \frac{2}{EI}\int_0^{\pi}\left[-\frac{PR}{\pi} + \frac{PR}{2}\sin\theta\right]\left[-\frac{R}{\pi} + \frac{R\sin\theta}{2}\right] R\, d\theta$$

$$= \frac{PR^3}{2\pi^2 EI}\left| 4\pi\cos\theta + \left(\frac{\theta - \frac{\sin 2\theta}{2}}{2}\right)\pi^2 + 4\theta \right|_0^{\pi}$$

$$\Delta = \frac{PR^3}{4\pi EI}\left[\pi^2 - 8\right]$$

= decrease in length of the vertical diameter.

Deflection of the Horizontal Diameter

Let us apply two equal and opposite horizontal loads W across the horizontal diameter as shown in Fig. 5.13 d.

$$0 < \theta < 90°, \quad M = M_o - \frac{P}{2}R\sin\theta - \frac{W}{2}(R - R\cos\theta)$$

$$\frac{\partial M}{\partial W} = -\frac{1}{2}(R - R\cos\theta)$$

Strain energy stored in part AEB

$$U = 2\int_0^{\pi/2}\frac{M^2 ds}{2EI}$$

Horizontal movement of E relative to A,

$$\Delta_{EA} = \frac{\partial U}{\partial W}$$

$$= \frac{2}{EI}\int_0^{\pi/2} M\frac{\partial M}{\partial W}ds$$

$$\frac{\partial U}{\partial W} = \frac{2}{EI}\int_0^{\pi/2}\left[M_o - \frac{PR}{2}\sin\theta - \frac{W}{2}(R - R\cos\theta)\right]\left[-\frac{1}{2}(R - R\cos\theta)\right]Rd\theta$$

$$= -\frac{R^2}{EI}\int_0^{\pi/2}\left[M_o(1-\cos\theta) - \frac{PR}{2}\left(\sin\theta - \frac{\sin 2\theta}{2}\right) - \frac{WR}{2}\left(1 - 2\cos\theta + \cos^2\theta\right)\right]d\theta$$

$$= \frac{R^2}{EI}\left[-M_o\left(\frac{\pi}{2} - 1\right) + \frac{PR}{2}\left(-\frac{1}{4}\right) + \frac{WR}{2}\left(\frac{\pi}{2} - 2 + \frac{\pi}{4}\right) - \frac{RP}{2}\left(-1 + \frac{1}{4}\right)\right]$$

$$\frac{\partial U}{\partial W} = \frac{R^2}{EI}\left[-M_o\frac{\pi}{2} + M_o + \frac{PR}{4} + \frac{WR}{2}\left(\frac{3\pi}{4} - 2\right)\right]$$

$$\left(\frac{\partial U}{\partial W}\right)_{W=0} = \Delta_{EA} = \frac{R^2}{EI}\left[M_o - M_o\frac{\pi}{2} + \frac{1}{4}PR\right]$$

or, $\Delta_{EA} = \frac{PR^3}{EI}\left[\frac{PR}{\pi} - \frac{PR}{4}\right]$ on substituting for M_o

$$= \frac{PR^3}{4\pi EI}[4 - \pi]$$

∴ Horizontal deflection of E relative to F

$$\Delta_{EF} = \frac{PR^3}{2\pi EI}[4 - \pi]$$

Example 5.9

Analyze a box frame for a culvert shown in Fig.5.14a using the strain energy method. Draw bending moment and shear force diagrams.

Solution

The frame and loads are symmetrical about a centre line passing through B as shown in Fig 5.14b. Let us cut the frame into two equal halves. The free body diagrams for various members are shown in Fig 5.14c. The shear is zero at B due to symmetry. There are two unknowns at B, axial thrust T and bending moment M_o. These two redundants can be determined using the strain energy method.

For minimum strain energy, $\frac{\partial U}{\partial M_o} = 0, \quad \frac{\partial U}{\partial T} = 0$

Member BC

$M = -M_o - 50x$ (Taking sagging moment as positive)

$$\frac{\partial M}{\partial M_o} = -1, \quad \frac{\partial M}{\partial T} = 0$$

$$\int M\frac{\partial M}{\partial M_o}\frac{dx}{EI} = \int_0^{2.5}(M_o + 50x)\frac{dx}{2EI} = \left| M_o x + 50\frac{x^2}{2}\right|_0^{2.5}\frac{1}{2EI}$$

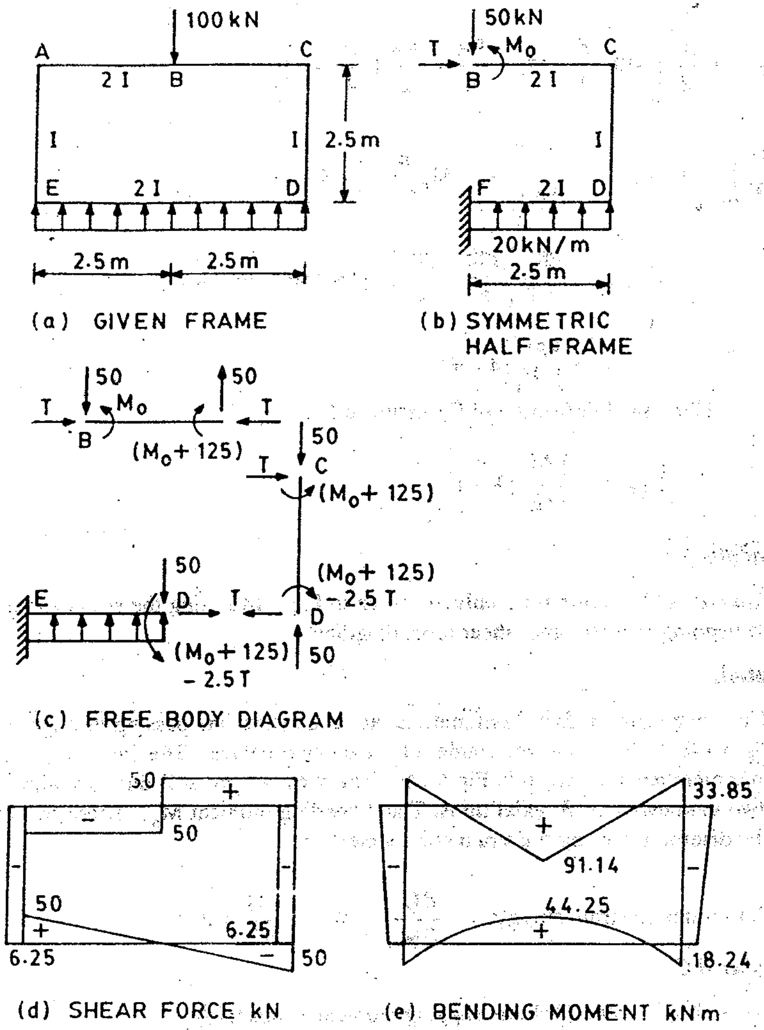


Fig. 5.14 Box frame

$$= (1.25\, M_o + 78.125)\,\frac{1}{EI} \qquad (1)$$

$$\int M \frac{\partial M}{\partial T}\frac{dx}{EI} = 0 \qquad (2)$$

Member C D

$$M = -Tx - (M_o + 125), \quad \frac{\partial M}{\partial M_o} = -1, \quad \frac{\partial M}{\partial T} = x$$

$$\int M \frac{\partial M}{\partial M_o}\frac{dx}{EI} = \int_0^{2.5}\left[-Tx + \left(M_o + 125\right)\right]\frac{dx}{EI} = \left|-\frac{Tx^2}{2} + M_o x + 125x\right|_0^{2.5}\frac{1}{EI}$$

$$= (-3.12\,T + 2.5\,M_o + 312.5)\,\frac{1}{EI} \qquad (3)$$

$$\int M \frac{\partial M}{\partial T}\frac{dx}{EI} = \int_0^{2.5}\left[Tx - \left(M_o + 125\right)\right]x\frac{dx}{EI} = \left|T\frac{x^3}{3} - \left(M_o + 125\right)\frac{x^2}{2}\right|_0^{2.5}\frac{1}{EI}$$

$$= [5.21\,T - 3.125\,(M_o + 125)]\,\frac{1}{EI} \qquad (4)$$

Member F D

$$M = -[(M_o + 125) - 2.5\,T] + 50x - 20\,x^2/2, \quad \frac{\partial M}{\partial M_o} = -1, \quad \frac{\partial M}{\partial T} = 2.5$$

$$\int M \frac{\partial M}{\partial M_o}\frac{dx}{EI} = \int_0^{2.5}\left[M_o + 125 - 50x + 10x^2 - 2.5T\right]\frac{dx}{2EI}$$

$$= \left|-2.5Tx + M_o x + 125x - 50\frac{x^2}{2} + 10\frac{x^3}{3}\right|_0^{2.5}\frac{1}{2EI}$$

$$= (1.25\,M_o + 104.17 - 3.125\,T)\,\frac{1}{EI} \qquad (5)$$

$$\int M \frac{\partial M}{\partial T}\frac{dx}{EI} = \int_0^{2.5}\left[2.5T - \left(M_o + 125\right) + 50x - 10x^2\right]2.5\frac{dx}{2EI}$$

$$= [7.81\,T - 3.125\,M_o - 260.42]\,\frac{1}{EI} \qquad (6)$$

$\frac{\partial U}{\partial M_o} = 0$ combining Eqs. 1, 3 and 5

or, $1.25\,M_o + 78.125 - 3.125\,T + 2.5\,M_o + 312.5 + 1.25\,M_o + 104.17 - 312.5\,T = 0$

or, $\quad 5\,M_o - 6.25\,T = -494.8 \qquad (7)$

$$\frac{\partial U}{\partial T} = 0 \quad \text{combining Eqs. 2, 4 and 6}$$

or, $$5.21\,T - 3.125\,M_o - 390.625 + 7.81\,T - 3.125\,M_o - 260.42 = 0$$

or, $$13.02\,T - 6.25\,M_o = 651.05 \tag{8}$$

Rearranging Eqs. 7 and 8 in matrix form gives,

$$\begin{bmatrix} 5 & -6.25 \\ -6.25 & 13.02 \end{bmatrix} \begin{Bmatrix} M_o \\ T \end{Bmatrix} = \begin{Bmatrix} -494.8 \\ 651.05 \end{Bmatrix}$$

or, $$\begin{Bmatrix} M_o \\ T \end{Bmatrix} = \begin{Bmatrix} -91.14 \\ 6.25 \end{Bmatrix}$$

Back Substitution in the values of M_C, M_D and M_E (Fig. 5.14c) gives

M_C = − 33.85 kNm hogging
M_D = − 18.24 kNm hogging
M_E = 44.25 kNm sagging

The resulting shear force and bending moment diagrams are shown in Figs 5.14 d and e.

Example 5.10

Analyze the frame shown in Fig. 5.15a and determine the contribution of axial, shear and bending deformations in the member forces. For all members,

EI = constant, AE = 100 EI, GA_e = 50 EI where EI is in kN meter units.

Solution

The strain energy due to bending, shear and thrust deformations is given by

$$U = \int \frac{1}{2}\frac{M^2 dx}{EI} + \int \frac{1}{2}\frac{V^2 dx}{A_e G} + \int \frac{1}{2}\frac{T^2 dx}{AE} \tag{5.9a}$$

The frame is statically indeterminate to a degree 3. The released structure is shown in Fig. 5.15b. The strain energy will be a function of M_A, M_D and R_{Dx}. Let us draw moment, shear and thrust diagrams due to the applied loads as well as each of the redundants. This will help in writing the strain energy expressions for each member of the portal frame. The bending moment, shear and thrust diagrams due to the applied loads and unit values of the redundants M_A, M_D and R_{Dx} are shown in Figs 5.15c to f. Axial tension is considered as positive while axial compression is considered as negative. Let us first evaluate basic integrals required for writing the expression for minimum strain energy.

Segment	AB	BC	CD
Origin	A	B	C
Limits	0 − L	0 − 2L	0 − L

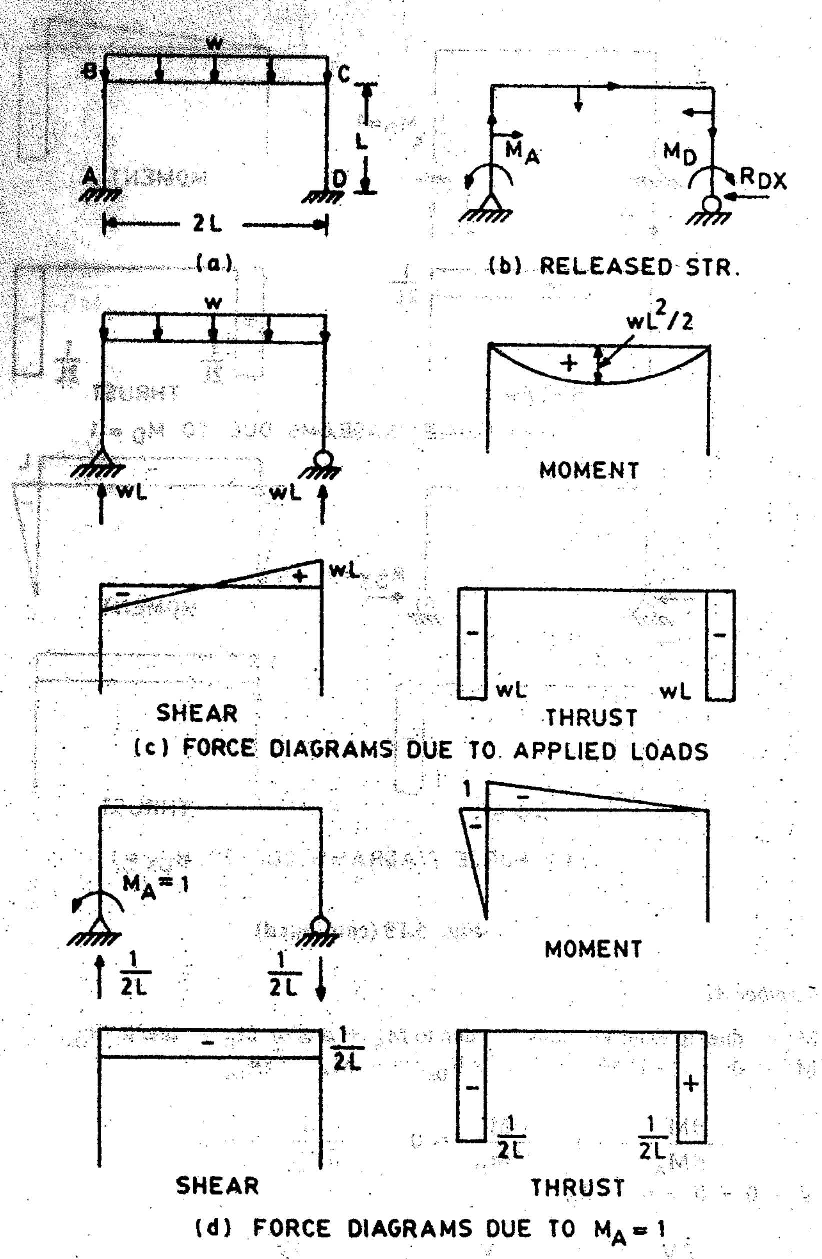
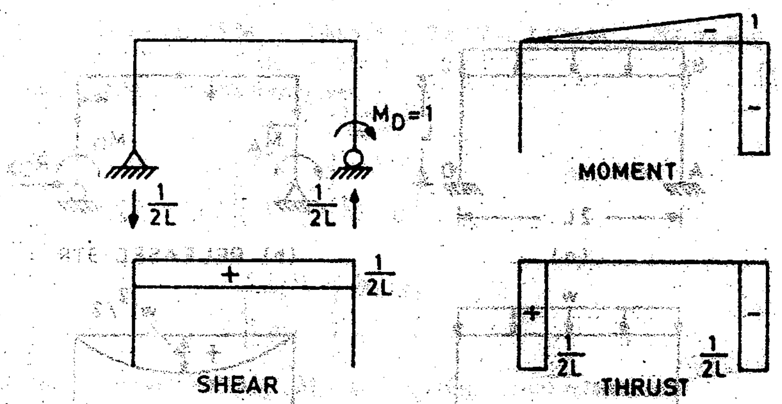
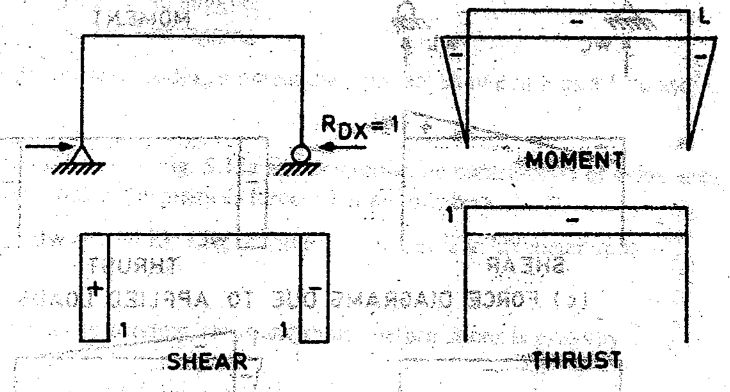


Fig. 5.15



(e) FORCE DIAGRAMS DUE TO $M_D = 1$



(f) FORCE DIAGRAMS DUE TO $R_{DX} = 1$

Fig. 5.15 (continued)

Member AB

M = due to external loads + due to M_A + due to M_D + due to R_{Dx}

$$M = 0 + (-1)M_A + 0 - xR_{Dx} = -M_A - xR_{Dx}$$

$$\frac{\partial M}{\partial M_A} = -1, \quad \frac{\partial M}{\partial M_D} = 0, \quad \frac{\partial M}{\partial R_{Dx}} = -x$$

$$V = 0 + 0 + 0 + R_{Dx}$$

$$\frac{\partial V}{\partial M_A} = 0, \quad \frac{\partial V}{\partial M_D} = 0, \quad \frac{\partial V}{\partial R_{Dx}} = 1$$

$$T = -wL - \frac{1}{2L}M_A + \frac{M_D}{2L} + 0$$

$$\frac{\partial T}{\partial M_A} = -\frac{1}{2L}, \quad \frac{\partial T}{\partial M_D} = \frac{1}{2L}, \quad \frac{\partial T}{\partial R_{Dx}} = 0$$

Member BC

$$M = \left(wLx - w\frac{x^2}{2}\right) - M_A + \frac{M_A x}{2L} - \frac{M_D}{2L}x - R_{Dx}L$$

$$\frac{\partial M}{\partial M_A} = -1 + \frac{x}{2L}, \quad \frac{\partial M}{\partial M_D} = -\frac{x}{2L}, \quad \frac{\partial M}{\partial R_{Dx}} = -L$$

$$V = -wL + wx - \frac{M_A}{2L} + \frac{M_D}{2L} + 0$$

$$\frac{\partial V}{\partial M_A} = -\frac{1}{2L}, \quad \frac{\partial V}{\partial M_D} = \frac{1}{2L}, \quad \frac{\partial V}{\partial R_{Dx}} = 0$$

$$T = 0 + 0 + 0 - R_{Dx}$$

$$\frac{\partial T}{\partial M_A} = 0, \quad \frac{\partial T}{\partial M_D} = 0, \quad \frac{\partial T}{\partial R_{Dx}} = -1$$

Member C D

$$M = 0 + 0 - M_D - LR_{Dx} + R_{Dx}x$$

$$\frac{\partial M}{\partial M_A} = 0, \quad \frac{\partial M}{\partial M_D} = -1, \quad \frac{\partial M}{\partial R_{Dx}} = -L + x$$

$$V = 0 + 0 + 0 - R_{Dx}$$

$$\frac{\partial V}{\partial M_A} = 0, \quad \frac{\partial V}{\partial M_D} = 0, \quad \frac{\partial V}{\partial R_{Dx}} = -1$$

$$T = -wL + \frac{M_A}{2L} - \frac{M_D}{2L} + 0$$

$$\frac{\partial T}{\partial M_A} = \frac{1}{2L}, \quad \frac{\partial T}{\partial M_D} = -\frac{1}{2L}, \quad \frac{\partial T}{\partial R_{Dx}} = 0$$

For minimum strain energy,

$$\frac{\partial U}{\partial M_A} = 0, \quad \frac{\partial U}{\partial M_D} = 0, \quad \frac{\partial U}{\partial R_{Dx}} = 0,$$

Let us first evaluate $\dfrac{\partial U}{\partial M_A}$

$$\int M\frac{\partial M}{\partial M_A}\frac{dx}{EI} = \int_0^L [-M_A - xR_{Dx}](-1)\frac{dx}{EI} + \int_0^{2L}\left[\left(wLx - \frac{wx^2}{2}\right) - M_A + \frac{M_A x}{2L} - \frac{M_D x}{2L} - R_{Dx}L\right]\left[-1+\frac{x}{2L}\right]\frac{dx}{EI}$$

$$\int M\frac{\partial M}{\partial M_A}\frac{dx}{EI} = \left[\frac{5}{3}M_A L + \frac{M_D L}{3} + \frac{3}{2}R_{Dx}L^2 - \frac{1}{3}wL^3\right]\frac{1}{EI} \quad (1)$$

$$\int V\frac{\partial V}{\partial M_A}\frac{dx}{A_eG} = 0 + \int_0^{2L}\left[-wL + wx - \frac{M_A}{2L} + \frac{M_D}{2L}\right]\left(-\frac{1}{2L}\right)\frac{dx}{A_eG} + 0$$

$$= \left[-w2L^2 + w2L^2 - M_A + M_D\right]\left(-\frac{1}{2L}\right)\frac{dx}{50EI}$$

$$= (M_A - M_D)\frac{1}{100EIL} \quad (2)$$

$$\int T\frac{\partial T}{\partial M_A}\frac{dx}{AE} = \int_0^L\left[-wL - \frac{M_A}{2L} + \frac{M_D}{2L}\right]\left(-\frac{1}{2L}\right)\frac{dx}{AE} + 0 + \int_0^L\left[-wL + \frac{M_A}{2L} - \frac{M_D}{2L}\right]\left(\frac{1}{2L}\right)\frac{dx}{AE}$$

$$= (M_A - M_D)\frac{1}{200EIL} \quad (3)$$

$\therefore$ Combining Eqs. 1, 2 and 3

$$\frac{\partial U}{\partial M_A} = \left[\frac{5}{3}M_A L + \frac{M_D L}{3} + \frac{3}{2}R_{Dx}L^2 - \frac{1}{3}wL^3 + \frac{M_A}{100L} + \frac{M_A}{200L} - \frac{M_D}{100L} - \frac{M_D}{200L}\right] = 0$$

or,
$$\left(\frac{5L}{3} + \frac{3}{200L}\right)M_A + \left(\frac{L}{3} - \frac{3}{200L}\right)M_D + \frac{3}{2}L^2R_{Dx} - \frac{wL^3}{3} = 0 \quad (4)$$

Now let us evaluate $\dfrac{\partial U}{\partial M_D}$

$$\int M\frac{\partial M}{\partial M_D}\frac{dx}{EI} = 0 + \int_0^{2L}\left[wLx - \frac{wx^2}{2} - M_A + \frac{M_A x}{2L} - \frac{M_D x}{2L} - R_{Dx}L\right]\left(-\frac{x}{2L}\right)\frac{dx}{EI} + \int_0^L(-M_D - R_{Dx}L + R_{Dx}x)(-1)\frac{dx}{EI}$$

$$= \left[\frac{M_A L}{3} + \frac{5}{3}M_D L + \frac{3}{2}R_{Dx}L^2 - \frac{1}{3}wL^3\right]\frac{1}{EI} \quad (5)$$

$$\int V\frac{\partial V}{\partial M_D}\frac{dx}{A_eG} = 0 + \int_0^{2L}\left[-wL + wx - \frac{M_A}{2L} + \frac{M_D}{2L}\right]\left(\frac{1}{2L}\right)\frac{dx}{A_eG} + 0$$

$$= (-M_A + M_D)\frac{1}{100EIL} \quad (6)$$

$$\int T\frac{\partial T}{\partial M_D}\frac{dx}{AE} = \int_0^L\left[-wL - \frac{M_A}{2L} + \frac{M_D}{2L}\right]\left(\frac{1}{2L}\right)\frac{dx}{AE} + 0 + \int_0^L\left[-wL + \frac{M_A}{2L} - \frac{M_D}{2L}\right]\left(-\frac{1}{2L}\right)\frac{dx}{AE}$$

$$= (-M_A + M_D)\frac{1}{200EIL} \quad (7)$$

Combining Eqs. 5, 6 and 7.

$$\frac{\partial U}{\partial M_D} = \left[\frac{M_A L}{3} + \frac{5}{3}M_D L + \frac{3}{2}R_{Dx}L^2 - \frac{1}{3}wL^3 - \frac{M_A}{100L} + \frac{M_D}{100L} - \frac{M_A}{200L} + \frac{M_D}{200L}\right] = 0$$

or,
$$\left(\frac{L}{3} - \frac{3}{200L}\right)M_A + \left(\frac{5L}{3} + \frac{3}{200L}\right)M_D + \frac{3}{2}L^2R_{Dx} - \frac{1}{3}wL^3 = 0 \quad (8)$$

Now let us evaluate $\dfrac{\partial U}{\partial R_{Dx}}$

$$\int M\frac{\partial M}{\partial R_{Dx}}\frac{dx}{EI} = \int_0^L(-M_A - xR_{Dx})(-x)\frac{dx}{EI} + \int_0^{2L}\left(wLx - \frac{wx^2}{2} - M_A + \right.$$

$$\left.\frac{M_A x}{2L} - \frac{M_{Dx}}{2L} - R_{Dx}L\right)(-L)\frac{dx}{EI} + \int_0^L(-M_D - LR_{Dx} + R_{Dx}x)(-L+x)\frac{dx}{EI}$$

$$\int M\frac{\partial M}{\partial R_{Dx}}\frac{dx}{EI} = \left[\frac{3}{2}M_A L^2 + \frac{3}{2}M_D L^2 + \frac{8}{3}R_{Dx}L^3 - \frac{2}{3}wL^4\right]\frac{1}{EI} \tag{9}$$

$$\int V\frac{\partial V}{\partial R_{Dx}}\frac{dx}{A_eG} = \int_0^L R_{Dx}1\frac{dx}{A_eG} + 0 + \int_0^L(-R_{Dx})(-1)\frac{dx}{A_eG}$$

$$= \frac{2R_{Dx}L}{A_eG} = \frac{R_{Dx}L}{25EI} \tag{10}$$

$$\int T\frac{\partial T}{\partial R_{Dx}}\frac{dx}{AE} = 0 + \int_0^{2L}(-R_{Dx})(-1)\frac{dx}{EI} + 0$$

$$= 2L\frac{R_{Dx}}{AE} = \frac{R_{Dx}L}{50EI} \tag{11}$$

Combining Eqs. 9, 10 and 11

$$\frac{\partial U}{\partial R_{Dx}} = \frac{3}{2}M_A L^2 + \frac{3}{2}M_D L^2 + \left(\frac{8}{3}L^3 + \frac{3}{50}L\right)R_{Dx} - \frac{2}{3}wL^4 = 0 \tag{12}$$

Arranging Eqs. 4, 8 and 12 in matrix form,

$$\begin{bmatrix} \left(\frac{5}{3}L + \frac{3}{200L}\right) & \left(\frac{L}{3} - \frac{3}{200L}\right) & \frac{3}{2}L^2 \\ & \left(\frac{5}{3}L + \frac{3}{200L}\right) & \frac{3}{2}L^2 \\ \text{Symmetric} & & \left(\frac{8}{3}L^3 + \frac{3}{50}L\right) \end{bmatrix}\begin{Bmatrix} M_A \\ M_D \\ R_{Dx} \end{Bmatrix} + \begin{Bmatrix} -\frac{1}{3}wL^3 \\ -\frac{1}{3}wL^3 \\ -\frac{2}{3}wL^4 \end{Bmatrix} = \{0\} \tag{13}$$

Let $L = 4$ m, $w = 30$ kN/m, the matrix reduces to

$$\begin{bmatrix} 6.670 & 1.329 & 24 \\ & 6.670 & 24 \\ \text{Symmetric} & & 171.067 \end{bmatrix}\begin{Bmatrix} M_A \\ M_D \\ R_{Dx} \end{Bmatrix} = \begin{Bmatrix} 640 \\ 640 \\ 5120 \end{Bmatrix} \tag{14}$$

or,

$$\begin{Bmatrix} M_A \\ M_D \\ R_{Dx} \end{Bmatrix} = \begin{Bmatrix} -62 \\ -62 \\ 47.3 \end{Bmatrix} \tag{15}$$

If the effect of shear and axial deformations is ignored, the expressions for minimum strain energy are obtained by using Eqs. 1, 5 and 9, that is,

$$\begin{bmatrix} \frac{5}{3}L & \frac{L}{3} & \frac{3}{2}L^2 \\ & \frac{5}{3}L & \frac{3}{2}L^2 \\ & & \frac{8}{3}L^3 \end{bmatrix}\begin{Bmatrix} M_A \\ M_D \\ R_{Dx} \end{Bmatrix} + \begin{Bmatrix} -\frac{1}{3}wL^3 \\ -\frac{1}{3}wL^3 \\ -\frac{2}{3}wL^4 \end{Bmatrix} = \{0\} \tag{16}$$

For $L = 4$ m, $w = 30$ kN/m, the matrix reduces to

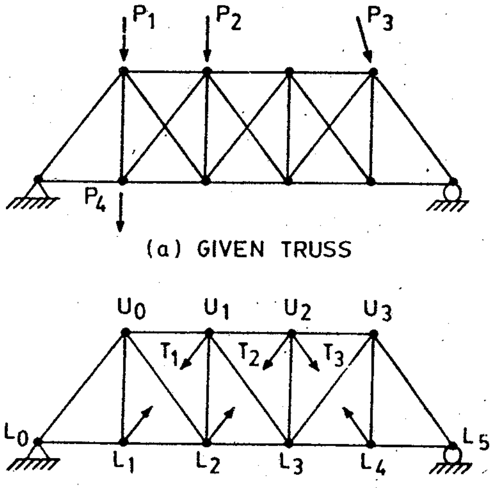
$$\begin{bmatrix} 6.667 & 1.334 & 24 \\ & 6.667 & 24 \\ & & 170.667 \end{bmatrix}\begin{Bmatrix} M_A \\ M_D \\ R_{Dx} \end{Bmatrix} = \begin{Bmatrix} 640 \\ 640 \\ 5120 \end{Bmatrix}$$

and the solution is

$$\begin{Bmatrix} M_A \\ M_D \\ R_{Dx} \end{Bmatrix} = \begin{Bmatrix} -64 \\ -64 \\ 48 \end{Bmatrix} \tag{17}$$

5.7 TRUSS

Castigliano's second theorem can be used to analyze a statically internally or externally indeterminate truss. Consider a truss shown in Fig. 5.16a subjected to loads P_1, P_2, P_3 etc. The truss is indeterminate to a degree three. Let us identify three redundant members such that if they are removed the truss should remain stable. Let T_1, T_2 and T_3 be the forces in these members. Let us remove the three members and replace them with a pair of pulling forces T_1, T_2 and T_3. This change will not alter forces in other



(b) RELEASED TRUSS

Fig. 5.16 Statically indeterminate truss

members of the truss which has now become perfect and statically determinate. The compatibility condition is that the increase in distance between the joints is equal to the elongation of the member itself. The statically determinate truss can be analyzed due to the external loads and forces T_1, T_2 and T_3 individually.

If a reaction is chosen as a redundant, the constraint support in the direction of the reaction is removed. A force is applied at the joint in the direction of the redundant reaction. The compatibility condition is that net deflection in the direction of the reaction must be zero.

Let the force in any member $F = F' + u_1 + u_2 + u_3$.

where, F' = member force due to external loads

u_1, u_2, u_3 = member forces due to T_1, T_2 and T_3, respectively

Strain energy in any member except those considered redundant

$$dU_i = \frac{F^2{}_i L_i}{2A_i E_i} \tag{5.17a}$$

Strain energy in a redundant member, $$dU_j = \frac{T^2{}_j L_j}{2A_j E_j} \tag{5.17b}$$

Total strain energy $U = \sum_i dU_i + \sum_j dU_j$

or, $$U = \sum (F' + u_1 + u_2 + u_3)^2 \frac{L_i}{2A_i E_i} + \sum \frac{T^2{}_j L_j}{2A_j E_j} \tag{5.18}$$

where, i represents i th member in the statically determinate truss, and j represents j th redundant member.

The value of F' is constant depending on the external loads. The values of T_1, T_2 and T_3 will adjust themselves so as to make the strain energy of the truss a minimum. Since T_1, T_2 and T_3 are variables,

$$\frac{\partial U}{\partial T_1} = \sum (F' + u_1 + u_2 + u_3)^2 \frac{\partial u_1}{\partial T_1} \frac{L_i}{A_i E_i} + \frac{T_1 L_1}{A_1 E_1} = 0$$

The rest of the terms containing T_2 and T_3 will become zero upon differentiation with respect to T_1

$\frac{\partial u_1}{\partial T_1}$ is the ratio between increments of u_1 and T_1, and therefore equal to u_1 / T_1.

or,

$$\sum (F' + u_1 + u_2 + u_3) \frac{L_i}{A_i E_i} \frac{u_1}{T_1} = -\frac{T_1 L_1}{A_1 E_1} \tag{5.19}$$

Similarly, U can be differentiated with respect to T_2 and T_3, and equated to zero. From these equations, the values of T_1, T_2 and T_3 can be obtained. It can be seen that Eq. 5.19 is similar to Eq. 3.5.

Deflections in a Truss

Castigliano's second theorem can be conveniently used to determine deflections in a indeterminate truss. Let the forces in the redundant members be T_1, T_2 and T_3 in the truss shown in Fig. 5.16. Let load P be applied at a joint where deflection is required. The truss can be made statically determinate by removing the three redundant members. The forces in the truss members can be determined in terms of T_1, T_2, T_3 and P. If U is the total strain energy of the truss, then

$$\left(\frac{\partial U}{\partial T_1}\right)_{P=0} = 0, \left(\frac{\partial U}{\partial T_2}\right)_{P=0} = 0, \left(\frac{\partial U}{\partial T_3}\right)_{P=0} = 0 \tag{5.20a}$$

and $$\left(\frac{\partial U}{\partial P}\right)_{P=0} = \Delta \tag{5.20b}$$

The solution of Eq. 5.20 gives values of the redundants as well as the desired deflection.

Example 5.11

Determine horizontal deflection of joint 1 of the truss shown in Fig. 5.17a by the strain energy method.

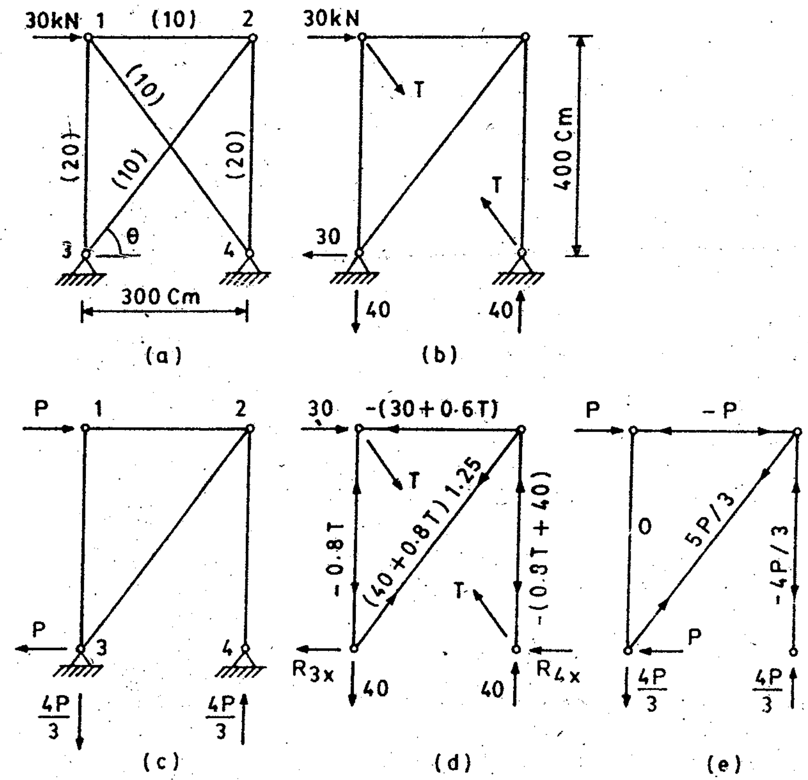


Fig. 5.17

Solution

Let us treat member 1 - 4 as a redundant and is assumed to carry a force T. By removing this member and replacing with the member force T, the truss becomes statically determinate as shown in Fig. 5.17b. Let us apply horizontal load P at joint 1 to calculate the horizontal displacement of this joint. Member forces F ' due to external loads, and u due to load P are shown in Figs. 5.17 d, e and Table 5.1.

$$\text{Strain energy } U = \int \frac{1}{2}\frac{F^2 dx}{AE}$$

$$\text{where, } F = (F' + u)$$

$$\frac{\partial U}{\partial T} = \int F \frac{\partial F}{\partial T}\frac{dx}{AE}$$

and
$$\frac{\partial U}{\partial P} = \int F \frac{\partial F}{\partial P}\frac{dx}{AE}$$

The integrals can be evaluated as follows using the values given in Table 5.1

Table 5.1 - Computation of member forces

Member	$\frac{L}{A}$	F '	u	F = F ' + u	$\frac{\partial F}{\partial T}$	$\frac{\partial F}{\partial P}$
1–2	30	– (30+0.6T)	– P	– [30+0.6T+P]	– 0.6	– 1
1–3	20	– 0.8 T	0	– 0.8 T	– 0.8	0
2–4	20	– (0.8T+40)	– 1.33 P	–[40+0.8T+1.33P]	– 0.8	– 1.33
1–4	50	T	0	T	1	0
2–3	50	(40+0.8T)1.25	1.67 P	[(40+0.8T)×1.25+1.67P]	1	1.67

$$\frac{\partial U}{\partial T} = \int F \frac{\partial F}{\partial T}\frac{dx}{AE} = -[30 + 0.6\,T + P](-0.6)\frac{30}{E} + (-0.8T)(-0.8) \times \frac{20}{E} -$$

$$[40 + 0.8\,T + 1.33P](-0.8)\frac{20}{E} + T \times 1 \times \frac{50}{E} +$$

$$[(40 + 0.8\,T)\,1.25 + 1.67\,P] \times 1 \times \frac{50}{E}$$

or,
$$\left(\frac{\partial U}{\partial T}\right)_{P=0} = 0$$

$136.4\,T = -3680$ or, $T = -26.98$ kN

Force in member 1 – 4 = – 26.98 kN , compression

$$\left(\frac{\partial U}{\partial P}\right) = -[30 + 0.6\,T + P](-1)\frac{30}{E} + (-0.8T)\,0 - [40 + 0.8T + 1.33P]$$

$$(-1.33)\frac{20}{E} + T \times 0 + (50 + T + 1.67P)\,1.67 \times \frac{50}{E}$$

or,
$$\left(\frac{\partial U}{\partial P}\right)_{P=0} = \frac{30}{E}(30 + 0.6\,T) + \frac{26.6}{E}(40 + 0.8T) + (50 + T)\;1.67 \times \frac{50}{E}$$

On substituting the value of T,

$$\left(\frac{\partial U}{\partial P}\right)_{P=0} = [6139 + 122.78\,(-26.98)]\frac{1}{E} = \frac{2826}{E}$$

$$\therefore \quad \Delta = \frac{2826}{E}, \qquad E = 2 \times 10^4 \text{ kN/cm}^2$$

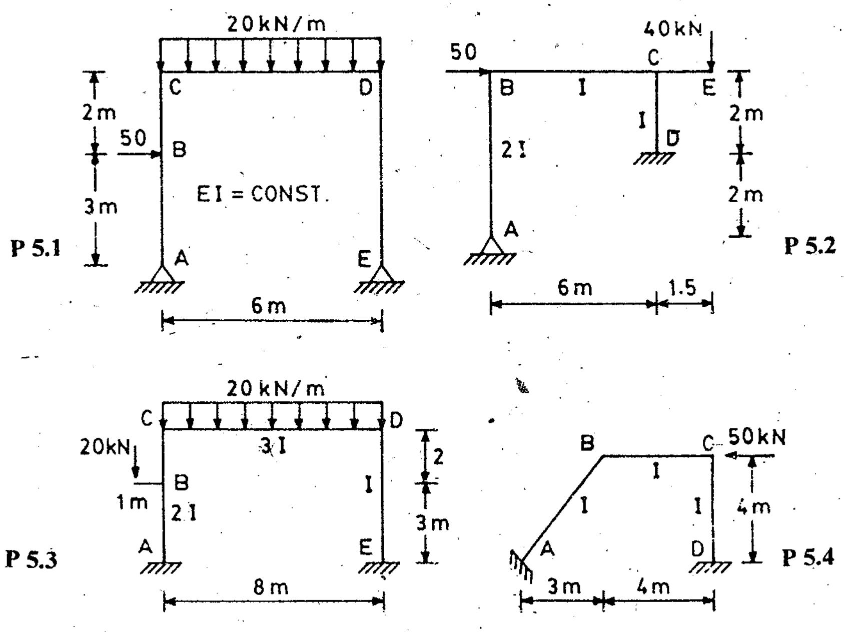
$$\Delta = 0.14 \text{ cm} = 1.4 \text{ mm}$$

PROBLEMS

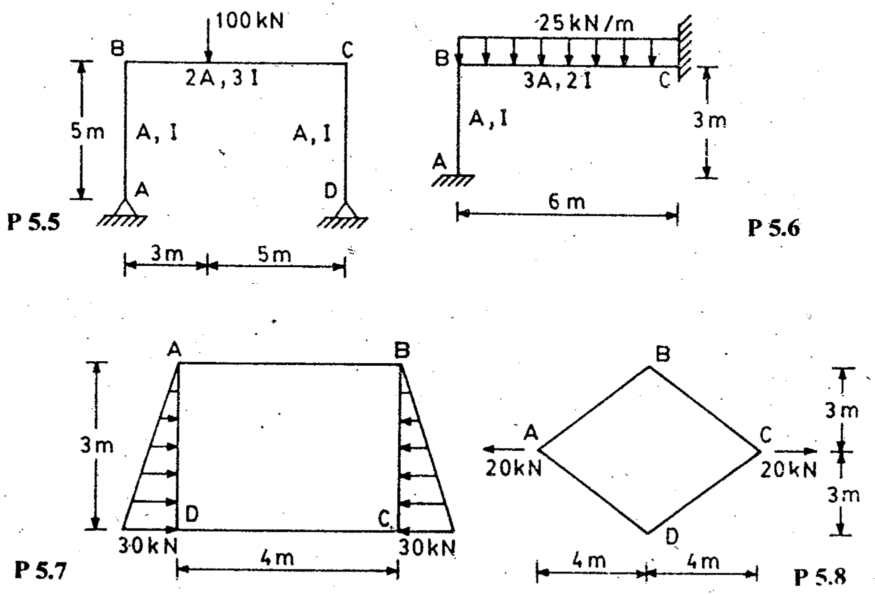
NOTE : *In the answer, all clockwise moments are shown positive and anti-clockwise moments are shown as negative.*

5.1 Analyze the continuous beams shown in Figs. P3.3, P3.5 and P3.6 using the strain energy method. Neglect axial and shear deformations. Draw shear force and bending moment diagrams. Take EI = constant unless specified otherwise.

5.2 Analyze the frame shown in Fig. P5.1 by treating R_{ex} as a redundant. Neglect axial and shear deformations. Draw shear force and bending moment diagrams. *[R_{ex} = 24.43 kN←, M_{cb} = – 27.84 kNm, M_{dc} = 122.14 kNm]*



Figs. P 5.1 - P 5.4



Figs. P 5.5 - P 5.8

5.3 Analyze the frame shown in Fig. P5.2 by treating R_{ax} and R_{ay} as redundants. Draw shear force and bending moment diagrams. *[R_{ax} = 8.14 kN←, R_{ay} = 5.30 kN↓, M_{ba} = – 32.53 kNm, M_{dc} = – 84.44 kNm]*

5.4 Reanalyze the frame shown in Fig P 5.2 by treating R_{dx} and R_{dy} as redundants. *[R_{dx} = 41.86 kN←, R_{dy} = 45.30 kN↑]*

5.5 Analyze the frame shown in Fig. P 5.3 by treating R_{ex}, R_{ey} and M_e as redundants. *[R_{ex} = 16.4 kN←, R_{ey} = 77.4 kN↑, M_e = – 28.64 kNm, M_{ca} = 74.07 kNm]*

5.6 Reanalyze the frame shown in Fig P5.3 by treating M_a, R_{ex} and M_e as redundants. *[M_a = 27.94 kNm, R_{ex} = 16.4 kN←, M_e = – 28.64 kNm, M_{de} = – 53.37 kNm]*

5.7 Analyze the frame shown in Fig. P5.4 by treating R_{ax}, R_{ay} and M_a as redundants. *[R_{ax} = 30.66 kN→, R_{ay} = 17.95 kN↑, M_a = 34.33 kNm]*

5.8 Determine force in member 2 - 4 of the truss hown in Fig P3.9. Area of each member = 15 cm². *[F_{2-4} = 4.62 kN]*

5.9 Analyze the truss shown in Fig P3.10. Take area of top and bottom chords = 100 cm², area of web members = 70 cm². Treat R_{6y} as redundants. *[R_{6y} = 285.5 kN↑]*

5.10 Analyze the truss shown in Fig.P3.11; assume members 2 -5, 2-7 to be the redundants. Figures on the members indicate the areas in cm^2.
(a) Treat members 5 - 2 and 2 - 7 as redundants.
(b) Treat members 1 - 6 and 6 - 3 as redundants.
[$F_{5-2} = -23.96$ kN, $F_{2-} = -11.19$ kN, $F_{1-6} = 126.3$ kN, $F_{6-3} = 121.40$ kN]

5.11 Analyze the truss shown in Fig P3.12. Treat member 2- 3 as internal redundant and R_{5y} as external redundants.
[$F_{2-3} = -59.94$ kN, $R_{5y} = 45.28$ kN↑]

5.12 Analyze the portal frames shown in Figs. P5.5 - P5.6 using the strain energy method. Determine the effect of axial and shear deformations. Take E = 200 GPa, $I = 200 \times 10^{-6}\ m^4$, $A = 150 \times 10^{-4}\ m^2$, $\nu = 0.25$.
[P5.5 : $M_{bc} = -41.65$ kNm, $R_{ax} = 8.33$ kN→]
[P5.6 : $R_{ax} = 18.4$ kN→, $R_{ay} = 65.4$ kN↑, $M_{ab} = 18.29$ kNm, $M_{bc} = -36.9$ kNm]

5.13 Analyze the closed frame shown in Fig.P 5.7 by making use of symmetry. Neglect axial and shear deformations. E and I are constant.
[$M_{ab} = -4.36$ kNm, $M_{dc} = 5.26$ kNm, $T_{ab} = -14.7$ kN, $T_{dc} = -30.3$ kN]

5.14 Analyze the closed frame shown in Fig. P 5.8 by making use of symmetry through B and D. *[$M_{bc} = 15$ kNm $= M_{cb}$, Horizontal thrust at B = 10 kN tensile]*

CHAPTER
six

COLUMN ANALOGY METHOD

6.1 INTRODUCTION

The column analogy method was proposed by Professor Hardy Cross in 1930 for the analysis of statically indeterminate structures having a maximum indeterminacy equal to 3. This method is useful for the analysis of fixed beams, arches and single cell open or closed frames. It is very convenient for the analysis of curved members and non-prismatic members. It is based on a mathematical similarity between the stresses created in a short column section subjected to eccentric load and the moments induced in a member by redundant reactions.

6.2 STRESS IN A COLUMN

For a proper understanding of the column analogy method, it is essential to recaptulate the formula for the stresses in a column subjected to bi-axial bending. Consider a column section subjected to an eccentric load P with respect to the two rectangular centroidal axes x-x and y-y as shown in Fig. 6.1. The stresses in the column are assumed to vary linearly with the x and y coordinates of various fibers. For equilibrium, the following conditions must be satisfied:

$$\Sigma F_z = 0$$
$$\text{or,} \quad P - \int \sigma\, dA = 0 \qquad (6.1a)$$

$$\Sigma M_x = 0$$
$$\text{or,} \quad P y_o - \int \sigma\, dA\, y = 0 \qquad (6.1b)$$

$$\Sigma M_y = 0$$
$$\text{or,} \quad P x_o - \int \sigma\, dA\, x = 0 \qquad (6.1c)$$

The moment of the internal stresses may also be written as :

$$M_x = \int \sigma\, dA\, y = P y_o \qquad (6.2a)$$
$$\text{and} \quad M_y = \int \sigma\, dA\, x = P x_o \qquad (6.2b)$$

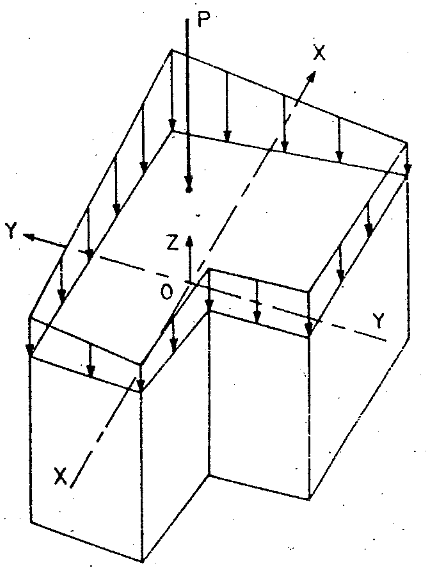


Fig. 6.1 A column section under load and stresses

General formula for stress at any section in a short column may be written as:

$$\sigma = \frac{P}{A} + \left[\frac{M_y I_x - M_x I_{xy}}{I_x I_y - I^2_{xy}}\right]x + \left[\frac{M_x I_y - M_y I_{xy}}{I_x I_y - I^2_{xy}}\right]y \qquad (6.3)$$

If one or both of the centroidal rectangular axes are axes of symmetry, then product of inertia I_{xy} of the cross-section is zero. In this case, Eq. 6.3 reduces to

$$\sigma = \frac{P}{A} + \frac{M_y}{I_y}x + \frac{M_x}{I_x}y \qquad (6.4)$$

6.3 DEVELOPMENT OF THE METHOD

Consider a fixed beam having uniform cross-section and a general loading as shown in Fig 6.2a. It can be made statically determinate by introducing a hinge at A and a roller at B. The moments M_A and M_B are treated as redundants. The bending moment diagram for this beam consists of two components :

(a) The free span moment diagram due to the applied loads (Fig.6.2b).
(b) End moment diagram due to fixity (Fig.6.2c)

The net moment diagram can be obtained by adding these two diagrams algebraically. The compatibility condition requires that :

(1) Slope of the beam at B relative to the tangent at A is zero.
(2) Deflection of the beam at B relative to the tangent at A is zero.

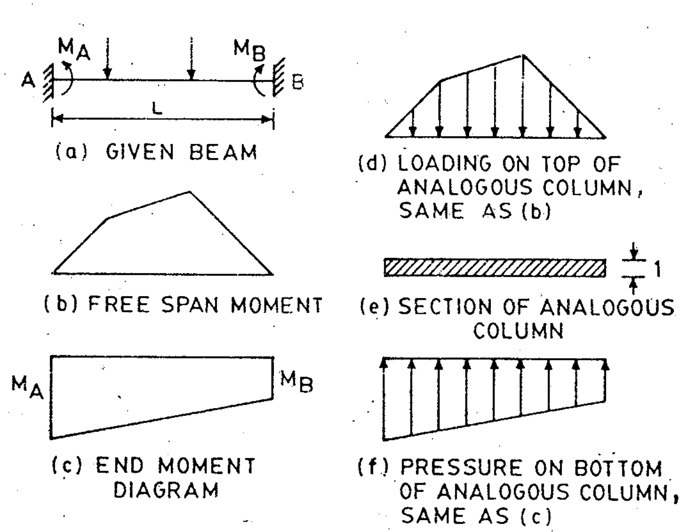


Fig. 6.2

The moment area theorem gives :

(1) The slope of the beam at B with respect to the tangent at A is equal to area of the moment diagram between A and B,

$$\theta = \int \frac{M_s ds}{EI} \qquad (6.5)$$

but $\theta = 0$ due to the compatibility condition hence the total area of the moment diagram between A and B must be zero. It follows that area of the free span moment diagram must be equal and opposite to the area of the end moment diagram.

(2) The deflection of the beam at B with respect to the tangent at A is equal to the moment of area of moment diagram between A and B about B,

$$\Delta_y = \int \frac{M_s\, ds\, x}{EI} \qquad (6.6)$$

but $\Delta_y = 0$ due to the compatibility condition, hence moment about B of the free span moment diagram must be equal and opposite to the moment about B of the end moment diagram. Because the two areas are equal, this requires that the centres of gravity of these two areas should coincide.

Now let us consider a column section of uniform width and length equal to length of the beam as shown in Fig 6.2 e. If the column section is loaded with the free span moment diagram M_s as shown in Fig. 6.2d, the stress in the column section will vary linearly from A to B as shown in Fig.6.2f. The stress at any section in a short column subjected to uniaxial bending is given by:

$$\sigma = \frac{P}{A} \pm \frac{M_y x}{I_y} \qquad (6.7)$$

Thus, for equilibrium in a column :

(1) The total stress on the section must be equal and opposite to the applied load, that is, area of the stress diagram must be equal and opposite to the area of the load diagram.

(2) The moment of the total applied load about any point should be equal and opposite to the moment of the total stress about the same point. Since areas of the load diagram and stress diagram are equal and opposite to each other, their centres of gravity must coincide.

Thus, it is apparent that the relation between the free span moment diagram and end moment diagram in a beam treating it as simply supported is the same as in a column between the applied load and the stresses created on its section and hence the name *column analogy*. It may be noted that width of the column at any section is equal to 1/EI of the corresponding section of the real structure. The height of the analogous column is not important and is considered to be some small unknown value.

It is important to know that the units of stress and hence of the fixed end moment are the same as those used for the free span moment. The units for A, I, x and y must, of course, be consistent.

Alternative Approach

A fixed beam can also be made statically determinate by removing one of the supports, say support B, and treating R_B and M_B as redundants. The free span moment and end moment diagrams for a cantilever beam are shown in Figs. 6.3b and 6.3c. The values of the redundants R_B and M_B are such that the slope and deflection of the cantilever at B are both zero.

For compatibility,

(1) Area of the moment diagram of Fig. 6.3b should be equal and opposite to the total area of Fig. 6.3c.

(2) The moment of area of the diagram of Fig. 6.3b about B should be equal and opposite to the moment of area of diagram of Fig. 6.3c about B.

The loading and the analogous column are shown in Figs. 6.3d and e. The stresses on the section are shown in Fig. 6.3f as given by Eq. 6.7.

Thus stress at any point of the column section gives the moment due to the redundants in the beam at that section. The net bending moment at that section in the beam is given by algebraic superposition of the free span moment and the corresponding stress in the column section.

6.4 SIGN CONVENTION

The following sign convention should be followed in the column analogy method:

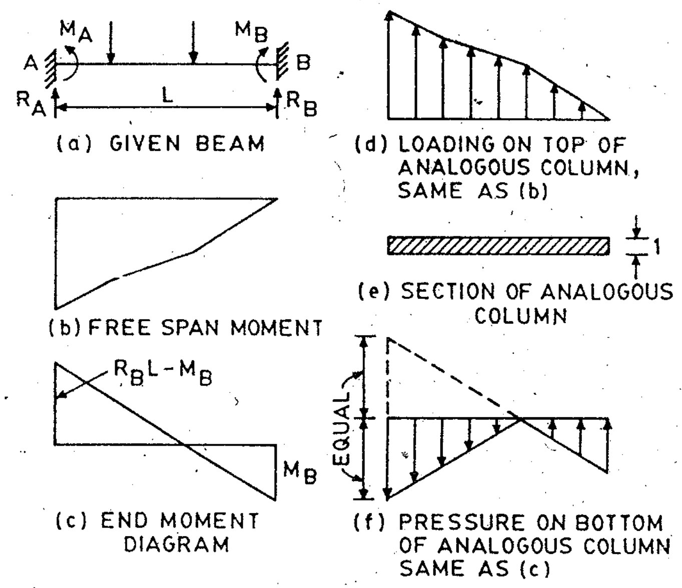


Fig. 6.3

1. The free span bending moment M_S is plotted on the compression face of the member. If the compression face is on the top or on the outside in the case of frames, the moment is considered positive and the load on the analogous column as a positive thrust.

Thus, sagging moments in the case of beams give a positive load on the analogous column.

2. Compressive stresses on the analogous column are taken as positive. Since a compressive stress in the column corresponds to a moment opposite in nature to the applied moment, it means compressive stress on the column section represents a hogging moment. It will be represented as M_i henceforth.

3. The net moment at any section will be found as the algebraic difference of the free span and redundant moments, that is,

$$M = M_s - M_i \qquad (6.8)$$

4. The distances x, y and eccentricity e should be taken with proper signs with respect to the coordinate axes passing through the centre of gravity of the analogous column.

6.5 ANALOGOUS COLUMN SECTIONS

Analogous column sections and the reference axes for various statically indeterminate structures having three, two or one degrees of indeterminacies are shown in Figs. 6.4, 6.5 and 6.6. If both supports are fixed as in the frames shown in Figs. 6.4a and 6.4b, then

the reference axes are taken through the centroid of the analogous column. The stress at any point in the column is given by Eq 6.3. If any one or both the centroidal reference axes are principal axes of inertia, the product of inertia I_{xy} is zero and the stress is given by Eq. 6.4.

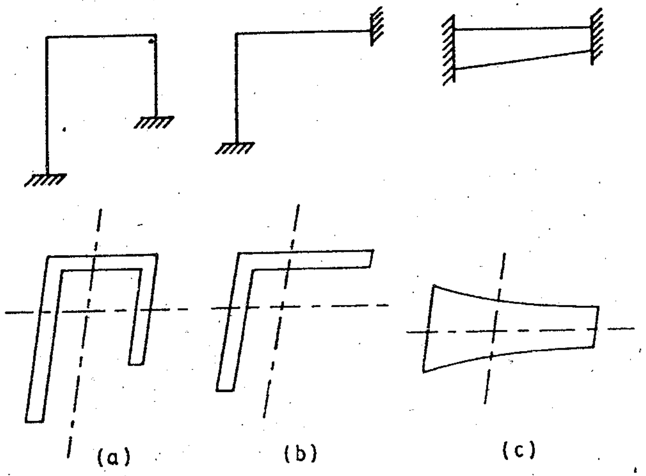


Fig. 6.4 Reference axes for analogous column sections - both ends fixed

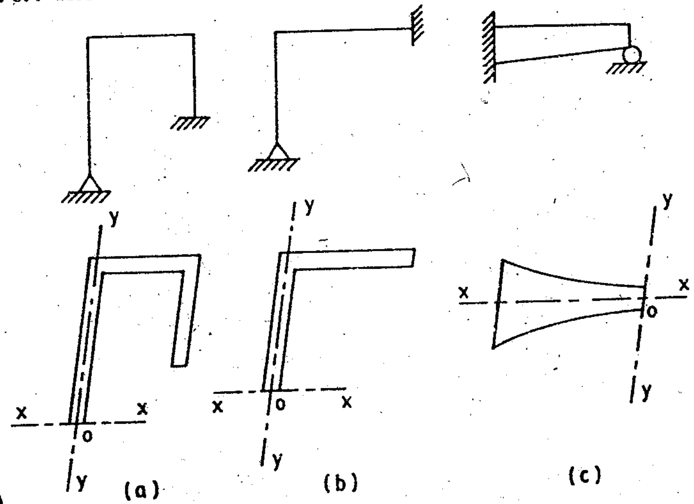


Fig. 6.5 Reference axes for analogous column sections-one end fixed and other hinged

If any one support is hinged, its moment of inertia is zero and therefore, width of the analogous column at this point is infinite. The area of the analogous column is infinite and is assumed to be concentrated at this point and therefore this is the centroid of the elastic areas. The reference axes are taken as shown in Fig. 6.5. Obviously, moment of inertia of this infinite area at the hinge is zero about both the x and y axes passing

through the hinge. The first term in Eq. 6.3 is equal to zero and three moments of inertia viz, I_{xx}, I_{yy} and I_{xy} are required to be calculated.

If both supports are hinged, the moment of inertia at both these sections will be zero and therefore, infinite area exist at these two points. The axis passing through these two points is one of the principal axes as shown in Fig. 6.6. $A = \infty$, $I_{xy} = 0$, and $I_y = \infty$, hence location of y- axis is immaterial. The moment of inertia of infinite areas about the x axis is zero because it passes through these two points. The stress at any point is given by the third term of Eq. 6.4.

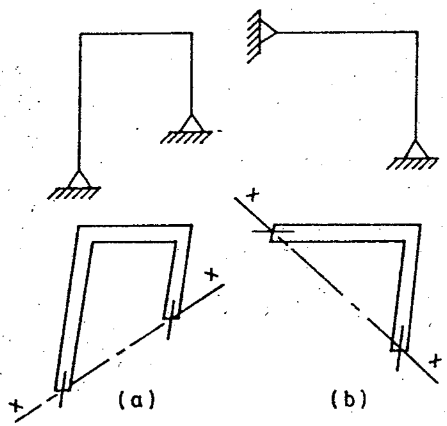


Fig. 6.6 Reference axes for analogous column section-both ends hinged

The following examples illustrate the application of the column analogy method.

6.6 FIXED END MOMENTS IN BEAMS OF UNIFORM CROSS-SECTION

Example 6.1

Determine the fixed end moments for the beam shown in Fig 6.7a by the column analogy method.

Solution

(a) *Solution assuming simple beam (Fig. 6.7b)*

The beam can be made statically determinate by treating M_A and M_B as redundants. The load on the simple beam due to uniform load is applied as a downward load on the top of the analogous column because this moment causes compression on the out side of the beam. The pressure along the base of the column is constant in this case.

The properties of the analogous column are :

Area $A = 1 \times L = L$

Moment of inertia about x – x axis $I_x = \dfrac{1 \times L^3}{12}$

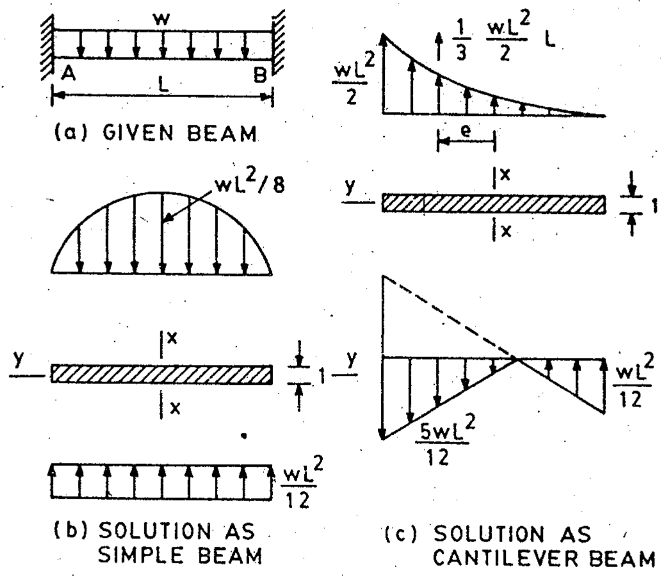


Fig. 6.7

Load on the column $P = \dfrac{2}{3} \times \dfrac{wL^2}{8} \times L = \dfrac{wL^3}{12}$

Its eccentricity about O = 0, hence moment about x – x axis = 0

∴ Stresses on the column section are :

$$\sigma_A = M_{Ai} = \frac{P}{A} + \frac{My}{I}$$

$$= \frac{\frac{wL^3}{12}}{L} = \frac{wL^2}{12} = M_{Bi}$$

Net moment at A = $M_s - M_{Ai} = 0 - \dfrac{wL^2}{12} = -\dfrac{wL^2}{12}$

Net moment at B = $M_s - M_{Bi} = 0 - \dfrac{wL^2}{12} = -\dfrac{wL^2}{12}$

The fixed end moments at A and B are both hogging.

(b) *Solution assuming cantilever beam (Fig. 6.7c)*

The beam can be made statically determinate by removing the support B. The load on the cantilever beam due to the uniform load is applied as an upward load on the top of

the analogous column because this moment causes compression on the inside of the beam.

$$M_i = \text{pressure} = \frac{P}{A} \pm \frac{My}{I}$$

For point A,

$$M_s = -\frac{wL^2}{2}$$

$$P = \frac{1}{3} \times \frac{wL^2}{2} \times L = \frac{wL^3}{6}$$

$$M = P\,e, \quad e = L/4, \quad I/y = 1 \times L^2/6$$

$$M_i = -\frac{\frac{wL^3}{6}}{1 \times L} - \frac{\left(\frac{wL^3}{6}\right)\left(\frac{L}{4}\right)}{1 \times \frac{L^2}{6}} = -\frac{wL^2}{6} - \frac{wL^2}{4} = -\frac{5wL^2}{12}$$

$$M_A = M_s - M_{Bi} = -\frac{wL^2}{2} - \left(-\frac{5wL^2}{12}\right) = -\frac{wL^2}{12}$$

For point B, $M_s = 0$, $I/y = -L^2/6$

$$M_i = -\frac{\frac{wL^3}{6}}{1 \times L} + \frac{\left(\frac{wL^3}{6}\right)\left(\frac{L}{4}\right)}{1 \times \frac{L^2}{6}} = \frac{wL^2}{12}$$

$$M_B = M_s - M_i = 0 - \frac{wL^2}{12} = -\frac{wL^2}{12} \qquad \text{O. K.}$$

Example 6.2

Determine the fixed end moments for the beam shown in Fig. 6.8a. by the column analogy method.

Solution

(a) *Solution assuming simple beam (Fig. 6.8a, b)*

The beam can be made statically determinate by treating M_A and M_B as redundants. The free span moment diagrams, analogous column section and end moment diagrams are shown in Fig. 6.8b.

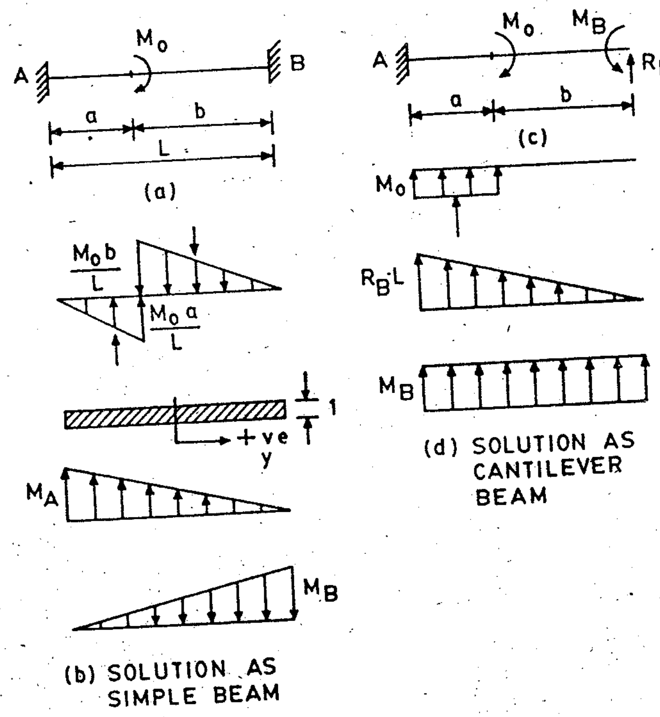


Fig. 6.8

For point A, $\quad M_S = 0$

$$P = -\frac{1}{2}a\left(M_o\frac{a}{L}\right)+\frac{1}{2}(L-a)\left(\frac{M_o b}{L}\right) = -\frac{M_o}{2L}a^2+\frac{M_o}{2L}(L-a)^2$$

$M = P_1 e_1 + P_2 e_2$, where e_1 and e_2 are measured from the centroidal axis.

$$e_1 = -\left(\frac{L}{2}-\frac{2}{3}a\right)$$

$$e_2 = \left(\frac{L}{2}-\frac{2}{3}b\right) = \frac{L}{2}-\frac{2}{3}(L-a) = \frac{4a-L}{6}$$

$y = -\,L/2, \qquad I = 1\times L^3/12$

$$M_i = \frac{P}{A}+\frac{Pey}{I}$$

$$M_i = \frac{M_o}{2L^2}\left[(L-a)^2-a^2\right]+\frac{(-)\dfrac{M_o a^2}{2L}\times(-)\left(\dfrac{L}{2}-\dfrac{2}{3}a\right)}{(-)\dfrac{L^2}{6}}+\frac{\dfrac{M_o}{2L}(L-a)^2\left(\dfrac{4a-L}{6}\right)}{(-)\dfrac{L^2}{6}}$$

$$= \frac{M_o}{2L^2}\left[L^2-2aL-\frac{6a^2}{L}\left(\frac{L}{2}-\frac{2a}{3}\right)-\frac{1}{L}(L-a)^2(4a-L)\right]$$

$$= \frac{M_o}{2L^3}\left(2L^3+6a^2L-8aL^2\right)$$

$$M_i = M_o\left(1-\frac{4a}{L}+\frac{3a^2}{L^2}\right)$$

$$\therefore \quad M_A = M_s - M_i = -M_o\left(1-\frac{a}{L}\right)\left(1-\frac{3a}{L}\right)$$

For point B,

$$M_i = \frac{M_o}{2L^2}\left[(L-a)^2-a^2\right]+\frac{(-)\dfrac{M_o a^2}{2L}\left(\dfrac{L}{2}-\dfrac{2}{3}a\right)}{\dfrac{L^2}{6}}+\frac{\dfrac{M_o}{2L}(L-a)^2\left(\dfrac{4a-L}{6}\right)}{\dfrac{L^2}{6}}$$

$$M_i = \frac{M_o}{2L^3}\left(-6a^2L+4aL^2\right) = M_o\frac{a}{L}\left(2-3\frac{a}{L}\right)$$

$$\therefore \quad M_B = M_S - M_i = -M_o\frac{a}{L}\left(2-3\frac{a}{L}\right)$$

(b) *Solution assuming cantilever beam (Fig. 6.8 c, d)*

The fixed beam can be made statically determinate by removing the support at B and treating R_B and M_B as redundants. The free span moment diagram, and end moment diagrams are shown in Fig. 6.8 d. The centroidal axis y - y passes through the mid-point of the column section at L/2.

For point A,

$$M_S = -M_o, \qquad P = -M_o a, \qquad y = -L/2$$

$$M = Pe, \qquad e = -\left(\frac{L}{2}-\frac{a}{2}\right) = -\left(\frac{L-a}{2}\right)$$

$$M_i = -\frac{M_o a}{L} + \frac{(-)M_o a(-)\frac{(L-a)}{2}}{(-)1 \times \frac{L^2}{6}}$$

$$= -M_o \frac{a}{L} - 3M_o \frac{a}{L^2}(L-a)$$

$$\therefore \quad M_A = M_s - M_i = -M_o + M_o \frac{a}{L} + 3M_o \frac{a}{L^2}(L-a)$$

or, $$M_A = -M_o\left(1-\frac{a}{L}\right)\left(1-\frac{3a}{L}\right) \qquad \text{O. K.}$$

For point B,

$$M_s = 0, \quad y = +\frac{L}{2}$$

$$M_i = -\frac{M_o a}{L} + \frac{M_o a(L-a)/2}{1 \times L^2/6} = -M_o \frac{a}{L} + 3M_o \frac{a}{L^2}(L-a)$$

$$\therefore \quad M_B = M_s - M_i = M_o \frac{a}{L} - 3M_o \frac{a}{L^2}(L-a)$$

or, $$M_B = -M_o \frac{a}{L}\left(2 - 3\frac{a}{L}\right) \qquad \text{O. K.}$$

6.7 STIFFNESS AND CARRY OVER FACTORS

The stiffness of a beam at end A is defined as the moment required to produce a unit slope at A when the end B is fixed. A moment M_B will be produced at B when the moment M_A is applied at A. The carry over factor from A to B is the ratio of the moment at the fixed end B to the moment applied at A under the above condition.

Consider a beam AB of uniform cross- section shown in Fig. 6.9a. If a clockwise moment is applied at A, the beam will deflect as shown in dotted line. The slope at A will be θ_A which is proportional to M_A , that is,

$$M_A \propto \theta_A$$
$$= k_A \theta_A$$

or, stiffness factor $$k_A = \frac{M_A}{\theta_A} \qquad (6.9)$$

The moment diagrams on the conjugate beam due to M_A and M_B are shown in Fig. 6.9b and the reaction θ_A in the conjugate beam is shown in Fig. 6.9c. Now if reaction on the conjugate beam is treated as loading on the analogous column (Fig. 6.9d) and

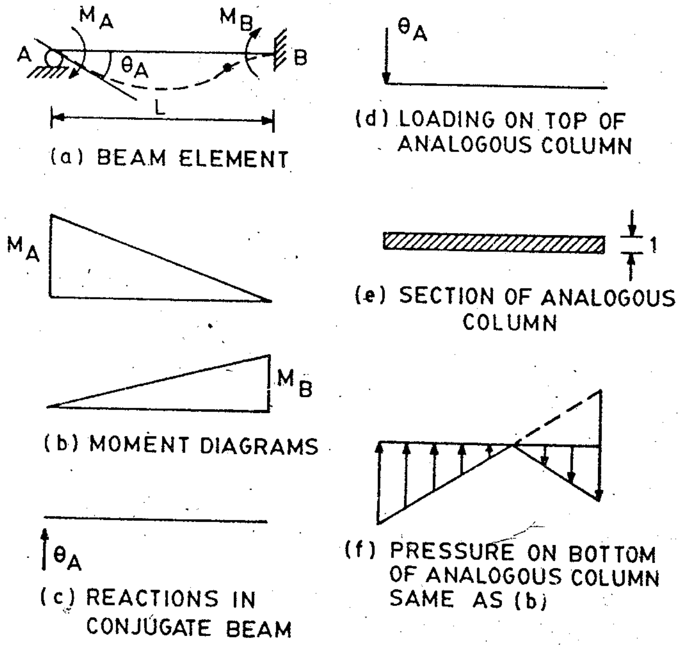


Fig. 6.9 Stiffness and carry over factors

M_A / EI and M_B / EI diagrams are considered as pressure diagrams on the bottom of the column, the column will still be in equilibrium. The M_A / EI diagram will be considered as positive while the M_B / EI diagram will be considered as negative. In this case, it is convenient to take the width of the analogous column as $1 / EI$ instead of unity. Thus, M_A and M_B can be found directly as the positive and negative pressures at A and B.

Area of the analogous column $= \frac{L}{EI}$, Moment of inertia $= \frac{L^3}{12EI}$

Downward load on column $= \theta_A$; Eccentricity $e = \frac{L}{2}$

Distance of extreme fiber $y = \pm \frac{L}{2}$

$$M_A = \sigma = \frac{P}{A} + \frac{My}{I}$$

$$= \frac{\theta_A}{\frac{L}{EI}} + \frac{\theta_A\left(\frac{L}{2}\right)\left(\frac{L}{2}\right)}{\left(\frac{L^3}{12EI}\right)} = \frac{4EI}{L}\theta_A$$

or, $$k_A = \frac{M_A}{\theta_A} = \frac{4EI}{L}$$

Similarly,

$$M_B = \sigma_B = \frac{P}{A} + \frac{My}{I} = \frac{\theta_A}{\frac{L}{EI}} + \frac{\theta_A\left(\frac{L}{2}\right)\left(-\frac{L}{2}\right)}{\left(\frac{L^3}{12EI}\right)} = -\frac{2EI}{L}\theta_A$$

or, $$M_B = -\frac{1}{2}M_A$$

The negative sign for M_B means that it is hogging. In fact, M_A and M_B are both clockwise causing sagging and hogging moments, respectively. Therefore, treating clockwise moments as positive,

Carry over factor from A to B, $$C_{AB} = \frac{M_B}{M_A} = \frac{1}{2} \qquad (6.10)$$

These results are very useful in the stiffness methods of analysis.

Example 6.3

Determine the stiffness at end B of the beam shown in Fig. 6.10, and the carry over factor from B to A.

Solution

(a) Properties of the analogous column section

$$A = \infty; \qquad I_y = \frac{1\times a^3}{3} + \frac{1\times b^3}{3}, \qquad I_x = \infty$$

(b) Stiffness factor k_B and carryover factor C_{BA} from B to A.

Let us apply a load θ_B at the right edge of the analogous column.

$$M_B = \text{pressure at B} = \frac{P}{A} \pm \frac{M_y}{I_y}x$$

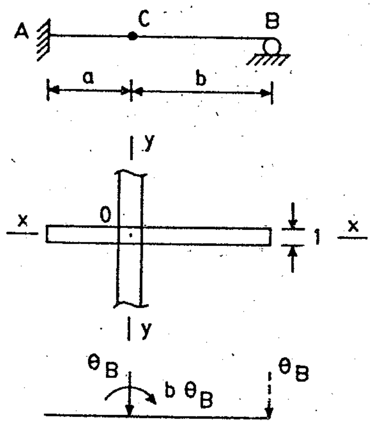


Fig. 6.10

or, $$M_B = \frac{\theta_B}{\infty} \pm \frac{b\theta_B x}{\frac{(a^3+b^3)}{3EI}} = \frac{3b^2EI}{(a^3+b^3)}\theta_B$$

$$\therefore \quad k_B = \frac{M_B}{\theta_B} = \frac{3b^2EI}{(a^3+b^3)}$$

and $$M_A = -\frac{b\theta_B a}{\frac{(a^3+b^3)}{3EI}} = -\frac{3abEI}{(a^3+b^3)}\theta_B$$

Taking clockwise moment acting on the beam element as positive for both ends,

$$C_{BA} = \frac{M_A}{M_B} = \frac{a}{b}$$

6.8 BEAMS WITH VARIABLE CROSS-SECTION

Example 6.4

Determine the fixed end moments for the beam shown in Fig. 6.11a by the column analogy method.

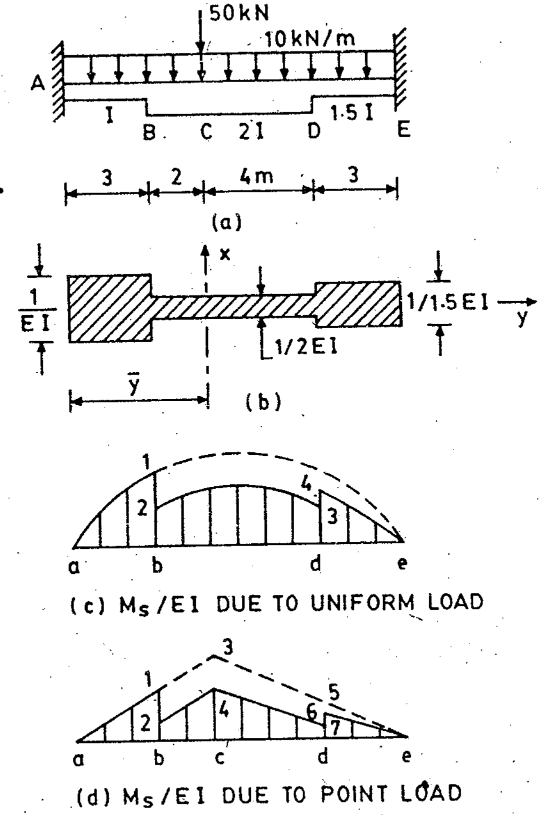


Fig. 6.11

Solution

(a) Properties of analogous column

The section of the beam is non-prismatic. The length of the analogous column section is equal to the span of the beam. Its width is equal to 1/EI as shown in Fig. 6.11b. Let EI = unity. The properties of the column section can be determined as follows:

$$\text{Area } A = 1 \times 3 + \frac{1}{2} \times 6 + \frac{1}{1.5} \times 3 = 8 \text{ units}$$

c.g. of the analogous column from A

$$\bar{y} = \frac{3 \times 1.5 + 3 \times 6 + 2 \times 10.5}{8} = 5.44 \text{ m}$$

Moment of inertia I_x about the centroidal axes

$$I_x = 1 \times \frac{3^3}{12} + 3 \times (5.44 - 1.5)^2 + \frac{1}{2} \times \frac{6^3}{12} + 3 \times (6 - 5.44)^2 + \frac{1}{1.5} \times \frac{3^3}{12} + 2 \times (10.5 - 5.44)^2$$

$$I_x = 111.47 \text{ units.}$$

The beam can be made statically determinate by introducing a hinge at support A and roller at support E. The M_S diagrams may be drawn separately for the uniform load and the point load for convenience.

(b) Analogous loading due to uniform load (Fig. 6.11c)

$$\text{Bending moment at B} = 60 \times 3 - 10 \times \frac{3^2}{2} = 135 \text{ kNm}$$

$$\therefore \quad \text{ordinate } 1 - b = \frac{M}{EI} = \frac{135}{EI} = 135 \text{ unit (since EI = 1)}$$

$$\text{and} \quad \text{ordinate } 2 - b = \frac{135}{2EI} = 67.5 \text{ units}$$

$$\text{Bending moment at D} = 60 \times 9 - 10 \times \frac{9^2}{2} = 135 \text{ kNm}$$

$$\therefore \quad \text{ordinate } 3 - d = \frac{M}{EI} = 67.5 \text{ units}$$

$$\text{and} \quad \text{ordinate } 4 - d = \frac{135}{1.5EI} = 90 \text{ units}$$

$$\text{Area of segment } a - b - 1, \; P_1 = \frac{wa^2}{12}(3L - 2a) \times \frac{1}{EI}$$

$$= 10 \times \frac{3^2}{12}(3 \times 12 - 2 \times 3) \times 1 = 225 \text{ units}$$

$$\text{c.g. of } a - b - 1 \text{ from A} = \frac{3 \times (4 \times 12 - 3 \times 3)}{2 \times (3 \times 12 - 2 \times 3)} = 1.95 \text{ m}$$

$$\text{Area of segment } 2 - b - d - 3, \; P_2 = \frac{w}{12}\left[3L\left(b^2 - a^2\right) - 2\left(b^3 - a^3\right)\right] \times \frac{1}{2EI}$$

$$= \frac{10}{12}\left[3 \times 12 \times \left(9^2 - 3^2\right) - 2\left(9^3 - 3^3\right)\right] \times \frac{1}{2} = 49.5 \text{ units}$$

c. g. of 2 – b – d – 3 from A $= 6$ m

Area of segment e – 4 – d, $P_3 = \dfrac{wa^2}{12}(3L - 2a) \times \dfrac{1}{1.5EI}$

$$= 10 \times \frac{3^2}{12}(3 \times 12 - 2 \times 3) \times \frac{1}{1.5} = 150 \text{ units}$$

c. g. of e – 4 – d from A $= 10.05$ m

(c) *Analogous loading due to the point load (Fig. 6.11d)*

Bending moment at B $= 29.17 \times 3 = 87.5$ kNm

∴ ordinate 1 – b $= \dfrac{M}{EI} = 87.5$ units

and ordinate 2 – b $= 43.75$ units

Bending moment at C $= 29.17 \times 5 = 145.85$ kNm

∴ ordinate 4 – c $= \dfrac{145.85}{2EI} = 72.93$ units

Bending moment at D $= 62.5$ kNm

∴ ordinate 7 – d $= 31.25$ units

ordinate 6 – d $= 41.67$ units

Area of segment a – 1 – b, $P_4 = \dfrac{1}{2} \times 3 \times 87.5 = 131.25$ units

c. g. of a – 1 – b from A $= 2$ m

Area of segment 2 – b – c – 4, $P_5 = \left(\dfrac{43.75 + 72.93}{2}\right) \times 2 = 116.68$ units

c. g. of 2 – b – c – 4 from A $= \dfrac{2}{3} \times \dfrac{(43.75 + 2 \times 72.93)}{(43.75 + 72.93)} + 3 = 4.08$ m

Area of segment 4 – c – d – 7, $P_6 = \left(\dfrac{72.93 + 31.25}{2}\right) \times 4 = 208.36$ units

c.g. of 2 – b – c – 4 from A $= \dfrac{4}{3} \times \dfrac{(72.93 + 2 \times 31.25)}{(72.93 + 31.25)} + 5 = 6.73$ m

Area of segment 7 – d – e, $P_7 = \dfrac{1}{2} \times 3 \times 41.67 = 62.5$ units

c. g. of 7 – d – e from A $= 10$ m

(d) *Net loading on the analogous column*

$$P = P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7$$
$$= 225 + 495 + 150 + 131.25 + 116.68 + 208.36 + 62.5 = 1388.8 \text{ units}$$

c.g. of the analogous loading from A

$$= (225 \times 1.95 + 495 \times 6 + 150 \times 10.05 + 131.25 \times 2 + 116.68 \times 4.08 + 208.36 \times 6.73 + 62.5 \times 10) / 1388.8$$

$$= \frac{7681.67}{1388.8} = 5.53 \text{ m}$$

Net moment of the loads about the centroidal axis

$$M = 225 \times (5.44 - 19.5) + 495 \times (5.44 - 6) + 150 \times (5.44 - 10.05) + 131.25 \times (5.44 - 2) + 116.68 \times (5.44 - 4.08) + 208.36 \times (5.44 - 6.73) + 62.5 \times (5.44 - 10)$$
$$= -127 \text{ units clockwise}$$

The net stress at any section can be determined from the following equation as shown in Table 6.1.

$$\sigma = \frac{P}{A} \pm \frac{My}{I_x} = \frac{1388.8}{8} \pm \frac{127y}{111.47} = 173.6 + 1.14y$$

TABLE 6.1 Computation of net moments

Point	M_s	y_m	P/A	1.14y	M_i kNm	$M = M_s - M_i$ kNm
A	0	– 5.44	173.6	– 6.20	167.4	– 167.40
B	222.50	– 2.44	173.6	– 2.78	170.8	51.68
C	320.88	– 0.44	173.6	– 0.50	173.1	147.80
D	197.50	+3.56	173.6	4.06	177.6	19.80
E	0	+6.56	173.6	7.48	181.1	– 181.10

Example 6.5

Determine stiffness of the beam shown in Fig. 6.12 a at its either end and also compute the carry over factor from A to B, and B to A.

Solution

(a) *Properties of analogous column section (Fig. 6.12b)*

Area of column section $= \dfrac{1}{2} \times 4 + 1 \times 4 = 6$ units

c.g. of section $\bar{y} = \dfrac{\frac{1}{2} \times 4 \times 2 + 1 \times 4 \times 6}{6} = \dfrac{28}{6} = 4.667$ from A

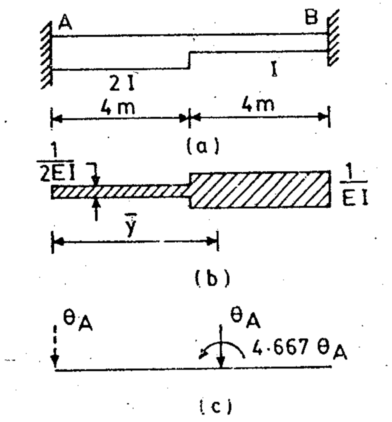


Fig. 6.12

$$I_x = \frac{1}{2}\times\frac{4^3}{12}+2\times(2.667)^2+\frac{1\times 4^3}{12}+4(2-0.667)^2$$

$$= 2.67+14.22+5.34+7.10 = 29.33\text{units}$$

(b) Stiffness factor k_A and carry over factor C_{AB} from A to B

Let us apply a load θ_A at the left edge of the analogous column as shown in Fig. 6.12c.

$$M_A = \text{pressure at A}$$

$$M_A = \frac{P}{A}+\frac{My}{I}$$

$$= \frac{\theta_A}{\frac{6}{EI}}+\frac{(4.667\theta_A)(4.667)}{\frac{29.33}{EI}} = EI\theta_A\left[\frac{1}{6}+0.7426\right]$$

or, $$M_A = \frac{EI}{L}\theta_A\left[\frac{8}{6}+8\times 0.7426\right] = 7.27\frac{EI}{L}\theta_A$$

Thus, $$k_A = 7.27\frac{EI}{L}$$

$$M_B = \text{pressure at B} = \frac{\theta_A}{\frac{6}{EI}}+\frac{(4.667\theta_A)(-3.333)}{\frac{29.33}{EI}} = EI\theta_A\left[\frac{1}{6}-0.53\right]$$

or, $$M_B = \frac{EI}{L}\theta_A\left[\frac{8}{6}-8\times 0.53\right] = -2.91\frac{EI}{L}\theta_A$$

∴ Taking clockwise moment acting on the beam element as positive for both ends,

$$C_{AB} = \frac{2.91}{7.27} = 0.4$$

(c) Stiffness factor K_B and carry over factor C_{BA} from B to A

A load θ_B is applied at the right edge of the analogous column (not shown).

$$M_B = \text{pressure at B}$$

$$= \frac{\theta_B}{\frac{6}{EI}}+\frac{(3.33\theta_B)(3.33)}{\frac{29.33}{EI}}$$

or, $$M_B = \frac{EI}{L}\theta_B\left[\frac{8}{6}+\frac{8\times 3.33^2}{29.33}\right] = 4.35\frac{EI}{L}\theta_B$$

$$k_B = 4.35\frac{EI}{L}$$

$$M_A = \text{pressure at A}$$

$$= \frac{\theta_B}{\frac{6}{EI}}-\frac{(3.33\theta_B)(4.667)}{\frac{29.33}{EI}}$$

or, $$M_A = \frac{EI}{L}\theta_B\left[\frac{8}{6}-\frac{8\times 3.33\times 4.667}{29.33}\right] = -2.91\frac{EI}{L}\theta_B$$

Again, using the same sign convention,

∴ $$C_{BA} = \frac{2.91}{4.35} = 0.67$$

6.9 PORTAL FRAMES WITH ONE AXIS OF SYMMETRY

Example 6.6

Analyze the portal frame shown in Fig. 6.13a by the column analogy method.

Solution

The frame may be treated as fixed at A and D and having a moment of inertia equal to zero. The frame is made statically determinate by placing support D on rollers. The

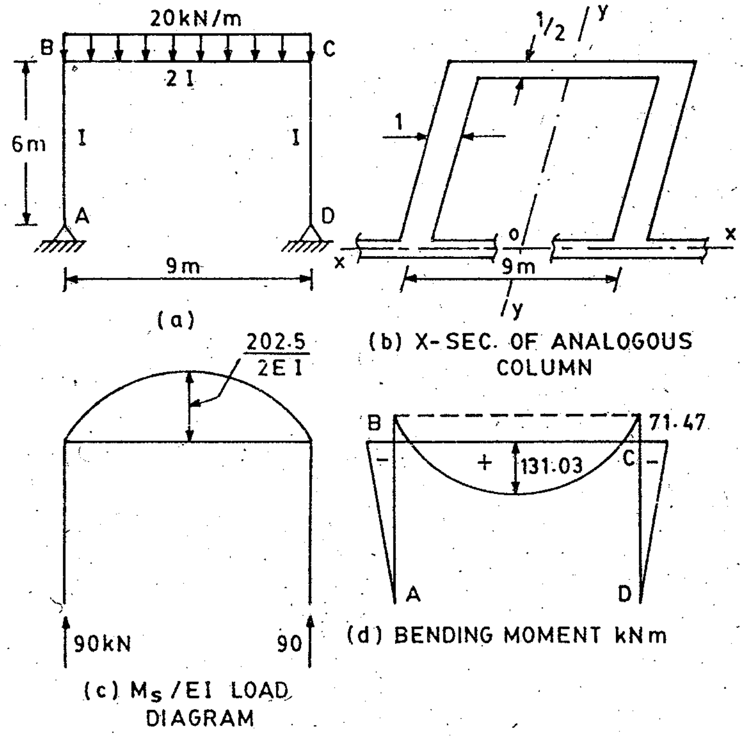


Fig. 6.13

analogous column is shown in Fig. 6.13b. The width of the sections at points A and D are infinite. Hence, area of the section is infinite and the centre of gravity of the section will act at mid-point of the line BC. The moment of inertia of the section about the axis y - y will be infinite. The moment of inertia about the axis x - x passing through the points A - D is given by :

$$I_x = \frac{9}{2EI} \times 6^2 + 2 \times \frac{1}{EI} \times \frac{6^3}{3} = \frac{306}{EI}$$

The free span moment diagram is shown in Fig. 6.13c.

$\therefore$ Load on the column section, $P = \frac{2}{3} \times \frac{202.5}{2EI} \times 9 = \frac{607.5}{EI}$

$M_y = Pe_x$, $e_x = 0$, $M_x = Pe_y$, $e_y = 6$ m

Stress at any point (x, y) on the section

$$\sigma = \frac{P}{A} + \frac{M_y}{I_y}x + \frac{M_x}{I_x}y \qquad (6.4)$$

$$\sigma = \frac{\frac{607.5}{EI}}{\infty} + \frac{\left[\frac{607.5}{EI}\right]0}{\infty}x + \frac{\left(\frac{607.5}{EI}\right)6y}{\frac{306}{EI}} = 11.91y$$

The stress is compressive throughout and varies linearly. Hence, the end moments are hogging.

Stress at A = 0
B = − 71.47 kNm
C = − 71.47 kNm
D = 0

The net bending moment diagram can now be drawn by subtracting the above values from the M_S diagram, and is shown in Fig. 6.13d.

Example 6.7

Analyze the gable frame shown in Fig. 6.14a by the column analogy method.

Solution

(a) Properties of the analogous column section (Fig.6.14b)

$$A = 2 \times 1 \times 5 + 2 \times \frac{1}{2} \times 7.211 = 17.211 \text{ units}$$

$$\bar{y} \text{ from AE} = \frac{2 \times 5 \times 2.5 + 7.211 \times 7}{17.211} = 4.385 \text{ m}$$

$$\sin\theta = \frac{4}{7.211} = 0.555, \quad \cos\theta = \frac{6}{7.211} = 0.832$$

The inclined member BC is shown in Fig. 6.14c. Let width = b

$$I'_x \text{ of member BC} = 2\int_0^{\frac{L}{2}} b\,dr\,(r\sin\theta)^2 = \int_0^L b r^2 \sin^2\theta\,dr$$

$$\text{or} \quad I'_x = \frac{bL^3}{12}\sin^2\theta = \frac{1}{2} \times 7.211^3 \times \frac{0.555^2}{12} = 4.812 \text{ units}$$

$$\text{Similarly } I'_y \text{ of member BC} = \frac{bL^3}{12}\cos^2\theta = \frac{1}{2} \times 7.211^3 \times \frac{0.832^2}{12} = 10.81 \text{ units}$$

Moment of inertia about the centroidal axis of the frame

$$I_x = 2 \times 1 \times \frac{5^3}{12} + 2 \times 5 \times (4.385 - 2.5)^2 + 2 \times 4.812 + 2 \times \left(\frac{1}{2} \times 7.211\right) \times 2.615^2$$

$$= 20.83 + 35.53 + 9.625 + 49.31 = 115.30 \text{ units}$$

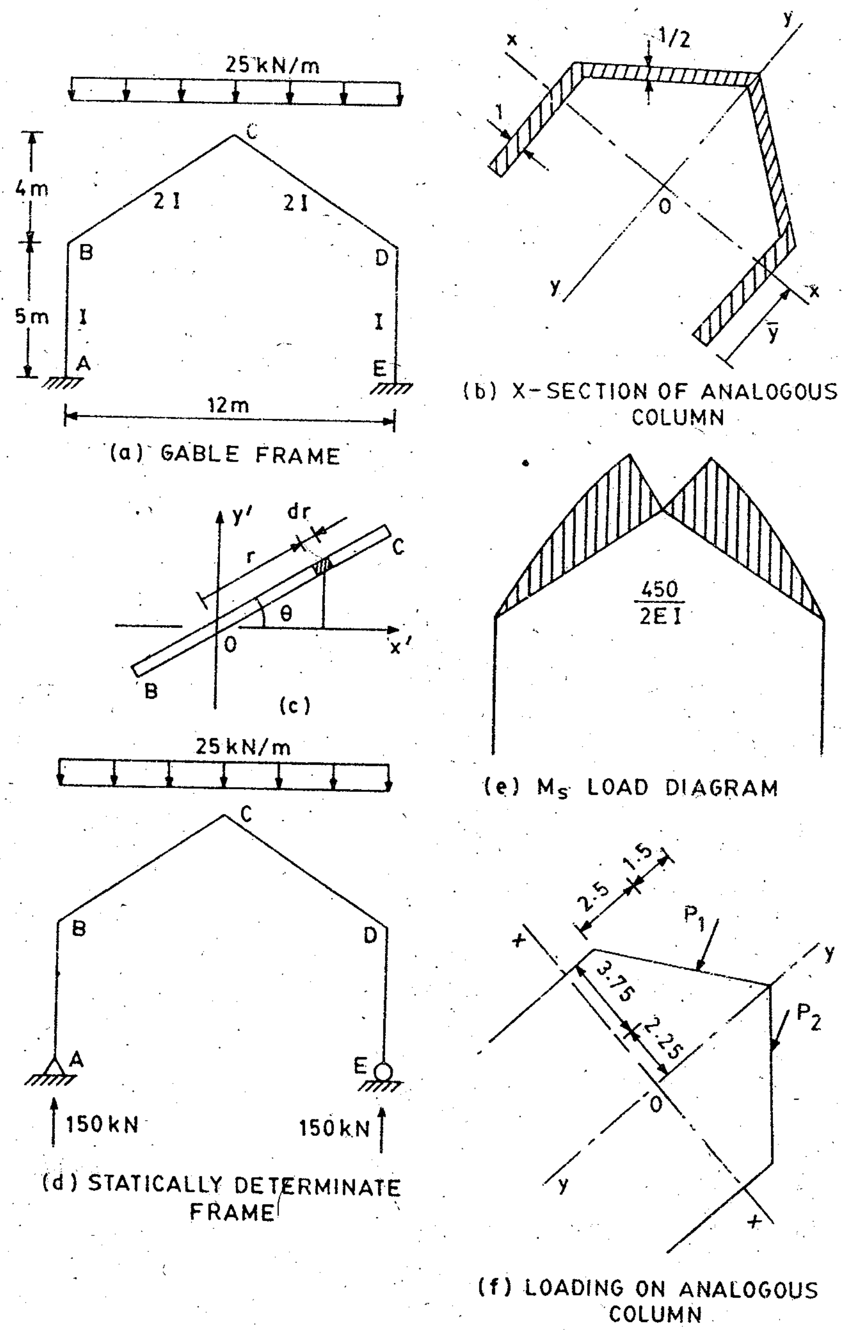


Fig. 6.14 Gable frame

$$I_y = 2 \times 1 \times 5 \times 6^2 + 2 \times (\frac{1}{2} \times 7.211) \times 3^2 + 10.81 = 381.63 \text{ units}$$

The statically determinate structure can be obtained in several ways. By introducing a hinge at A and a roller at E result in a symmetrical free span moment diagram and thus symmetrical loads on the analogous column as shown in Figs. 6.14d, e and f.

$$\text{Load on BC } P_1 = \frac{2}{3} \times 7.211 \times 225 = 1081.65 \text{ downward}$$

$$= \text{load on CD, } P_2$$

$$\text{Total load } P = 2163.3$$

$$\text{c.g. of load } P_1 \text{ from B} = \frac{5L}{16} = \frac{5 \times 12}{16} = 3.75 \text{ m}$$

eccentricity along y axis, $e_y = 2.5 + (5 - 4.385) = 3.115$ m

eccentricity along x axis, $e_x = 0$

Hence: $M_x = P\,e_y = 2163.3 \times 3.115 = 6738.7$ and $M_y = 0$

The stress at any point in the analogous column is given by

$$\sigma = \frac{P}{A} + \frac{M_x}{I_x} y$$

$$= \frac{2163.3}{17.211} + \frac{6738.7 \times y}{115.3} = 125.69 + 58.44\,y$$

Sections B, C, D will be in compression and A, E in tension due to the action of the moment M_x alone. The computations are shown in Table 6.2.

TABLE 6.2 Computation of net moments

Point	M_s	y_m	P/A	$\frac{My}{I}$	M_i kNm	$M = M_s - M_i$ kNm
A	−4.385	0	125.69	−256.26	−130.57	130.57
B	0.615	0	125.69	35.94	161.63	−161.63
C	4.615	450	125.69	269.70	395.39	54.60
D	0.615	0	125.69	35.94	161.63	−161.63
E	−4.385	0	125.69	−256.26	−130.57	130.57

6.10 CLOSED FRAMES WITH ONE AXIS OF SYMMETRY

Example 6.8

Analyze the closed rectangular frame shown in Fig. 6.15a by the column analogy method.

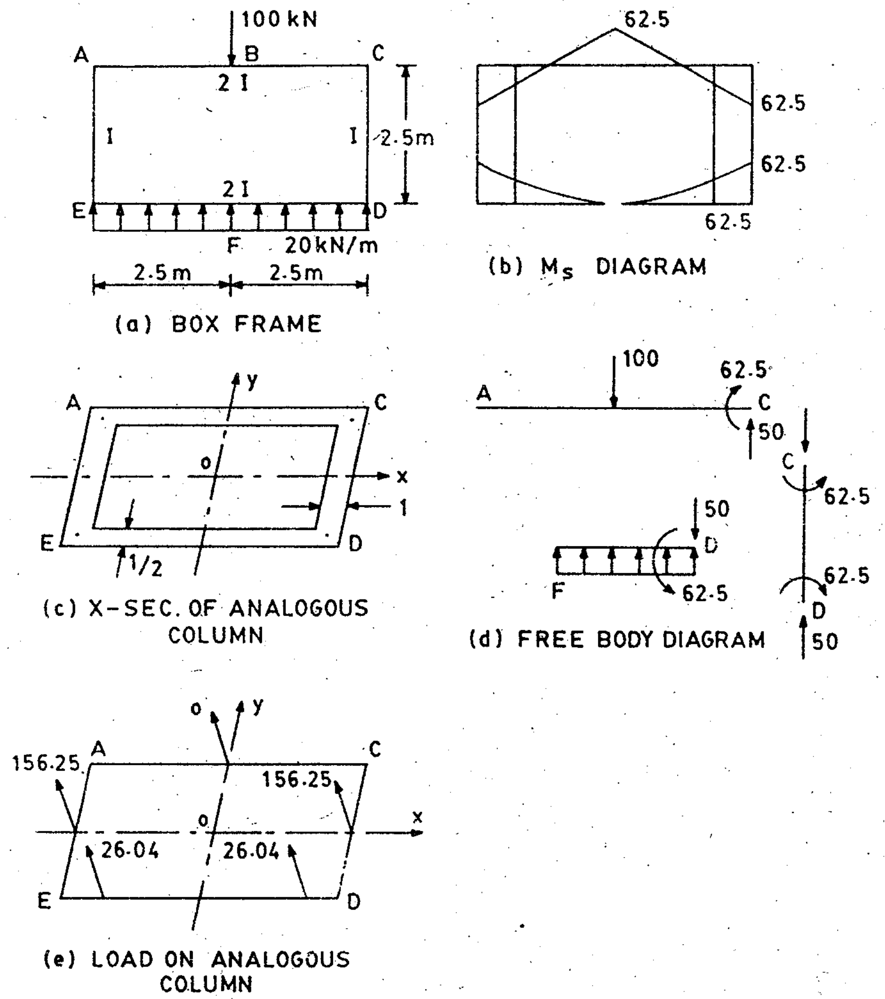


Fig. 6.15 Box frame

Solution

Let the frame be made statically determinate by making a cut in the member ED at F. Under such a condition, the M_s diagram is shown in Fig. 6.15b. It is hogging on sides ED, CD. The load is a pull as there is tension on outside.

Properties of the Analogous Section (Fig.6.15c)

$$\text{Area } A = 2 \times \frac{1}{2} \times 5 + 2 \times 1 \times 2.5 = 10 \text{ units}$$

$$I_{xx} = 2 \times 1 \times \frac{2.5^3}{12} + 2 \times \frac{1}{2} \times 5 \times (1.25)^2 = 10.42 \text{ units}$$

$$I_{yy} = 2 \times \frac{1}{2} \times \frac{5^3}{12} + 2 \times (1 \times 2.5) \times 2.5^2 = 41.67 \text{ units}$$

Free body diagram of the released frame is shown in Fig. 6.15d. The loads on the analogous column section are shown in Fig. 6.15e.

$$P = 2 \times \frac{1}{3} \times 2.5 \times \frac{62.5}{2} + 2 \times 2.5 \times 62.5 + \frac{62.5}{2} \times 5 - \frac{1}{2} \times 5 \times \frac{125}{2}$$
$$= 52.08 + 312.5 + 156.25 - 156.25 = 364.58 \text{ (pull)}$$

$$M_x = P e_y = 2 \times 26.04 \times \frac{2.5}{2} = 65.10 \text{ anti- clockwise}$$

(viewed towards positive x- axis)

$$M_y = 0, \ (= P e_x \text{ but } e_x = 0)$$

$$\text{Net stress at any point} = \frac{P}{A} + \frac{M_x}{I_x} y$$

$$= -\frac{364.58}{10} + \frac{65.1}{10.42} y$$

$$\text{Stress at A, C} = -36.45 + \frac{65.1 \times 1.25}{10.42} = -28.64 \text{ kNm (sagging)}$$

$$\text{Stress at D, E} = -36.45 - \frac{65.1 \times 1.25}{10.42} = -44.26 \text{ kNm (sagging)}$$

∴ Net end moment at A, C $= -62.5 + 28.64 = -33.86$ kNm

D, E $= -62.5 + 44.26 = -18.24$ kNm

Alternative Solution

Let us make the frame statically determinate by providing hinges at each of the four corners. The M_s diagram is shown in Fig. 6.16. It creates compression on outside face. Hence, load is a thrust.

$$P = \frac{1}{2} \times 5 \times \frac{125}{2} + \frac{2}{3} \times 5 \times \frac{62.5}{2} = 260.42 \text{ units}$$

$$M_x = 156.25 \times 1.25 - 104.17 \times 1.25 = 65.1 \text{ anti-clockwise}$$

(viewed towards positive x-axis)

$$M_y = 0$$

$$\text{Net stress at any point} = \frac{260.42}{10} \pm \frac{65.1}{10.42} y$$

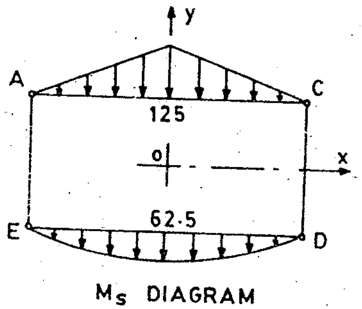


Fig. 6.16 Box frame - alternate solution

Stress at A and C = $26.04 + \dfrac{65.1 \times 1.25}{10.42} = 33.85$ kNm (hogging)

Stress at D and E = $26.04 - 7.81 = 18.23$ kNm (hogging)

The net moment diagram can now be obtained by algebraic addition of the above stresses (end moments) with the free span moment values. O. K.

6.11 PORTAL FRAMES WITH NO SYMMETRY

Example 6.9

Analyze the portal frame with fixed supports shown in Fig. 6.17a by column analogy method.

Solution

(a) *Properties of analogous column (Fig. 6.17b)*
Taking moments about the line AC

$$\bar{x} = \frac{\frac{1}{2} \times 7 \times 3.5 + 1 \times 4 \times 7}{1 \times 7 + \frac{1}{2} \times 7 + 1 \times 4} = \frac{40.25}{14.5} = 2.78 \text{ m}$$

Taking moments about a horizontal line passing through A,

$$\bar{y} = \frac{1 \times 7 \times 3.5 + \frac{1}{2} \times 7 \times 7 + 1 \times 4 \times 5}{1 \times 7 + \frac{1}{2} \times 7 + 1 \times 4} = \frac{69}{14.5} = 4.76 \text{ m}$$

$$A = 1 \times 7 + \frac{1}{2} \times 7 + 1 \times 4 = 14.5 \text{ units}$$

$$I_x = \frac{1 \times 7^3}{12} + 1 \times 7 \times (4.76 - 3.5)^2 + \frac{1}{2} \times 7 \times (7 - 4.76)^2 + \frac{1 \times 4^3}{12} + 1 \times 4 \times (5 - 4.76)^2$$

$$I_x = 62.82$$

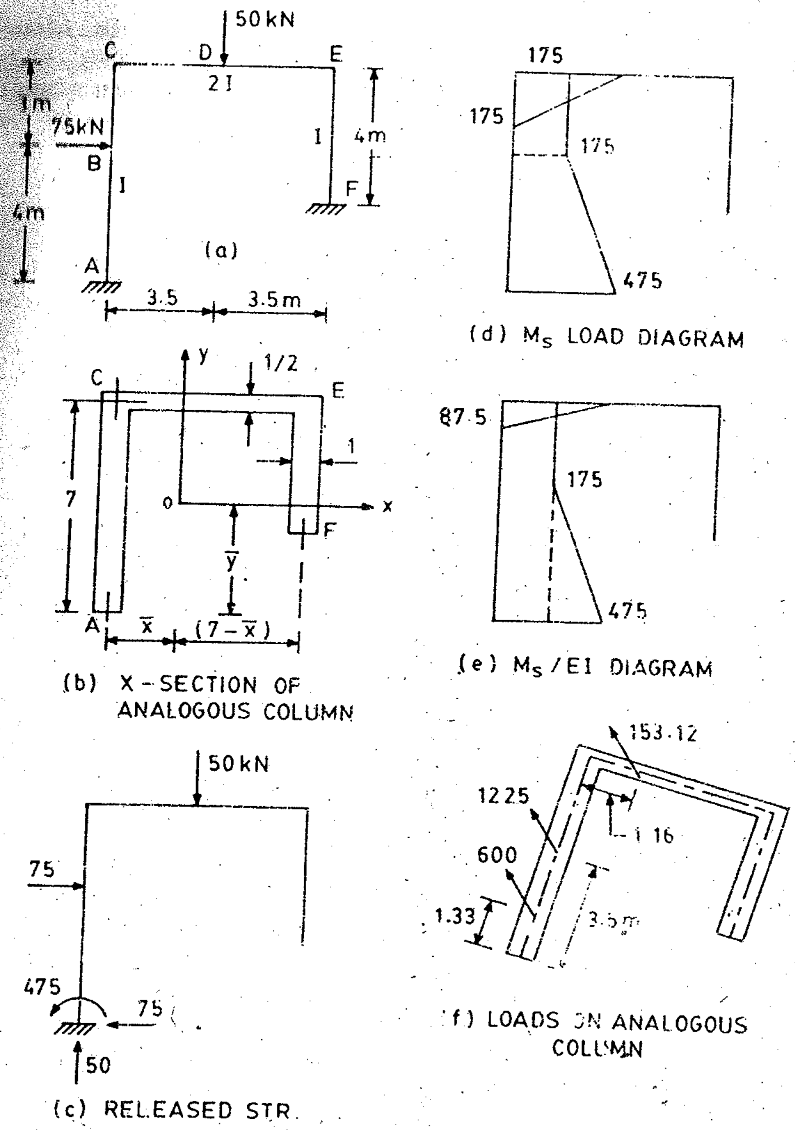


Fig. 6.17 Unsymmetrical portal frames with fixed ends

$$I_y = 1 \times 7 \times 2.78^2 + 1 \times 4 \times (7 - 2.78)^2 + \frac{1}{2} \times \frac{7^3}{12} + \frac{1}{2} \times 7 \times (3.5 - 2.78)^2 = 141.45$$

$$I_{xy} = 1 \times 4 \times (7 - 2.78)(5 - 4.76) + \frac{1}{2} \times 7 \times (3.5 - 2.78)(7 - 4.76) + 1 \times 7 \times (-2.78)(3.5 - 4.76)$$

$$= 34.21$$

The frame is made statically determinate by removing the support F. The released structure is shown in Fig. 6.17c. The M_s and M_s/EI diagrams are shown in Figs. 6.17 d and e. The loads on the analogous column are shown in Fig. 6.17 f and are tabulated in Table 6.3. Since the M_s diagram causes tension on the outside of the frame, the load represents pull.

TABLE 6.3 Load on the analogous column

P	e_x	e_y	$M_y = Pe_x$	$M_x = Pe_y$
– 600.00	– 2.78	– 3.43	1668.0	2058.0
– 1225.00	– 2.78	– 1.26	3405.5	1543.5
– 153.12	– 1.62	2.24	248.0	– 343.0
Σ – 1978.12			Σ 5321.5	Σ 3258.5

$$M_x' = M_x - M_y \frac{I_{xy}}{I_y} = 3258 - 5321 \times \frac{34.21}{141.45} = 1971$$

$$M_y' = M_y - M_x \frac{I_{xy}}{I_x} = 5321 - 3258 \times \frac{34.21}{62.82} = 3546$$

$$I_x' = I_x - \frac{I^2_{xy}}{I_y} = 62.82 - \frac{34.21^2}{141.45} = 54.54$$

$$I_y' = I_y - \frac{I^2_{xy}}{I_x} = 141.45 - \frac{34.21^2}{62.82} = 122.80$$

The net moments in the frame are shown in Table 6.4

TABLE 6.4 Moments at points A, C, E and F

Point	x m	y m	M_S	P/A	$\frac{M_y'}{I_y'}x$	$\frac{M_x'}{I_x'}y$	M_i	$M = M_s - M_i$
A	– 2.78	– 4.76	– 475	–1978/14.5 = – 36.41	– 80.27	– 172.00	– 388.70	– 86.40
C	– 2.78	2.24	– 175	– 36.41	– 80.27	80.95	– 135.70	– 39.30
E	4.22	2.24	0	– 36.41	121.85	80.95	+ 66.40	– 66.40
F	4.22	– 1.76	0	– 36.41	121.85	– 63.60	– 78.15	78.15

(b) Alternative Solution

The frame can be made statically determinate by treating M_A, R_{Fx} and R_{Fy} as redundants, that is, a hinge is introduced at A and a roller at F, as shown in Fig. 6.18 a.

The M_s and M_s / EI diagrams are shown in Fig. 6.18b and c. The analogous loading is shown in Fig. 6.18 d, and is tabulated in Table 6.5. Since the M_s diagram causes compression on the outside of the frame, the load P represents thrust.

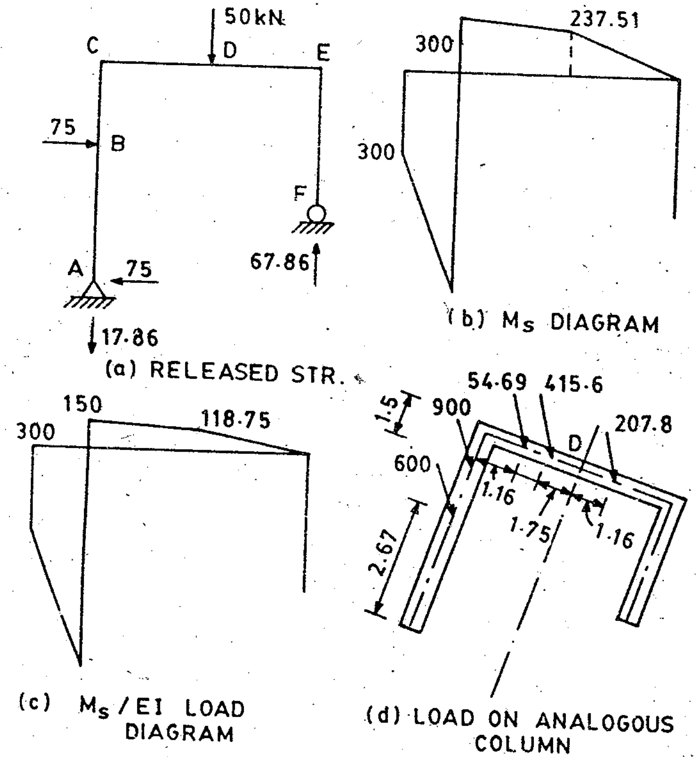


Fig. 6.18 Alternate solution

TABLE 6.5 Load on the analogous column

P	e_x	e_y	$M_y = Pe_x$	$M_x = Pe_y$
207.80	1.88	2.24	– 390.66	465.47
415.60	– 1.01	2.24	– 419.76	930.95
54.69	– 1.62	2.24	– 88.60	122.50
900.00	– 2.78	0.75	– 2502.00	675.00
600.00	– 2.78	– 2.09	– 1668.00	– 1254.00
Σ2178			– 4287.70	939.92

$$M_x = M_x - M_y \frac{I_{xy}}{I_y} = 939.92 + 4287.7 \times \frac{34.21}{141.45} = 1976.9$$

$$M_y{}' = M_y - M_x \frac{I_{xy}}{I_x} = -4287.7 - 939.92 \times \frac{34.21}{62.82} = -4800$$

The net moments in the frame are computed as shown in Table 6.6.

TABLE 6.6 Moments at points A, C, E and F

Point	x m	y m	M_s	P/A	$\frac{M_y'}{I_y'}x$	$\frac{M_x'}{I_x'}y$	M_i	$M = M_s - M_i$
A	– 2.78	– 4.76	0	150.2*	108.67**	– 172.5‡	86.4	– 86.4
C	– 2.78	2.24	300	150.2	108.67	81.2	340	– 40
E	– 4.22	2.24	0	150.2	– 165	81.2	66.4	– 66.4
F	– 4.22	– 1.76	0	150.2	– 165	– 63.6	– 78.6	78.4

$* \ \frac{2178}{14.5} = 150.2, \quad ** \ \frac{-4800(-2.78)}{122.8} = 108.67, \quad ‡ \ \frac{1976.9(-4.76)}{54.54} = -172.5$

The two alternative released structures give identical results within the round-off errors.

Example 6.10

Figure 6.19a shows a portal frame with hinged supports A and D. Draw bending moment diagram by column analogy method.

Solution

The frame has two hinged supports and, therefore, one degree of statical indeterminacy. The hinges in the analogous column will have infinite area passing through the points A and D. The axes system is chosen along AD and at its mid section as shown in Fig. 6.19b. Such an axes represents principal axes and, therefore, $I_{xy} = 0$.

also, $\quad A = \infty$ and $I_y = \infty$

For segment AB, $\quad I_x = \int_0^L b\, ds (S \sin 45)^2$

$$= \int_0^L b S^2 ds (\sin 45)^2$$

$$I_x = b \frac{L^3}{3} \times \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{6} bL^3$$

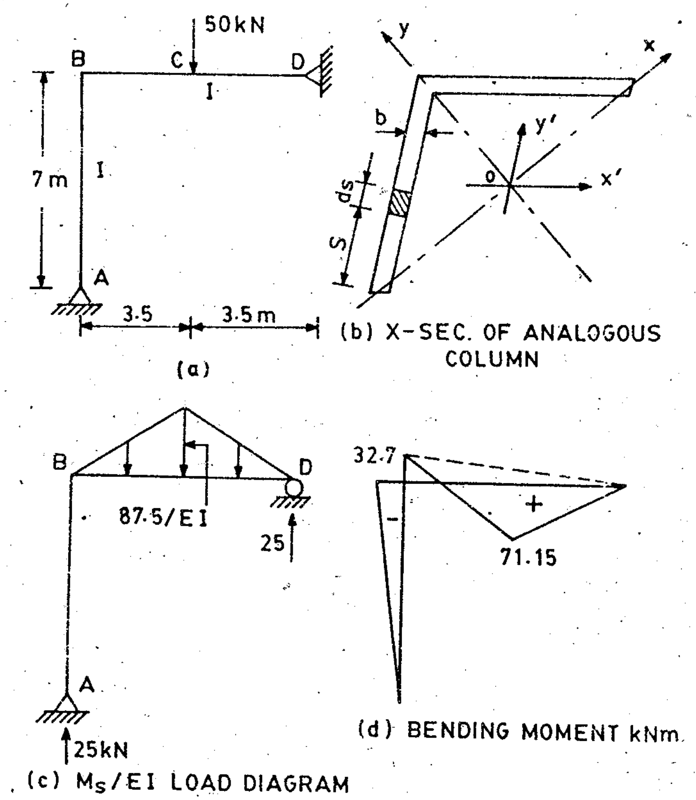


Fig. 6.19 Unsymmetrical portal frame with hinged ends

Similarly, for BD, $\quad I_x = \frac{bL^3}{6}$

Total moment of inertia $I_x = \frac{bL^3}{6} + \frac{bL^3}{6} = \frac{bL^3}{3}$

The frame is made statically determinate by introducing a roller at end D. The M_s diagram is shown in Fig. 6.19c.

$$\text{Total Load } P = \frac{1}{2} \times 87.5 \times 7 = 306.25$$

$$e_y = 3.5 \sin 45 = 2.47 \text{ m}$$

$$\sigma = \frac{P}{A} + \frac{M_x}{I_x} y = \frac{M_x}{I_x} y \quad \text{since } A = \infty$$

$$\therefore \quad \sigma = \frac{306.25 \times 2.47}{1 \times \frac{7^3}{3}}\, y \quad \text{or,} \quad \sigma = 6.616\, y$$

The stresses can be computed as shown in Table 6.7. The net bending moment diagram is shown in Fig. 6.19d.

TABLE 6. 7 Net moments at A, B, C and D

Point	y	M_S	$\frac{M_x}{I_x}.y = M_i$	$M = M_S - M_i$ kNm
A	0	0	0	0
B	4.950	0	32.70	32.70
C	2.475	87.5	16.35	71.15
D	0	0	0	0

PROBLEMS

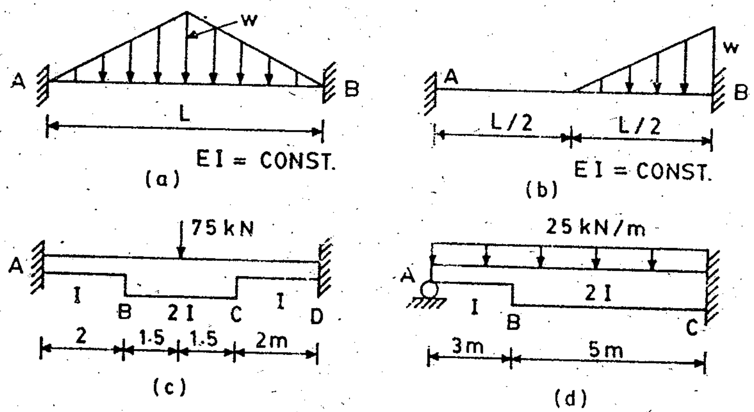
6.1 Determine the fixed end moments for the beams shown in Figs. P 6.1 a - d by treating the beams as (i) simply supported, and (ii) cantilever,

(P6.1a) : $M_A = -\frac{5}{96} wL^2$, $\quad M_B = \frac{5}{96} wL^2$

(P6.1b) : $M_A = -0.00725\, wL^2$, $\quad M_B = 0.024\, wL^2$

(P6.1c) : $M_A = -55.40$ kNm, $\quad M_D = 55.40$ kNm

(P6.1d) : $R_A = 58.39$ kN ↑, $\quad M_C = 332.88$ kNm

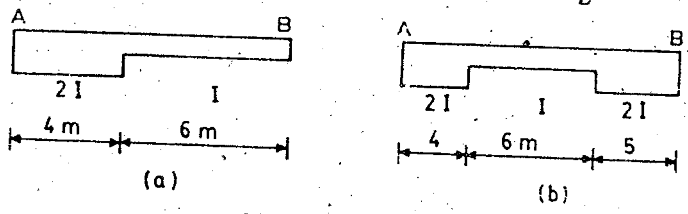


Figs. P 6.1

6.2 Determine the stiffness and carry over factors for beam elements with variable moment of inertia shown in Figs. P 6.2 a and b.

(P6.2a) : $K_A = 6.93\frac{EI}{L}$, $\quad CO_{AB} = 0.42$, $\quad K_B = 4.35\frac{EI}{L}$, $\quad CO_{BA} = 0.68$

(P6.2b) : $K_A = 6.9\frac{EI}{L}$, $\quad CO_{AB} = 0.61$, $\quad K_B = 7.26\frac{EI}{L}$, $\quad CO_{BA} = 0.58$



Figs. P 6.2

6.3 Analyze the portal frames shown in Figs. P 6.3a - d. Draw bending moment and shear force diagrams by treating:

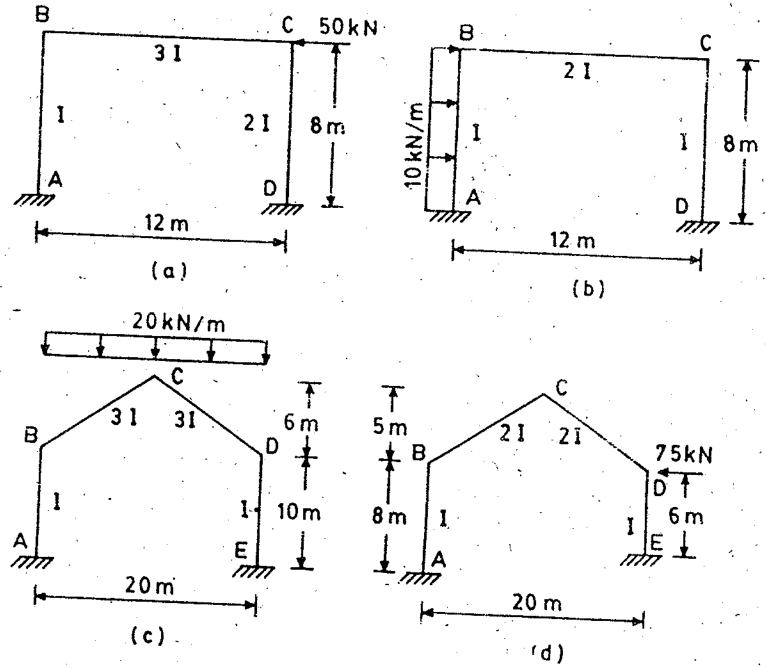
(a) the frames as simply supported,

(b) by removing support D in frames in Figs. P6.3a and P6.3b,

(c) by removing support A in frame in Fig. P6.3d.

(P6.3a) $R_{AX} = 19.80$ kN→, $\quad R_{AY} = 14.58$ kN↑, $M_A = 83.33$ kNm,
$M_D = 141.67$ kNm

(P6.3b) $R_{DX} = 17$ kN ←, $\quad R_{DY} = 7.9$ kN↑, $\quad M_D = -77.94$ kNm,
$M_A = -147.28$ kNm



Figs. P 6.3

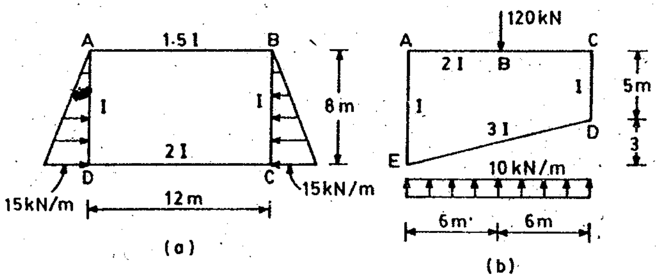
(P6.3c) R_{AX} = 61.53 kN→, R_{AY} = 200 kN↑, M_A = 258.88 kNm,
M_{CB} = −272.4 kNm

(P6.3d) R_{AX} = 17 kN→, R_{AY} = 9.48 kN↑, M_A = 87.41 kNm,
M_{DE} = 141 kNm, M_E = 207 kNm

6.4 Analyze the closed frames with one axis of symmetry shown in Figs. P 6.4a - b. Draw bending moment, shear force and thrust diagrams
(a) by introducing a cut at the midspan of top girder of each frame
(b) by introducing a hinge at the corners of each frame.

(P6.4a) *Thurst in AB = − 29.95 kN (compressive), Moment at the midspan of AB = 15.80 kNm (tension outside),* M_{CB} = −63.8 kNm

(P6.4b) *c.g. from E = (5.458 m, 5.14 m),* I_X = 170.9, I_Y = 576.75,
I_{XY} = 62.72 *units*
M_A = −28.86 kNm, M_B = 329.46 kNm, M_C = − 32.22 kNm,
M_D = −21.66 kNm, M_E = − 12.14 kNm

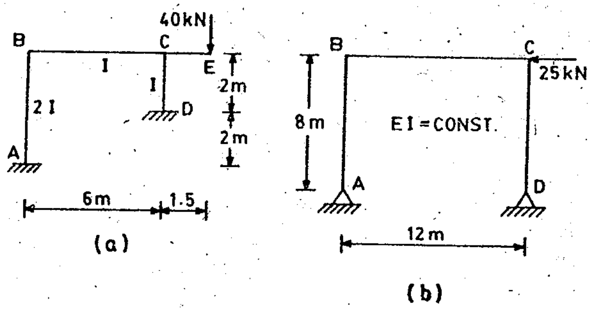


Figs. P 6.4

6.5 Analyze the frames shown in Figs. P 6.5 a and b and draw bending moment and shear force diagrams.

(P6.5a) M_A = −21.78 kNm, M_{BC} = 17.21 kNm, M_{CD} = 30.54 kNm,
M_D = −11.08 kNm

(P6.5b) R_{AX} = 12.5 kN→, R_{AY} = 16.67 kN↑, M_{BA} = 100 kNm,
M_{CB}= −100 kNm



Figs. P6.5

CHAPTER
seven

INFLUENCE COEFFICIENT METHOD

7.1 INTRODUCTION

The use of flexibility matrix provides a systematic method for the analysis of a structure having a high degree of static indeterminacy. The matrix approach is more convenient for use on computers. There are two flexibility matrix approaches in vogue:

(i) system approach
(ii) member approach

In the system approach, the flexibility matrix is developed directly for the structure as a whole, which is also known as the *influence coefficient method.* This method is good for small structures only. In the member approach, the flexibility matrix of the structure is developed through the flexibility matrices of the constituent elements of the structure and using the transformation matrices. The member approach is more amenable for a computer than the system approach, and can be used for solution of very large structures.

In this chapter only the system approach is discussed. The basic steps of the influence coefficient method are the same as discussed for the flexibility method in Chapter 1.

Each displacement component can be determined by using the Castigliano's second theorem or the unit load method. Both these methods involve complicated integrations and are prone to errors. There is a graphical method which can be conveniently used to integrate expressions to determine the displacements. Once the various displacements are known, a compatibility equation must be written for each redundant in the structure. These equations will be in terms of all the redundants. This implies that the action of one redundant will influence the displacement associated with the compatibility equation of another redundant. Thus, the redundants can be determined by the solution of all the linear simultaneous compatibility equations.

Instead of applying each of the redundant loads, a unit load is applied in the direction of each redundant. The displacements due to the unit loads are the influence coefficients. The actual displacement due to a redundant R is simply R times the

displacement due to the unit load. The compatibility equation (1.5 as derived in Chapter 1) can be reproduced as :

$$-\{\Delta_{ro}\} + [F]\{R\} = \{\Delta_r\} \qquad (7.1)$$

or,

$$-\mathbf{\Delta}_{ro} + \mathbf{F}\,\mathbf{R} = \mathbf{\Delta}_r$$

7.2 SIGN CONVENTION

In the influence coefficient method, right hand side axes system is used as shown in Figure 7.1. The y_3 – axis is aligned with the longitudinal axis of the member, and the y_1 – axis is perpendicular to the plane of the paper. It may be either upward or downward depending upon the direction of y_2 – axis. The right hand side rule may be stated as follows :

Fingers of the right hand are pointed towards positive y_2 - axis and are curled towards positive y_3–axis. The direction of the thumb gives the positive direction of y_1–axis.

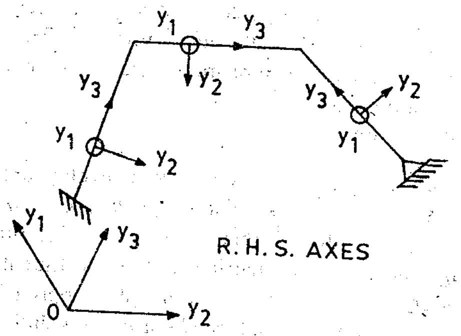


Fig. 7.1 Right hand side axes system for a member

Figure 7.2 a shows the positive direction of y_1 – axis according to the right hand side axes system. The force functions are represented by x_i, where the subscript i may be 1, 2 or 3, that is,

1 = bending moment,
2 = shear force, and
3 = axial force.

Bending moment x_1 is positive if it causes tension on the positive y_2 side; shear force x_2 is positive if the positive face of an element moves along the positive y_2 axis and the negative side moves along the negative y_2 axis; the axial force x_3 is positive if it causes tension in the element. The positive forcing functions are shown in Figs. 7.2b, c and d.

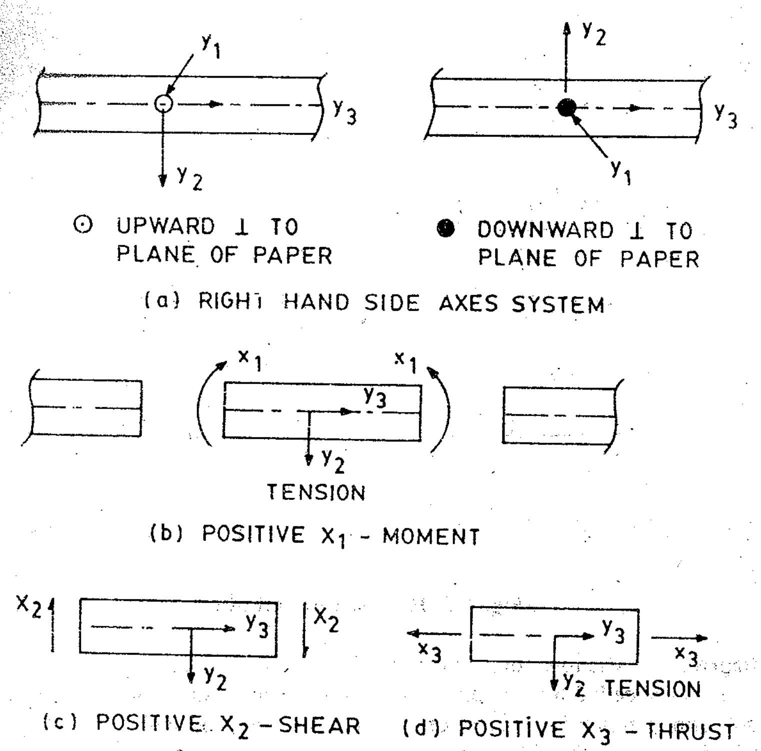


Fig. 7.2 Sign convention for force functions

A statically indeterminate structure is made statically determinate by introducing one or more releases as shown in Fig. 7.3. Corresponding to a given release, appropriate bi-actions or pair of redundants p_i are introduced as shown in the same figure. The subscript i represents bending moment, shear force or axial force, as before.

7.3 FORCE DIAGRAMS

Once a structure is made statically determinate, the next step is to determine displacements or discontinuities at a release in the direction of the redundant or bi-actions due to applied loads as well as redundant or bi- actions. Discontinuities can be evaluated if the various force diagrams are available. These are classified as :

(i) x – diagrams
(ii) h – diagrams

x – diagrams : The force diagrams in a released structure due to applied loading are called x_{io} diagrams, where the subscript i may be 1, 2 or 3. The subscripts 1, 2 and 3 represent bending moment, shear force and axial force; and subscript o represents external loading. If shear or axial deformation is ignored, the corresponding force diagram need not be drawn.

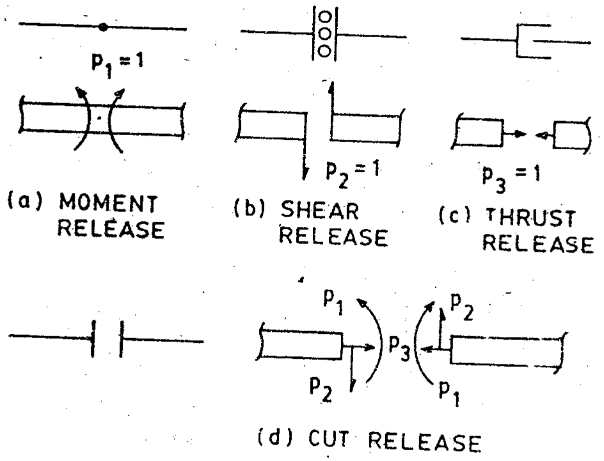


Fig. 7.3 Release discontinuities

h – diagrams : The force diagrams in a released structure due to unit redundant or unit bi – actions are called h or h_{ij} diagrams, that is force i caused due to unit redundant j. The subscript i represents the force function and may be equal to 1, 2, or 3. The subscript j represents the redundant.

Total displacements or discontinuities due to applied loads in the released structure corresponding to bi – actions can be calculated as :

$$\Delta_{jo} = \int \frac{h_{ij}\, x_{io}\, dy_3}{EI} \tag{7.2}$$

Total discontinuity due to bi – actions in the released structure corresponding to bi – actions can be calculated as :

$$f_{jk} = \int \frac{h_{ij}\, h_{ik}\, dy_3}{EI} \tag{7.3}$$

The flexibility matrix [F] is a square matrix consisting of influence coefficient elements, f_{jk}

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots\cdots & f_{1\alpha} \\ f_{21} & f_{22} & f_{23} & \cdots\cdots & f_{2\alpha} \\ \vdots & & & \cdots\cdots & \\ f_{\alpha 1} & f_{\alpha 2} & f_{\alpha 3} & \cdots\cdots & f_{\alpha\alpha} \end{bmatrix}_{\alpha_s \times \alpha_s} \tag{7.4}$$

where α_s = degree of static indeterminacy.

The influence coefficient due to the combined effect of flexure, shear and axial deformations can be easily determined as follows :

$$f_{jk} = \int \frac{h_{1j}\, h_{1k}\, dy_3}{EI} + \int \frac{h_{2j}\, h_{2k}\, dy_3}{A_e G} + \int \frac{h_{3j}\, h_{3k}\, dy_3}{AE} \tag{7.5}$$

Once the bi – actions or redundants are determined using Eq. 7.1, the net force {x} at any section in the structure can be computed using the equation :

$$\{x\} = \{x_o\} + \{H\}^T \{p\} \tag{7.6a}$$

or,

$$\mathbf{x} = \mathbf{x_o} + \mathbf{H^T p} \tag{7.6b}$$

where $\{x_o\}$ = force values at the section under consideration due to the external loads

$\{H\}^T$ = h_{ij} values at the section under consideration due to unit bi – action

$\{p\}$ = bi – actions

7.4 GRAPHICAL METHOD OF INTEGRATION

The deflection due to unit load method can be written as :

$$\Delta = \int \frac{Mm\, ds}{EI} \tag{7.7}$$

where, M = bending moment due to applied loads

m = bending moment due to unit load

Mds represents a small element of the M-diagram and m represents the corresponding ordinate of the m – diagram. Hence, displacement Δ can be rewritten as :

$$\Delta = \frac{\text{Area of the M – diagram} \times \text{Ordinate of the m – diagram at the c.g. of the M – diagram}}{EI} \tag{7.8}$$

Alternatively,

$$\Delta = \frac{\text{Area of the m – diagram} \times \text{Ordinate of the M – diagram at the c.g. of the m – diagram}}{EI} \tag{7.9}$$

In case a structural member is non-prismatic, it should be divided into a sufficient number of parts so that the M – or m – diagram is continuous and E I is constant in each part. It should be noted that M need not be a bending moment diagram. It may be any forcing function diagram such as a axial force or shear force. Table 7.1 gives the values of Eq.7.8 or 7.9 for common shapes of force functions. This table can be directly used for computing displacements once the diagrams due to various force functions are drawn for a given structure.

Thus, Eqs.7.3 and 7.5 can be evaluated directly using the Table 7.1.

Table 7.1 Evaluation of volume integrals $\int_0^L m_i\, m_j\, dx$

m_j \ m_i	e, L	e, L	e, L	e, f, L	e, g, f, L	Parabola e, L	Parabola e, L	Parabola e, L
a, L	Lae	$\frac{Lae}{2}$	$\frac{Lae}{2}$	$\frac{La(e+f)}{2}$	$\frac{Lag}{2}$	$\frac{2Lae}{3}$	$\frac{2Lae}{3}$	$\frac{Lae}{3}$
a, L	$\frac{Lae}{2}$	$\frac{Lae}{3}$	$\frac{Lae}{6}$	$\frac{La(2e+f)}{6}$	$\frac{ag(L+f)}{6}$	$\frac{Lae}{3}$	$\frac{Lae}{4}$	$\frac{Lae}{12}$
a, L	$\frac{Lae}{2}$	$\frac{Lae}{6}$	$\frac{Lae}{3}$	$\frac{La(e+2f)}{6}$	$\frac{ag(L+e)}{6}$	$\frac{Lae}{3}$	$\frac{5Lae}{12}$	$\frac{Lae}{4}$
a, b, L	$\frac{Le(a+b)}{2}$	$\frac{Le(b+2a)}{6}$	$\frac{Le(a+2b)}{6}$	$\frac{Lb(e+2f)}{6}+\frac{La(f+2e)}{6}$	$\frac{ag(L+f)}{6}+\frac{bg(L+e)}{6}$	$\frac{L(a+b)e}{3}$	$\frac{L(3b-a)e}{12}$	$\frac{L(a+3b)}{12}$
c, d, L	$\frac{Le(c+d)}{2}$	$\frac{Le(2c+d)}{6}$	$\frac{Le(c+2d)}{6}$	$\frac{Lc(2e+f)}{6}+\frac{Ld(e+2f)}{6}$	$\frac{cg(L+f)}{6}+\frac{dg(L+e)}{6}$	$\frac{L(c+d)e}{3}$	$\frac{L(3c+5d)e}{12}$	$\frac{L(c+3d)e}{12}$

7.5 ILLUSTRATIVE EXAMPLES

Example 7.1

Develop the influence coefficients for the beam shown in Fig.7.4a.

Solution

The beam is statically indeterminate to a degree 3. It can be made statically determinate by introducing a hinge at A and a roller at B. The axes and the three bi - actions p_1, p_2 and p_3 are shown in Fig. 7.4b. Let us draw moment (h_{1j}), shear (h_{2j}) and thrust (h_{3j}) diagrams due to each of the three redundants (j = 1, 2, 3) as shown in Figs. 7.4c, d and e. The influence coefficients, that is, discontinuities due to unit bi - actions can be evaluated using the graphical integration technique.

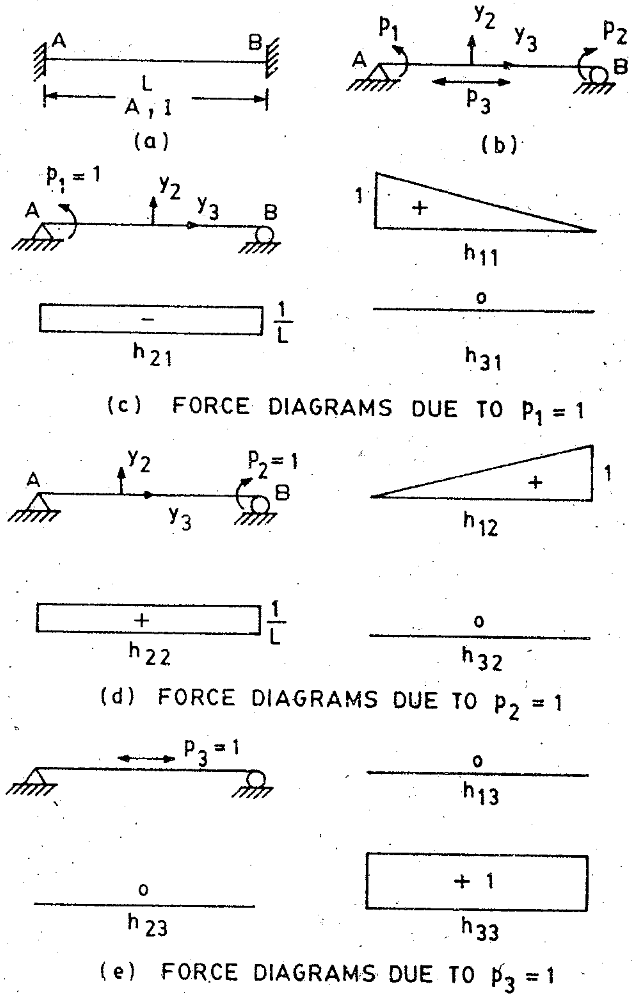


Fig. 7.4 Fixed beam

$$f_{jk} = \int \frac{h_{1j}\, h_{1k}\, dy_3}{EI} + \int \frac{h_{2j}\, h_{2k}\, dy_3}{A_e G} + \int \frac{h_{3j}\, h_{3k}\, dy_3}{AE}$$

$$f_{11} = \int \frac{h_{11}\, h_{11}\, dy_3}{EI} + \int \frac{h_{21}\, h_{21}\, dy_3}{A_e G} + \int \frac{h_{31}\, h_{31}\, dy_3}{AE}$$

$$= \frac{1 \times 1 \times L}{3EI} + \frac{\frac{1}{L} \times L \times \frac{1}{L}}{A_e G} + 0 = \frac{L}{3EI} + \frac{1}{LA_e G}$$

$$f_{12} = \int \frac{h_{11}\, h_{12}\, dy_3}{EI} + \int \frac{h_{21}\, h_{22}\, dy_3}{A_e G} + \int \frac{h_{31}\, h_{32}\, dy_3}{AE}$$

$$= \frac{1 \times 1 \times L}{6EI} - \frac{\frac{1}{L} \times L \times \frac{1}{L}}{A_e G} + 0 = \frac{L}{6EI} - \frac{1}{LA_e G}$$

$$= f_{21}$$

$$f_{13} = \int \frac{h_{11}\, h_{13}\, dy_3}{EI} + \int \frac{h_{21}\, h_{23} dy_3}{A_e G} + \int \frac{h_{31}\, h_{33}\, dy_3}{AE}$$

$$= 0 = f_{31}$$

$$f_{22} = \int \frac{h_{12}\, h_{12}\, dy_3}{EI} + \int \frac{h_{22}\, h_{22}\, dy_3}{A_e G} + \int \frac{h_{32}\, h_{32}\, dy_3}{AE}$$

$$= \frac{1 \times 1 \times L}{3EI} + \frac{\frac{1}{L} \times L \times \frac{1}{L}}{A_e G} + 0 = \frac{L}{3EI} + \frac{1}{LA_e G}$$

$$f_{23} = \int \frac{h_{12}\, h_{13}\, dy_3}{EI} + \int \frac{h_{22}\, h_{23}\, dy_3}{A_e G} + \int \frac{h_{32}\, h_{33}\, dy_3}{AE}$$

$$= 0 = f_{32}$$

$$f_{33} = \int \frac{h_{13}\, h_{13}\, dy_3}{EI} + \int \frac{h_{23}\, h_{23}\, dy_3}{A_e G} + \int \frac{h_{33}\, h_{33}\, dy_3}{AE}$$

$$= 0 + 0 + \frac{1 \times 1 \times L}{AE} = \frac{L}{AE}$$

The influence coefficient matrix may now be written as:

$$F = \begin{bmatrix} \frac{L}{3EI} + \frac{1}{LA_e G} & \frac{L}{6EI} - \frac{1}{LA_e G} & 0 \\ \frac{L}{6EI} - \frac{1}{LA_e G} & \frac{L}{3EI} + \frac{1}{LA_e G} & 0 \\ 0 & 0 & \frac{L}{AE} \end{bmatrix} \qquad \text{(i)}$$

This is always a symmetric matrix. If shear deformation is ignored, the h_{2j} diagrams become zero, and the influence coefficient matrix can be written as:

$$F = \begin{bmatrix} \frac{L}{3EI} & \frac{L}{6EI} & 0 \\ \frac{L}{6EI} & \frac{L}{3EI} & 0 \\ 0 & 0 & \frac{L}{AE} \end{bmatrix} \qquad \text{(ii)}$$

If axial deformation is also ignored, the influence coefficient matrix can be written as:

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{(iii)}$$

Alternative Solution

The beam can be made statically determinate by removing the support B, that is, introducing a cut at B. The axes and the three bi-actions are shown in Fig. 7.5b. Let us again draw moment (h'_{1j}), shear (h_{2j}) and thrust (h_{3j}) diagrams due to each of the three redundants ($j = 1, 2, 3$) as shown in Figs. 7.5 c, d and e. The influence coefficients are :

$$f_{11} = \frac{1 \times 1 \times L}{EI} = \frac{L}{EI}$$

$$f_{12} = -\frac{1 \times 1 \times L}{2EI} = -\frac{L}{2EI} = f_{21}$$

$$f_{13} = 0 = f_{31}$$

$$f_{22} = \frac{1 \times 1 \times L}{3EI} + \frac{1 \times 1 \times L}{A_e G} + 0 = \frac{L}{3EI} + \frac{L}{A_e G}$$

$$f_{23} = 0 = f_{32}$$

$$f_{32} = \frac{L}{AE}$$

The influence coefficient matrix may now be written as:

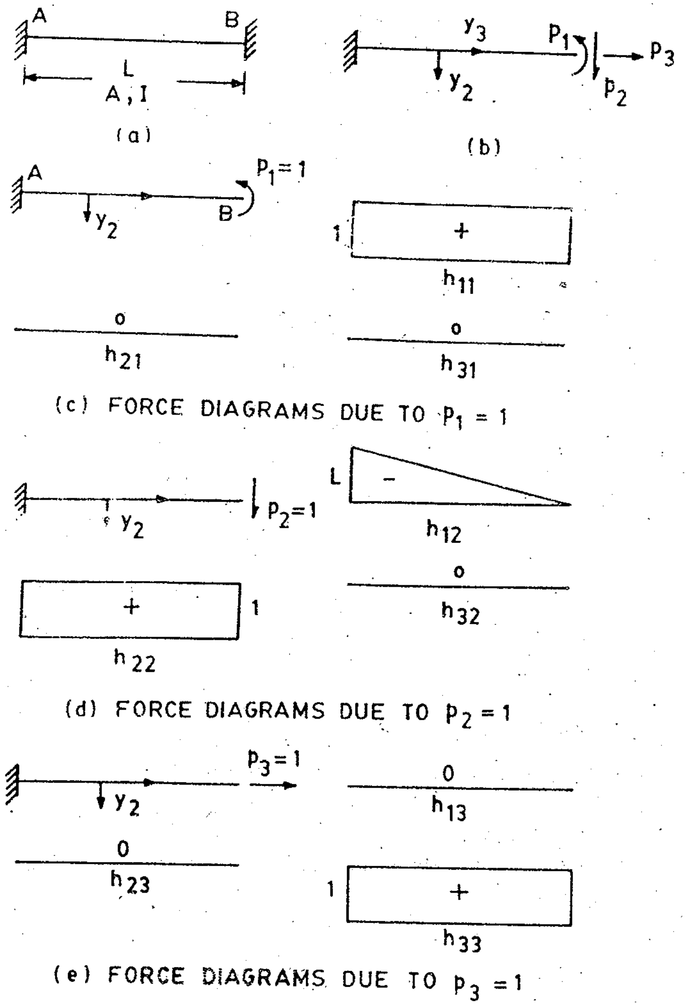


Fig. 7.5 Fixed beam - alternative solution

$$F = \begin{bmatrix} \dfrac{L}{EI} & -\dfrac{L}{2EI} & 0 \\ -\dfrac{L}{2EI} & \dfrac{L}{3EI}+\dfrac{L}{A_eG} & 0 \\ 0 & 0 & \dfrac{L}{AE} \end{bmatrix} \qquad \text{(iv)}$$

If shear deformation is ignored, the influence coefficient matrix becomes:

$$F = \begin{bmatrix} \dfrac{L}{EI} & -\dfrac{L}{2EI} & 0 \\ -\dfrac{L}{2EI} & \dfrac{L}{3EI} & 0 \\ 0 & 0 & \dfrac{L}{AE} \end{bmatrix} \qquad \text{(v)}$$

If axial deformation is also ignored, the influence coefficient matrix becomes:

$$F = \frac{L}{6EI}\begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix} \qquad \text{(vi)}$$

It may be seen that the influence coefficient matrices obtained by using the two alternative solutions are different. The stress resultant however, always remain the same.

Example 7.2

Determine the flexibility influence coefficients due to the combined effect of bending, shear and thrust deformations in the frame shown in Fig. 7.6a.
Take $A_eG = 50\,EI$, $AE = 100\,EI$

Solution

The frame is indeterminate to a degree 3. A cut release is introduced at the support D to make it statically determinate and stable. The axis system and the unit bi-actions p_1 to p_3 are as shown in Fig. 7.6 c. Since it is desired to include the effect of shear and axial deformations, all the three force diagrams are drawn due to each unit bi-action as shown in Figs. 7.6 d to f.

The influence coefficients can be evaluated as follows :

$$f_{11} = \int \frac{h_{11}\,h_{11}\,dy_3}{EI} + \int \frac{h_{21}\,h_{21}\,dy_3}{A_eG} + \int \frac{h_{31}\,h_{31}\,dy_3}{AE}$$

$$= \frac{1}{EI}\left[1\times 4 + 1\times\frac{6}{2} + 1\times 4\right] + \frac{1}{50EI}[0] + \frac{1}{100EI}[0] = \frac{11}{EI}$$

$$f_{12} = \int \frac{h_{11}\,h_{12}\,dy_3}{EI} + \int \frac{h_{21}\,h_{22}\,dy_3}{A_eG} + \int \frac{h_{31}\,h_{32}\,dy_3}{AE}$$

$$= (-)\frac{1}{EI}\left[\frac{1}{2}\times 4\times 4\times 2 + \frac{4\times 6}{2}\right] + 0 + 0 = -\frac{28}{EI}$$

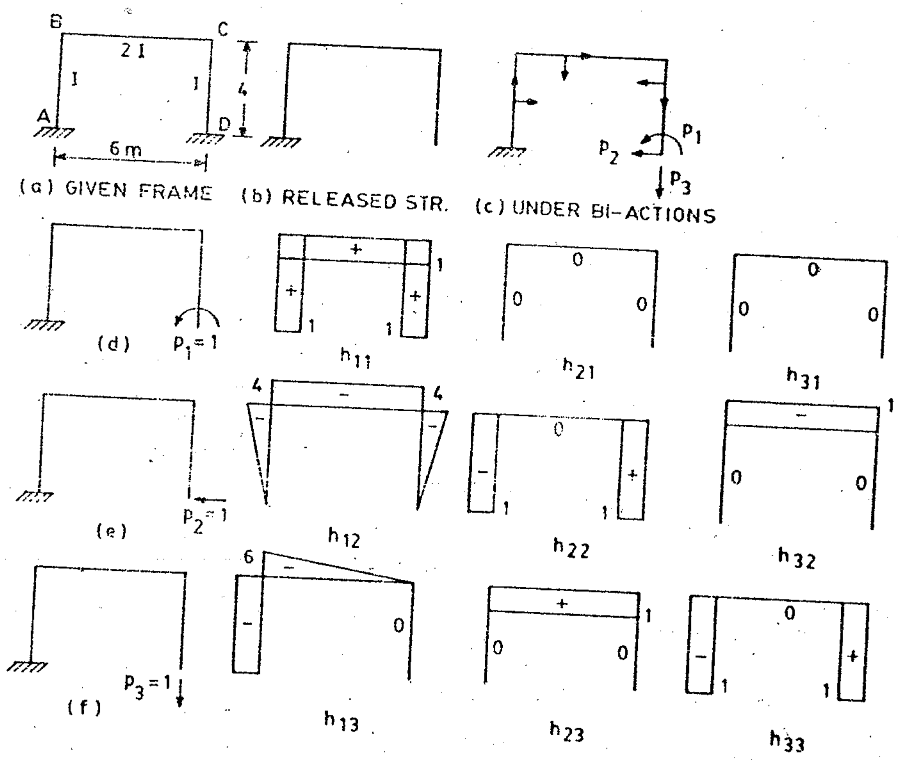


Fig. 7.6 Fixed end portal frame - force diagrams

$$f_{13} = \int \frac{h_{11}\, h_{13}\, dy_3}{EI} + \int \frac{h_{21}\, h_{23}\, dy_3}{A_e G} + \int \frac{h_{31}\, h_{33}\, dy_3}{AE}$$

$$= -\frac{1}{EI}\left[6 \times 4 + \frac{1}{2} \times 6 \times \frac{6}{2}\right] + 0 + 0 = -\frac{1}{EI}(33)$$

$$f_{21} = f_{12}$$

$$f_{22} = \int \frac{h_{12}\, h_{12}\, dy_3}{EI} + \int \frac{h_{22}\, h_{22}\, dy_3}{A_e G} + \int \frac{h_{32}\, h_{32}\, dy_3}{AE}$$

$$= \frac{1}{EI}\left[\frac{1}{2} \times 4 \times 4 \times \frac{2}{3} \times 4 \times 2 + \frac{4 \times 6}{2} \times 4\right] + \frac{1}{50EI}[1 \times 4 + 1 \times 4] + \frac{1}{100EI}\left[\frac{1 \times 6}{2}\right]$$

$$= \frac{1}{EI} \times \left[\frac{128}{3} + 48\right] + \frac{8}{50EI} + \frac{3}{100EI} = \frac{90.86}{EI}$$

For member BC, $A_e G = 50E\,(2I) = 100\,E\,I$ and $AE = 100\,E\,(2I) = 200\,E\,I$

$$f_{23} = \int \frac{h_{12}\, h_{13}\, dy_3}{EI} + \int \frac{h_{22}\, h_{23}\, dy_3}{A_e G} + \int \frac{h_{32}\, h_{33}\, dy_3}{AE}$$

$$= \frac{1}{EI}\left[-\frac{1}{2} \times 4 \times 4 \times (-6) + \frac{1}{2}(-6) \times 6 \times \frac{(-4)}{2}\right] + 0 + 0 = \frac{84}{EI}$$

$$f_{31} = f_{13}$$

$$f_{32} = f_{23}$$

$$f_{33} = \int \frac{h_{13}\, h_{13}\, dy_3}{EI} + \int \frac{h_{23}\, h_{23}\, dy_3}{A_e G} + \int \frac{h_{33}\, h_{33}\, dy_3}{AE}$$

$$= \frac{1}{EI}\left[6 \times 4 \times 6 + \frac{1}{2} \times 6 \times 6 \times \frac{2}{3} \times 6 \times \frac{1}{2}\right] + \frac{1}{50EI}\left[\frac{1 \times 6}{2}\right] + \frac{1}{100EI}[1 \times 4 + 1 \times 4]$$

$$= \frac{1}{EI}[180] + \frac{3}{50EI} + \frac{8}{100EI} = \frac{180.14}{EI}$$

The influence coefficient matrix F is

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ & f_{22} & f_{23} \\ \text{Symm} & & f_{33} \end{bmatrix}$$

$$F = \frac{1}{EI}\begin{bmatrix} 11 & -28 & -33 \\ & 90.86 & 84 \\ \$ & & 180.14 \end{bmatrix}$$ [$ represents symmetry]

If shear and axial deformations are ignored, the h_{2j} and h_{3j} stress diagrams will be zero. Hence, the second and third terms in each of the f_{ij} expressions would be zero. The influence coefficient matrix F becomes

$$\mathbf{F} = \frac{1}{EI}\begin{bmatrix} 11 & -28 & -33 \\ & 90.67 & 84 \\ \$ & & 180 \end{bmatrix}$$

Alternative Solution I

The frame can be made statically determinate and stable by introducing a cut in beam as shown in Fig. 7.7 b. The axes system and bi-actions p_1 to p_3 are also shown in the same figure. Let us draw moment (h_{1j}), shear (h_{2j}) and thrust (h_{3j}) diagrams due to each of the three unit bi-actions (j = 1, 2 and 3) as shown in Figs. 7.7 c, d and e. The influence coefficients can be evaluated as follows:

$$f_{11} = \frac{1}{EI}\left[1\times4+\frac{1\times6}{2}+1\times4\right]+0+0=\frac{11}{EI}$$

$$f_{12} = 0 = f_{21}$$

$$f_{13} = -\frac{1}{EI}\left[\frac{4\times4\times2}{2}\right]=-\frac{16}{EI} = f_{31}$$

$$f_{22} = \frac{1}{EI}\left[3\times4\times3+\frac{3\times3\times3}{3}\times2+3\times4\times3\right]+\frac{1}{A_e G}\left[\frac{1\times6}{2}\right]+\frac{1}{AE}[1\times4\times2]$$

$$= \frac{1}{EI}[90]+\frac{6}{100EI}+\frac{8}{100EI}=\frac{90.14}{EI}$$

$$f_{23} = 0 = f_{32}$$

$$f_{33} = \frac{1}{EI}\left[\frac{4\times4\times4}{3}\times2\right]+\frac{1}{A_e G}[1\times4\times2]+\frac{1}{AE}\left[\frac{1\times6}{2}\right]$$

$$= \frac{128}{3EI}+\frac{8}{50EI}+\frac{6}{200EI}=\frac{42.86}{EI}$$

The influence coefficient matrix can be written as:

$$\mathbf{F} = \frac{1}{EI}\begin{bmatrix} 11 & 0 & -16 \\ & 90.14 & 0 \\ \$ & & 42.86 \end{bmatrix}$$

If shear and axial deformations are ignored, the influence coefficient matrix can be written as:

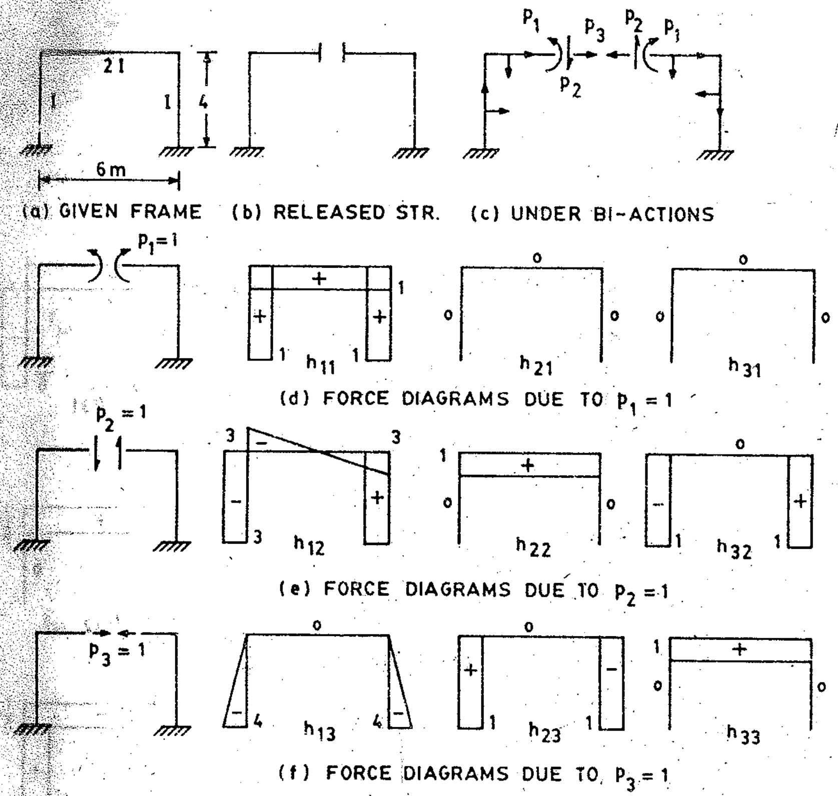


Fig. 7.7 Fixed end portal frame - alternative solution I

$$\mathbf{F} = \frac{1}{EI}\begin{bmatrix} 11 & 0 & -16 \\ & 90 & 0 \\ \$ & & 42.67 \end{bmatrix}$$

Alternative Solution II

The portal frame can be made statically determinate and stable by introducing three hinges, one at each support and one in the beam BC. The axes system and bi-actions

p_1, p_2 and p_3 are shown in Fig.7.8c. Let us draw the moment, shear and thrust diagrams due to unit bi-actions as shown in Figs. 7.8 d, e and f. The influence coefficients can be evaluated as follows :

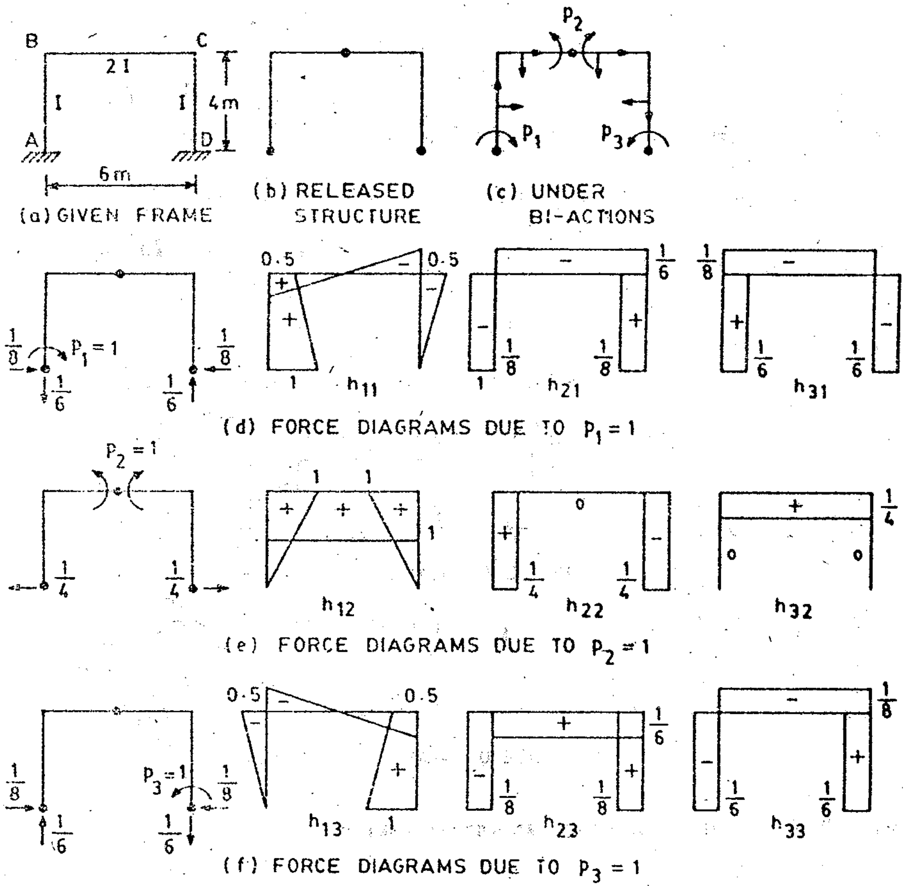


Fig. 7.8 Fixed end portal frame - alternative solution II

$$f_{11} = \frac{1}{EI}\left[\frac{4}{6}(2\times1+2\times0.5+0.5+0.5)+0.5\times\frac{0.5\times3\times2}{2\times3}+\frac{0.5\times0.5\times4}{3}\right]$$

$$+\frac{1}{A_eG}\left[\frac{1}{8}\times\frac{1}{8}\times4\times2+\frac{1}{6}\times\frac{1}{6}\times\frac{6}{2}\right]+\frac{1}{AE}\left[\frac{1}{6}\times\frac{1}{6}\times4\times2+\frac{1}{8}\times\frac{1}{8}\times\frac{6}{2}\right]$$

$$= \frac{3.25}{EI}+\frac{0.208}{A_eG}+\frac{0.269}{AE} = \frac{3.25}{EI}+\frac{0.208}{50EI}+\frac{0.269}{100EI}=\frac{3.727}{EI}$$

$$f_{12} = \frac{1}{EI}\left[\frac{1\times4}{6}(2\times0.5+1)-0.5\times1\times\frac{4}{3}\right]+\frac{1}{A_eG}\left[-\frac{1}{8}\times\frac{1}{4}\times4\times2\right]+$$

$$\frac{1}{AE}\left[-\frac{1}{8}\times\frac{1}{4}\times\frac{6}{2}\right]$$

$$= \frac{2}{EI}-\frac{0.25}{A_eG}-\frac{0.09375}{AE} = \frac{2}{EI}-\frac{0.25}{50EI}-\frac{0.09375}{100EI}=\frac{1.9856}{EI} = f_{21}$$

$$f_{13} = \frac{1}{EI}\left[-\frac{0.5\times4}{6}(2\times0.5+1)\times2-0.5\times0.5\times\frac{3}{3}\times\frac{2}{2}\right]+$$

$$\frac{1}{A_eG}\left[\frac{1}{8}\times\frac{1}{8}\times4\times2+\frac{1}{6}\times\frac{1}{6}\times\frac{6}{2}\right]+\frac{1}{AE}\left[-\frac{1}{6}\times\frac{1}{6}\times4\times2+\frac{1}{8}\times\frac{1}{8}\times\frac{6}{2}\right]$$

$$= -\frac{1.583}{EI}+\frac{0.208}{A_eG}-\frac{0.175}{AE}=-\frac{1.580}{EI} = f_{31}$$

$$f_{22} = \frac{1}{EI}\left[1\times1\times\frac{4}{3}\times2+\frac{1\times6}{2}\right]+\frac{1}{A_eG}\left[\frac{1}{4}\times\frac{1}{4}\times4\times2\right]+\frac{1}{AE}\left[\frac{1}{4}\times\frac{1}{4}\times\frac{6}{2}\right]$$

$$= \frac{5.667}{EI}+\frac{0.5}{A_eG}+\frac{0.187}{AE}=\frac{5.679}{EI}$$

$$f_{23} = \frac{1}{EI}\left[-0.5\times\frac{4}{3}+\frac{1\times4}{6}(2\times0.5+1)\right]+\frac{1}{A_eG}\left[-\frac{1}{4}\times\frac{1}{8}\times4\times2\right]+$$

$$\frac{1}{AE}\left[-\frac{1}{4}\times\frac{1}{8}\times\frac{6}{2}\right]$$

$$= \frac{0.667}{EI}-\frac{0.25}{A_eG}-\frac{0.0937}{AE}=\frac{0.661}{EI}$$

$$= f_{32}$$

$$f_{33} = f_{11} \text{ by symmetry}$$

The influence coefficient matrix may now be written as:

$$\mathbf{F} = \frac{1}{EI}\begin{bmatrix} 3.727 & 1.986 & -1.580 \\ & 5.679 & 0.661 \\ \$ & & 3.727 \end{bmatrix}$$

If shear and axial deformations are ignored, the F matrix becomes:

$$\mathbf{F} = \frac{1}{EI}\begin{bmatrix} 3.25 & 2 & -1.583 \\ & 5.667 & 0.667 \\ \$ & & 3.250 \end{bmatrix}$$

Thus, it can be inferred that the influence coefficient matrix depends on the choice of releases and axes system. However, the final solution of any structure will remain unchanged.

Example 7.3

A joint is suspended by three springs as shown in Fig. 7.9a. It is subjected to a vertical load of W. Determine the forces in the springs.

Solution

Three springs are meeting at joint O. Hence, the joint O is statically indeterminate by degree 1. Let us introduce a thrust release in the member BO. The released joint and axes system are shown in Fig. 7.9b. From geometry,

$$\sin\alpha = \frac{L_2}{L_1}, \quad \cos\alpha = \frac{a}{L_1}$$

$$\sin\beta = \frac{L_2}{L_3}, \quad \cos\beta = \frac{b}{L_3}$$

Let us apply load W at joint O in the released structure. The axial force diagram x_{30} due to external load W is shown in Fig. 7.9c.

For equilibrium

$$P_{10}\cos\alpha = P_{30}\cos\beta$$

and,

$$P_{10}\sin\alpha + P_{30}\sin\beta = W$$

or,

$$P_{10} = \frac{W\cos\beta}{\sin(\alpha+\beta)} \quad \text{tension}$$

$$P_{30} = \frac{W\cos\alpha}{\sin(\alpha+\beta)} \quad \text{tension}$$

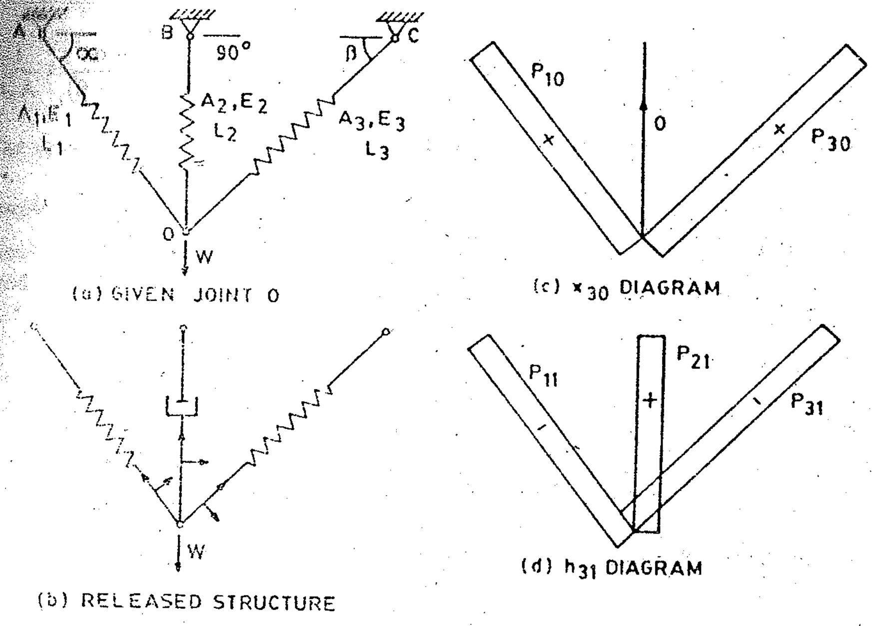


Fig. 7.9 Indeterminate joint

Let us write equilibrium equations under the application of a unit bi-action $p_1 = 1$ applied at O along OB,

$$P_{11}\cos\alpha = P_{31}\cos\beta$$

and,

$$P_{11}\sin\alpha + P_{31}\sin\beta = 1$$

or,

$$P_{11} = \frac{\cos\beta}{\sin(\alpha+\beta)} \quad \text{compression}$$

$$P_{31} = \frac{\cos\alpha}{\sin(\alpha+\beta)} \quad \text{compression}$$

and

$$P_{21} = 1$$

The axial force diagram h_{31} due to unit bi-action is shown in Fig. 7.9d. The subscript 3 indicates axial force and subscript 1 indicates the number of redundancy under consideration.

Δ_{10} = displacement at O in the direction of the release due to external load

$$= \int \frac{h_{31}\, x_{30}}{AE}\, dy_3$$

$$= -\left[\frac{W\cos\beta}{\sin(\alpha+\beta)} \times \frac{\cos\beta}{\sin(\alpha+\beta)} \times \frac{L_1}{A_1E_1} + 0 + \frac{W\cos\alpha}{\sin(\alpha+\beta)} \times \frac{\cos\alpha}{\sin(\alpha+\beta)} \times \frac{L_3}{A_3E_3}\right]$$

using the geometrical integration technique .

$$\text{or, } \Delta_{10} = \left[\frac{W\cos^2\beta}{\sin^2(\alpha+\beta)}\frac{L_1}{A_1E_1} + \frac{W\cos^2\alpha}{\sin^2(\alpha+\beta)}\frac{L_3}{A_3E_3}\right](-) \qquad \text{(i)}$$

f_{11} = Influence coefficient due to a unit bi-action

$$f_{11} = \int \frac{h_{31}\, h_{31}}{AE}\, dy_3$$

$$= \left[\frac{\cos^2\beta}{\sin^2(\alpha+\beta)}\frac{L_1}{A_1E_1} + \frac{L_2}{A_2E_2} + \frac{\cos^2\alpha}{\sin^2(\alpha+\beta)}\frac{L_3}{A_3E_3}\right] \qquad \text{(ii)}$$

Compatibility equation gives, $\Delta_{10} + f_{11}\, p_1 = 0$

$$\text{or,} \qquad p_1 = -\frac{\Delta_{10}}{F_{11}}$$

The redundant force in member OB is given by

$$p_1 = \frac{\left[\dfrac{W\cos^2\beta}{\sin^2(\alpha+\beta)}\dfrac{L_1}{A_1E_1} + \dfrac{W\cos^2\alpha}{\sin^2(\alpha+\beta)}\dfrac{L_3}{A_3E_3}\right]}{\left[\dfrac{\cos^2\beta}{\sin^2(\alpha+\beta)}\dfrac{L_1}{A_1E_1} + \dfrac{L_2}{A_2E_2} + \dfrac{\cos^2\alpha}{\sin^2(\alpha+\beta)}\dfrac{L_3}{A_3E_3}\right]} \qquad \text{(iii)}$$

Member forces

Force in member AO, T_1 is given by

$$T_1 = P_{10} + P_{11}\, p_1$$

$$= \frac{W\cos\beta}{\sin(\alpha+\beta)} - \frac{\cos\beta}{\sin(\alpha+\beta)}\, p_1 \qquad \text{(iv)}$$

Force in member BO, $T_2 = 0 + P_{21}\, p_1 = p_1$ (v)

Force in member CO, T_3 is given by

$$T_3 = P_{30} + P_{31}\, p_1$$

$$= \frac{W\cos\alpha}{\sin(\alpha+\beta)} - \frac{\cos\alpha}{\sin(\alpha+\beta)}\, p_1 \qquad \text{(vi)}$$

The value of p_1 from Eq.(iii) can be substituted in Eqs. (iv), (v) and (vi).

Example 7.4

Analyze a two span beam ABC shown in Fig. 7.10a using the influence coefficient method.

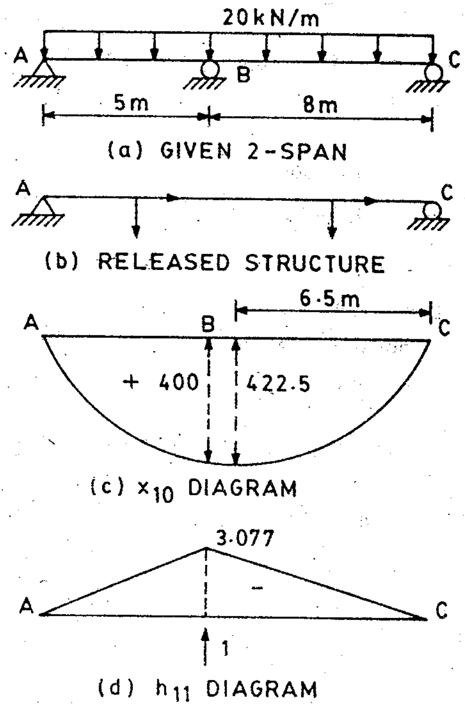


Fig. 7.10

Solution

The two span beam is statically indeterminate by degree 1. It can be made statically determinate by introducing a thrust release at B, that is, removing the support B. The x_{10} diagram is a parabola as shown in Fig. 7.10c. The h_{10} diagram due to a unit bi-action at

B is triangular as shown in Fig. 7.10d. The discontinuities at the release due to the applied load and the redundant bi-action can be evaluated as follows:

$$\Delta_{10} = \int \frac{h_{11}\, x_{10}}{EI}\, dy_3$$

ordinate of the x_{10} diagram. at $\quad B = 130 \times 5 - 20 \times \dfrac{5^2}{2} = 400$

Area ADB $= \dfrac{20 \times 5^2}{12}(3 \times 13 - 2 \times 5) = 1208.34$

c.g. from A $= \dfrac{2.5(4 \times 13 - 3 \times 5)}{(3 \times 13 - 2 \times 5)} = 3.19$ m

Ordinate of the h_{11} diagram at the c.g. of the segment ADB

$= (-)\ 1.963$

Area ADC $= \dfrac{2}{3} \times 422.5 \times 13 = 3661.667$

$\therefore$ Area BDC $= 3661.667 - 1208.34 = 2453.327$

c.g. of area BDC from A

$$= \frac{3661.667 \times 6.5 - 1208.34 \times 3.19}{3661.667 - 1208.34} = 8.13 \text{ m}$$

$\therefore$ c.g. of area BDC from C $= 13 - 8.13 = 4.87$ m

Ordinate of the h_{11} diagram at the c.g. of the segment BDC = (–) 1.873

$$\therefore \quad \Delta_{10} = -[1208.34 \times 1.963 + 2453.327 \times 1.873]\frac{1}{EI} = -\frac{6967.36}{EI}$$

$$f_{11} = \int \frac{h_{11} h_{11}}{EI}\, dy_3$$

$$= \frac{1}{EI}\left[\frac{1}{2} \times 5 \times 3.077 \times \frac{2}{3} \times 3.077 + \frac{1}{2} \times 8 \times 3.077 \times \frac{2}{3} \times 3.077\right] = \frac{41.028}{EI}$$

For compatibility $\quad \Delta_{10} + f_{11}\, R_B = 0$

or, $\quad -6967.36 + 41.028\, R_B = 0 \quad$ or, $\quad R_B = 169.8$ kN

This is the same reaction as obtained by the method of consistent deformation in Example 3.2.

Example 7.5

Analyze the continuous beam supported on three elastic springs as shown in Fig. 7.11a using the influence coefficient method.

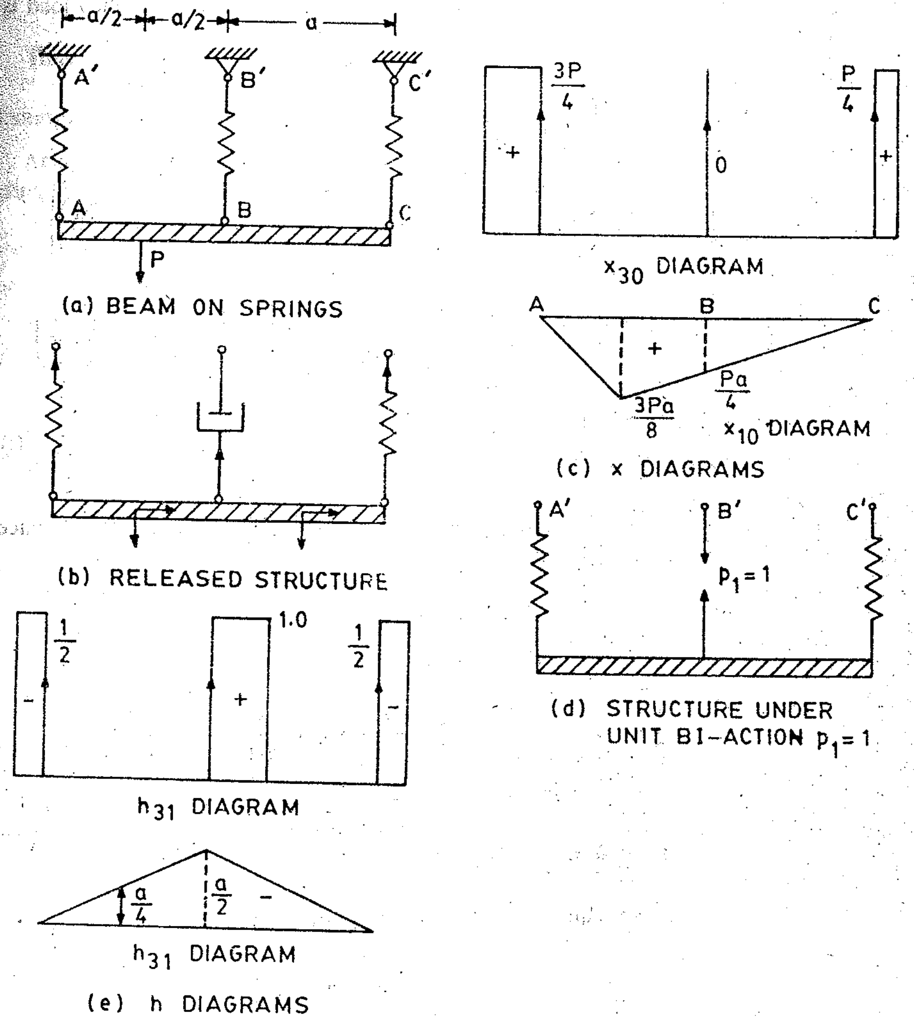


Fig. 7.11

Solution

The beam is statically indeterminate by degree 1. A thrust release is introduced in the middle spring to make it a statically determinate structure as shown in Fig. 7.11b. The springs AA', BB' and CC' carry only axial forces, while the beam ABC carry bending

moment. The bending moment x_{10} and axial force x_{30} diagrams due to the external load are shown in Fig. 7.11c. The structure under the unit bi-actions p_1 is shown in Fig. 7.11d. The bending moment h_{11} and axial force h_{31} diagrams due to the unit bi-actions $p_1 = 1$ are shown in Fig. 7.11e.

Discontinuity at the release in the direction of the release due to the applied load is determined as follows:

$$\Delta_{10} = \int \frac{h_{11}x_{10}}{EI}\,dy_3 + \int \frac{h_{31}x_{30}}{AE}\,dy_3$$

$$= \frac{1}{EI}\left[\frac{1}{2}\times\frac{a}{2}\times\left(-\frac{a}{4}\right)\times\frac{2}{3}\times\frac{3Pa}{8}+\frac{(-)a}{12}\left\{\frac{3}{8}Pa\left(\frac{2a}{4}+\frac{a}{2}\right)+\frac{Pa}{4}\left(\frac{a}{4}+\frac{2a}{2}\right)\right\}\right.$$

$$\left.+\frac{1}{2}\times a\times\left(-\frac{a}{2}\right)\times\frac{2}{3}\times\frac{Pa}{4}\right]+\frac{3P}{4}\times\frac{L}{A_1E_1}\times\left(-\frac{1}{2}\right)+\frac{P}{4}\times\frac{L}{A_3E_3}\times\left(-\frac{1}{2}\right)$$

$$= -\left[\frac{11}{96}\frac{Pa^3}{EI}+\frac{3}{8}\frac{PL}{A_1E_1}+\frac{1}{8}\frac{PL}{A_3E_3}\right] \qquad \text{(i)}$$

Discontinuity due to the bi-actions in the direction of the bi-actions or the influence coefficient is determined as follows:

$$f_{11} = \int \frac{h_{11}h_{11}}{EI}\,dy_3 + \int \frac{h_{31}h_{31}}{AE}\,dy_3$$

$$= \frac{2}{EI}\left(\frac{1}{2}\times a\times\frac{a}{2}\times\frac{2}{3}\times\frac{a}{2}\right)+\frac{1}{2}\times\frac{L}{A_1E_1}\times\frac{1}{2}+\frac{L}{A_2E_2}+\frac{1}{2}\times\frac{L}{A_3E_3}\times\frac{1}{2}$$

or,

$$f_{11} = \frac{a^3}{6EI}+\frac{1}{4}\frac{L}{A_1E_1}+\frac{L}{A_2E_2}+\frac{L}{4A_3E_3} \qquad \text{(ii)}$$

The compatibility condition requires $\Delta_{10} + f_{11}\,p_1 = 0$

or,

$$p_1 = \frac{\dfrac{11}{96}\dfrac{Pa^3}{EI}+\dfrac{3}{8}\dfrac{PL}{A_1E_1}+\dfrac{1}{8}\dfrac{PL}{A_3E_3}}{\dfrac{a^3}{6EI}+\dfrac{1}{4}\dfrac{L}{A_1E_1}+\dfrac{L}{A_2E_2}+\dfrac{1}{4}\dfrac{L}{A_3E_3}} \qquad \text{(iii)}$$

$= R_2$, reaction in the middle spring

The reactions in the outer two springs can be determined using the equilibrium equations:

$$\Sigma F_y = 0,\quad R_1 + R_2 + R_3 = P \qquad \text{(iv)}$$

and,

$$\Sigma M_c = 0,\quad R_1 \times 2a + R_2 \times a - P \times 1.5a = 0$$

or,

$$2R_1 + R_2 - 1.5P = 0 \qquad \text{(v)}$$

Thus forces in various members of the structure are known.

Example 7.6

Analyze the beam shown in Fig. 7.12a. due to settlement of support B by 10 mm using the influence coefficient method.

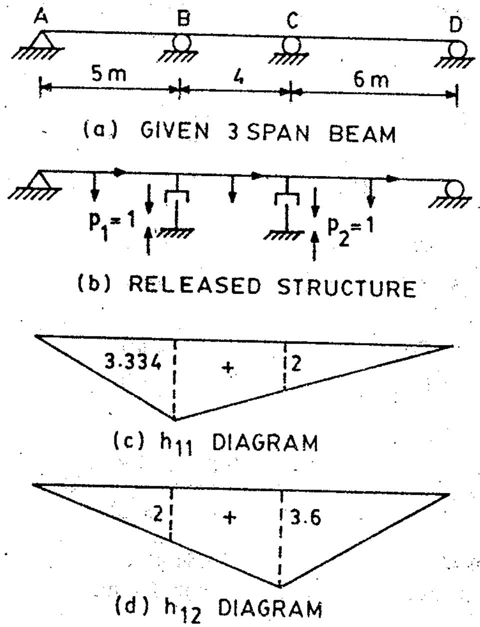


Fig. 7.12

Solution

The continuous beam is statically indeterminate to degree 2. Let us introduce thrust releases at supports B and C. The net vertical deflection at B is 10 mm and at C is zero. Since there are no external loads, x_{10} diagram is zero. The force diagrams due to the bi-actions at B and C are shown in Figs. 7.12c and d. Let us compute the influence coefficients.

$$f_{11} = \int \frac{h_{11}h_{11}}{EI}\, dy_3 \;=\; \frac{1}{3} \times 3.334 \times 15 \times \frac{3.334}{EI} = \frac{55.577}{EI}$$

$$f_{12} = \int \frac{h_{11}h_{12}}{EI}\, dy_3 \;=\; \frac{1}{EI}\left[\frac{1}{2} \times 5 \times 3.334 \times 1.334 + \frac{4}{6}\left[3.334(4+3.6) + 2(2+7.2)\right] + \frac{1}{3} \times 6 \times 2 \times 3.6\right]$$

$$= \frac{54.69}{EI}$$

It is important to note that the geometric integration of h_{11} over h_{12} should be done in three parts.

$$f_{22} = \int \frac{h_{12}h_{12}}{EI}\, dy_3 \;=\; \left[\frac{1}{3} \times 3.6 \times 15 \times 3.6\right]\frac{1}{EI} = \frac{64.8}{EI}$$

Compatibility requires, $\quad [F]\,\{p\} + \{\Delta_0\} = \{\Delta\}$

or, $$\frac{1}{EI}\begin{bmatrix} 55.577 & 54.69 \\ 54.69 & 64.8 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} + 0 = \begin{Bmatrix} 0.01 \\ 0 \end{Bmatrix}$$

or, $$\begin{bmatrix} 55.577 & 54.69 \\ 54.69 & 64.8 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 400 \\ 0 \end{Bmatrix}$$

or, $$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 42.46 \\ -35.84 \end{Bmatrix} \text{kN}$$

Thus reaction R_B is 42.46 kN ↑ and R_C is 35.84 kN↓. This is the same result as obtained by the method of consistent deformation in Example 3.10.

Example 7.7

Analyze the portal frame shown in Fig. 7.13 a using the influence coefficient method. Ignore axial and shear deformations.

Solution

The frame is statically indeterminate to a degree 2. It can be made statically determinate by removing the hinge support A. The released frame and the axes system are shown in Fig. 7.13b. The moment diagram x_{10} due to the applied load is shown in Fig. 7.13c and those due to the bi-actions are shown in Figs. 7.13d and e. The discontinuity due to the applied loads can be calculated as follows:

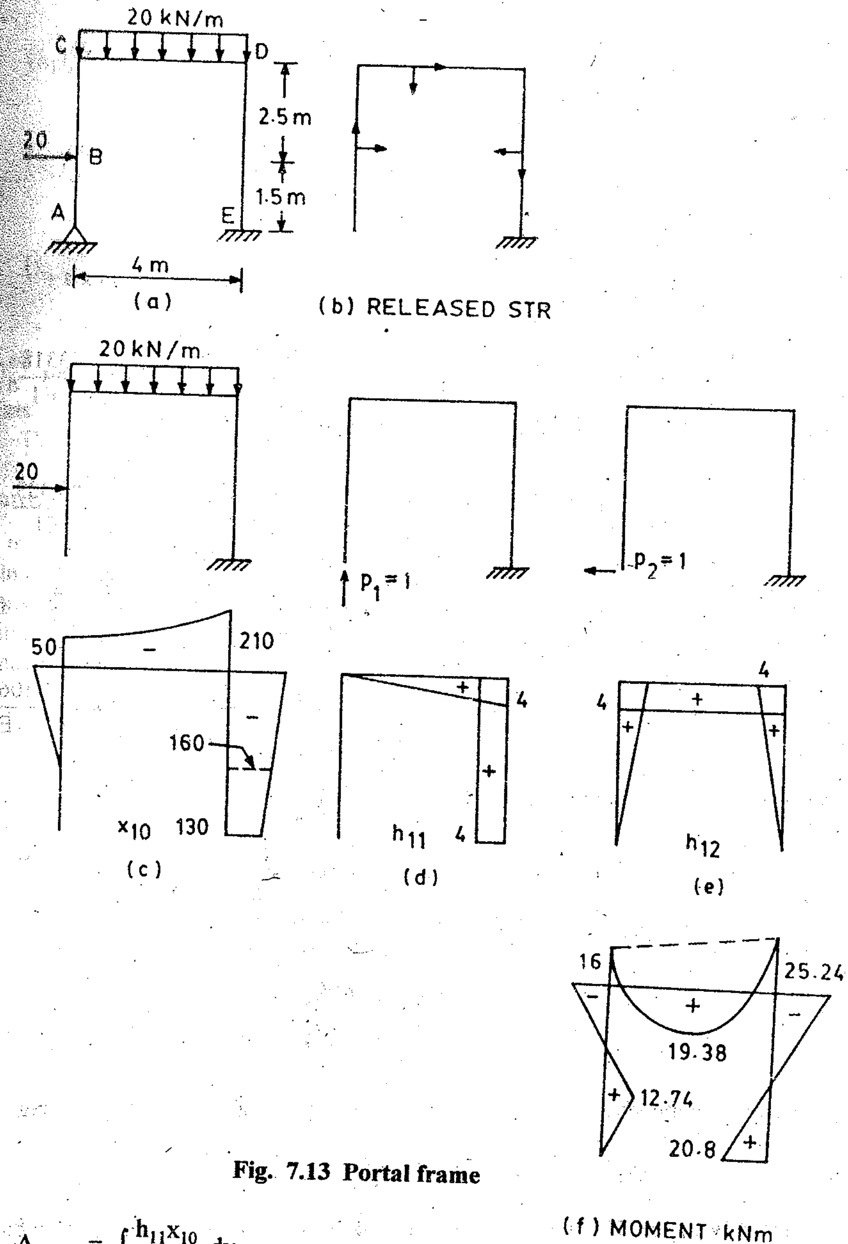


Fig. 7.13 Portal frame

$$\Delta_{10} = \int \frac{h_{11}x_{10}}{EI}\, dy_3$$

$$= -\frac{1}{EI}\left[\frac{1}{2} \times 4 \times 4 \times 50 + \frac{1}{3} \times 160 \times 4 \times 3 + \left(\frac{210+130}{2}\right) \times 4 \times 4\right]$$

$$= -\frac{3760}{EI}$$

$$\Delta_{20} = \int \frac{h_{12}x_{10}}{EI} dy_3$$

$$= -\frac{1}{EI}\left[\frac{1}{2}\times 2.5\times 50\times 3.167+50\times 4\times 4+\frac{1}{3}\times 160\times 4\times 4+\right.$$

$$\left. 130\times 4\times 2+\frac{1}{2}\times 80\times 4\times \frac{2}{3}\times 4\right] = -\frac{3318}{EI}$$

The influence coefficients can be calculated as follows :

$$f_{11} = \int \frac{h_{11}h_{11}}{EI} dy_3 = \frac{1}{EI}\left[\frac{1}{2}\times 4\times 4\times \frac{2}{3}\times 4+4\times 4\times 4\right] = \frac{85.333}{EI}$$

$$f_{12} = \int \frac{h_{11}h_{12}}{EI} dy_3 = \frac{1}{EI}\left[\frac{1}{2}\times 4\times 4\times 4+\frac{1}{2}\times 4\times 4\times 4\right] = \frac{64}{EI} = f_{21}$$

$$f_{22} = \int \frac{h_{12}h_{12}}{EI} dy_3 = \frac{1}{EI}\left[2\times \frac{1}{2}\times 4\times 4\times \frac{2}{3}\times 4+4\times 4\times 4\right] = \frac{106.67}{EI}$$

The compatibility condition requires, $\Delta_o + \mathbf{F}\,\mathbf{p} = \mathbf{0}$

or,

$$\begin{Bmatrix} -3760 \\ -3318 \end{Bmatrix} + \begin{bmatrix} 85.33 & 64 \\ 64 & 106.67 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \mathbf{0}$$

or,

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 37.70 \\ 8.49 \end{Bmatrix} \text{ kN}$$

The stress components can be evaluated at different sections i of the frame using

$$(x_1)_i = (x_0)_i + \mathbf{H}^T \mathbf{p}$$

The row vector $\mathbf{H}^T$ gives the h_{ij} values at the section under consideration due to the unit bi-actions.

$$(x_1)_B^{AC} = 0 + \{0 \quad 1.5\}\begin{Bmatrix} 37.7 \\ 8.49 \end{Bmatrix} = 12.74 \text{ kNm}$$

$$(x_1)_C^{AC} = -50 + \{0 \quad 4\}\begin{Bmatrix} 37.7 \\ 8.49 \end{Bmatrix} = -16 \text{ kNm}$$

$$(x_1)_D^{CD} = -210 + \{4 \quad 4\}\begin{Bmatrix} 37.7 \\ 8.49 \end{Bmatrix} = -25.24 \text{ kNm}$$

$$(x_1)_E^{DE} = -130 + \{4 \quad 0\}\begin{Bmatrix} 37.7 \\ 8.49 \end{Bmatrix} = 20.8 \text{ kNm}$$

$(x_1)_B^{AC}$ represents the value of the stress component x_1 at B in the span AC. $(x_o)_i$ represents the corresponding stress component in the released structures at the same section. The resulting bending moment diagram is shown in Fig. 7.13 f.

Alternatively

The frame can be made statically determinate by introducing a roller at support E as shown in Fig. 7.14a. The moment diagrams x_0 and h are shown in Figs. 7.14b, c and d. The moment diagram x_{10} on the beam CD can be split in a triangle and parabola as shown in Fig. 7.14e for ease in computations. Discontinuities due to the applied loads can be computed as follows:

$$\Delta_{10} = \int \frac{h_{11}x_{10}}{EI} dy_3$$

$$= \frac{1}{EI}\left[\frac{1}{2}\times 30\times 4\times \frac{1}{3}+\frac{2}{3}\times 40\times 4\times \frac{1}{2}\right] = \frac{73.34}{EI}$$

$$\Delta_{20} = \int \frac{h_{12}x_{10}}{EI} dy_3$$

$$= \frac{(-1)}{EI}\left[\frac{1}{2}\times 30\times 1.5\times \frac{2}{3}\times 1.5+30\times 2.5\times 2.75+\frac{1}{2}\times 30\times 4\times 4+\right.$$

$$\left.\frac{2}{3}\times 40\times 4\times 4\right] = -\frac{895.42}{EI}$$

The influence coefficients can be evaluated as follows:

$$f_{11} = \int \frac{h_{11}h_{11}}{EI} dy_3 = \frac{1}{EI}\left[\frac{1}{2}\times 1\times 4\times \frac{2}{3}+1\times 4\times 1\right] = \frac{5.33}{EI}$$

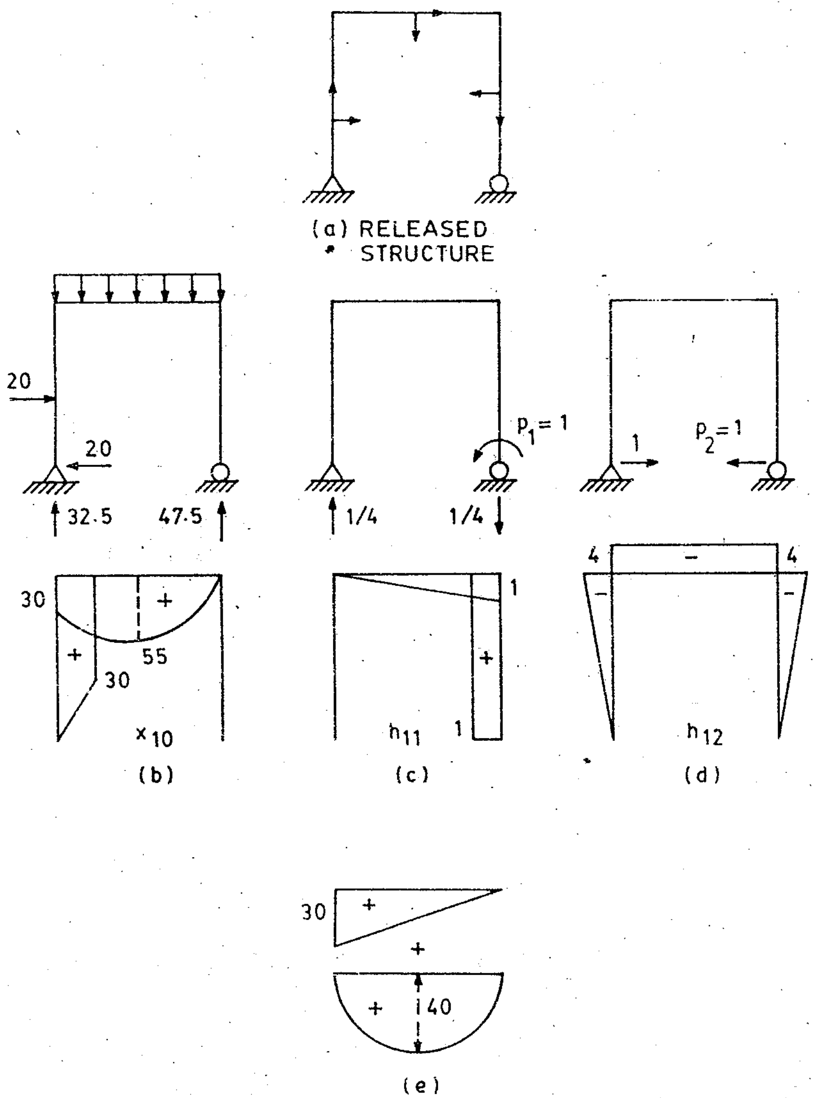


Fig. 7.14 Portal frame - alternative solution

$$f_{12} = \int \frac{h_{11}\, h_{12}}{EI}\, dy_3 = \frac{(-1)}{EI}\left[\frac{1}{2}\times 1\times 4\times 4+\frac{1}{2}\times 4\times 4\times 1\right] = -\frac{16}{EI}$$

$$= f_{21}$$

$$f_{22} = \int \frac{h_{12}\, h_{12}}{EI}\, dy_3 = \frac{106.67}{EI}$$

Compatibility conditions requires that: $\Delta_o + \mathbf{F}\ \mathbf{p} = \mathbf{0}$

or,
$$\begin{Bmatrix} 73.34 \\ 895.42 \end{Bmatrix} + \begin{bmatrix} 5.33 & -16 \\ -16 & 106.67 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = 0$$

or,
$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 20.80 \\ 11.50 \end{Bmatrix} \begin{matrix} \text{kNm} \\ \text{kN} \end{matrix}$$

The stress resultants at different sections can be calculated using

$$(x_1)_i = (x_0)_i + H_i^{\,T}\, p$$

Thus,
$$(x_1)_C^{AC} = 30 + \{0 \quad -4\}\begin{Bmatrix} 20.80 \\ 11.50 \end{Bmatrix} = -16 \text{ kNm}$$

$$(x_1)_D^{CD} = 0 + \{1 \quad -4\}\begin{Bmatrix} 20.80 \\ 11.50 \end{Bmatrix} = -25.20 \text{ kNm}$$

$$(x_1)^{CD}_{\text{midspan}} = 55 + \{0.5 \quad -4\}\begin{Bmatrix} 20.80 \\ 11.50 \end{Bmatrix} = 19.40 \text{ kNm}$$

Example 7.8

Analyze the continuous beam shown in Fig. 7.15a using the influence coefficient method. The flexibility of each spring is $f_0 = L^3/EI$, Take EI = constant.

Solution

The structure is statically indeterminate to a degree 2. The released structure is shown in Fig. 7.15b The force diagrams x_{10} and x_{30} due to the applied loads are shown in Fig. 7.15c, and those due to the bi-actions p_1 and p_2 are shown in Figs. 7.15d and e. It should be noted that the beam ABCD carries bending moment where as the springs BE and CF carry axial thrust. The discontinuities can be evaluated as follows:

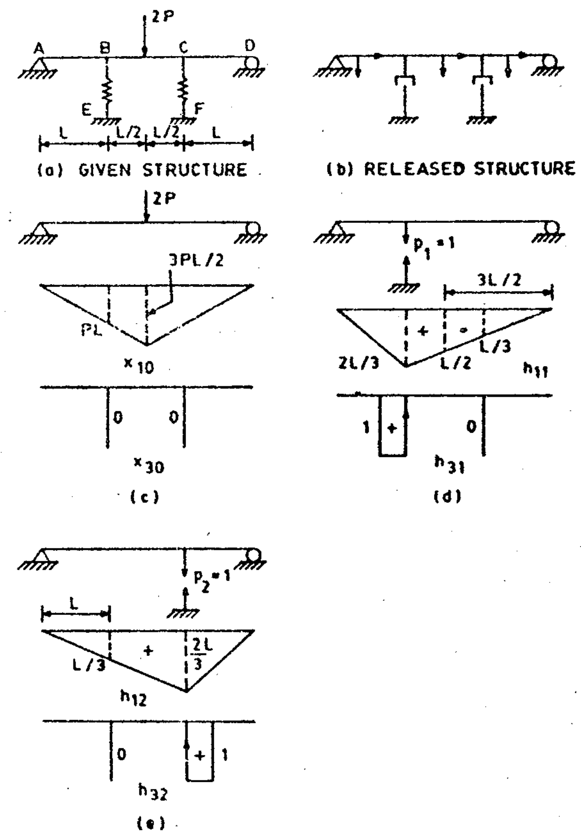


Fig. 7.15

$$\Delta_{10} = \int \frac{h_{11}\, x_{10}}{EI}\, dy_3 + h_{31}\, f'_3\, x_{30}$$

$$= \frac{1}{3EI} \times L \times PL \times \frac{2L}{3} + \frac{L}{12EI}\left[PL\left(\frac{4}{3}L + \frac{L}{2}\right) + \frac{3PL}{2}\left(\frac{2}{3}L + L\right)\right] +$$

$$\frac{L}{2EI} \times \frac{3PL}{2} \times \frac{L}{2} + 0$$

$$= \frac{23}{24}\,\frac{PL^3}{EI}$$

$\Delta_{20} = \Delta_{10}$ due to symmetry in the structure.

$$f_{11} = \int \frac{h_{11}\, h_{11}}{EI}\, dy_3 + h_{31}\, f'_3\, h_{31}$$

$$= \frac{3L}{3EI} \times \frac{2}{3}L \times \frac{2}{3}L + 1 \times \frac{L^3}{EI} \times 1 = \frac{13}{9}\,\frac{L^3}{EI}$$

$f_{22} = f_{11}$ due to symmetry

$$f_{12} = \int \frac{h_{11}\, h_{12}}{EI}\, dy_3 + h_{31}\, f'_3\, h_{32} + h_{31}\, f''_3\, h_{32}$$

$$= \frac{2L}{3EI} \times \frac{2}{3}L \times \frac{L}{3} + \frac{L}{6EI}\left[\frac{2}{3}L\left(\frac{2L}{3} + \frac{2L}{3}\right) + \frac{L}{3}\left(\frac{L}{3} + \frac{4L}{3}\right)\right] + 0 + 0$$

$$= \frac{7}{18}\,\frac{L^3}{EI}$$

The compatibility condition requires $\mathbf{F\,p} + \Delta_0 = 0$

or,

$$\begin{bmatrix} \frac{13}{9} & \frac{7}{18} \\ \frac{7}{18} & \frac{13}{9} \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} + \begin{Bmatrix} \frac{23}{24} \\ \frac{23}{24} \end{Bmatrix} P = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or,

$$p_1 = -\frac{23}{44}P = p_2$$

Reaction $R_A = \frac{21}{44}P$ and hence, moment at $B = \frac{21}{44}PL$

Example 7.9

Analyze the portal frame with inclined legs as shown in Fig. 7.16a using the influence coefficient method. Neglect axial and shear deformations. Take EI = constant.

Solution

The frame is statically indeterminate to a degree 3. A cut is introduced at the midspan of BC as shown in Fig. 7.16b. The local axes system is also shown in the same figure. The bending moment diagrams x_{10} due to external load, and h_{11}, h_{12} and h_{13} due to unit bi-actions are shown in Figs. 7.16c to 7.16j.

Discontinuity due to the applied load at the point of cut off and along the directions of the bi-actions can be calculated as follows:

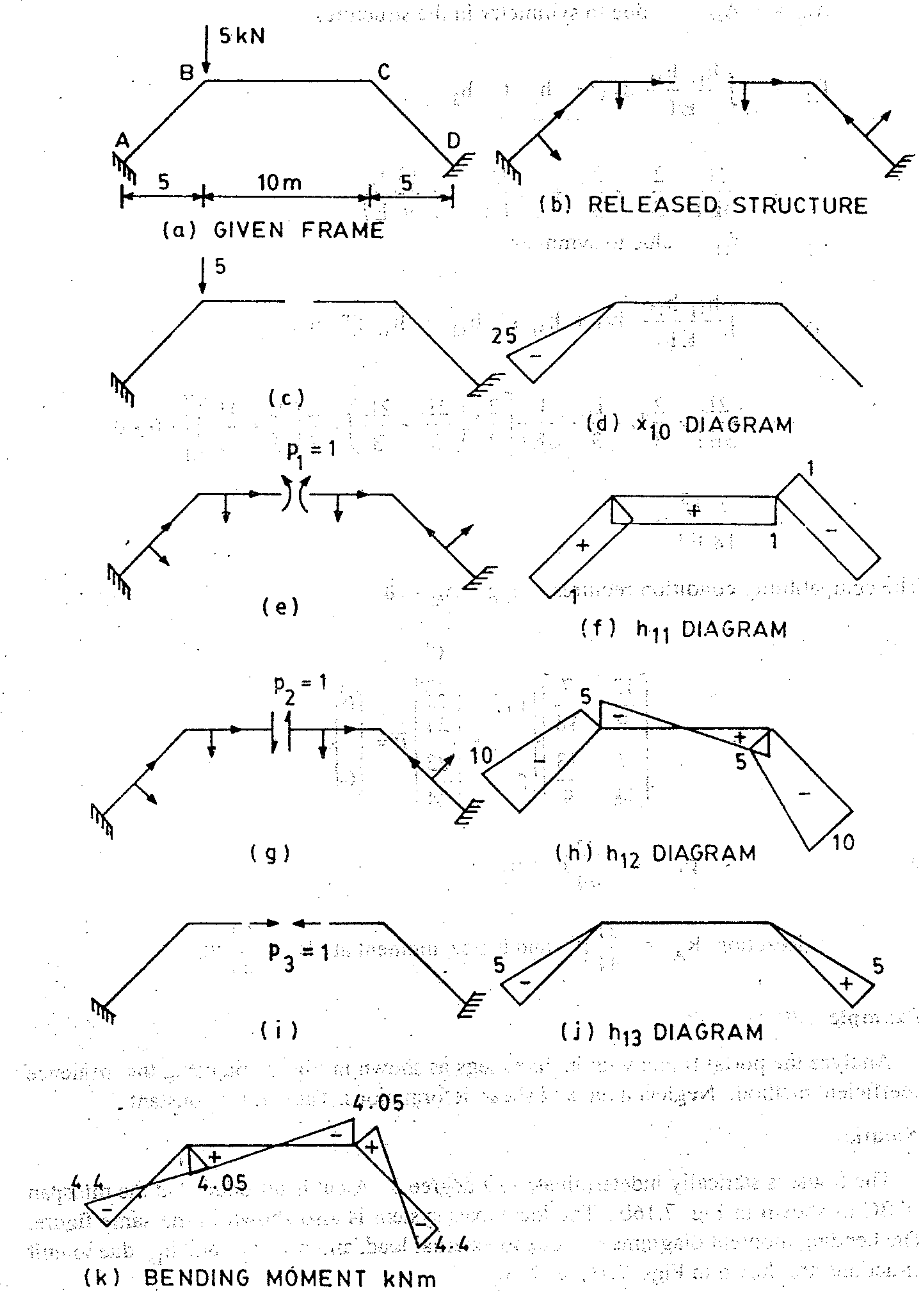


Fig. 7.16 Portal frame with inclined legs

$$\Delta_{10} = \int \frac{h_{11}\, x_{10}}{EI}\, dy_3 = \frac{-25\times 5\sqrt{2}\times 1}{2EI} = -\frac{88.4}{EI}$$

$$\Delta_{20} = \int \frac{h_{12}\, x_{10}}{EI}\, dy_3 = \frac{25\times 5\sqrt{2}}{2EI}\left(\frac{20}{3}+\frac{5}{3}\right) = \frac{736.5}{EI}$$

$$\Delta_{30} = \int \frac{h_{13}\, x_{10}}{EI}\, dy_3 = \frac{25\times 5\sqrt{2}}{2EI}\times\frac{10}{3} = \frac{294.6}{EI}$$

The influence coefficient or the discontinuities due to the bi-actions at the point of cut-off along the directions of the bi-actions can be calculated as follows:

$$f_{11} = \int \frac{h_{11}\, h_{11}}{EI}\, dy_3 = \left[5\sqrt{2}\times 1\times 1+10\times 1\times 1+5\sqrt{2}\times 1\times 1\right]\frac{1}{EI} = \frac{24.14}{EI}$$

$$f_{12} = \int \frac{h_{11}\, h_{12}}{EI}\, dy_3 = 0 = f_{21}$$

$$f_{13} = \int \frac{h_{11}\, h_{13}}{EI}\, dy_3 = \frac{2}{EI}\times\left(-\frac{5}{2}\times 5\sqrt{2}\times 1\right)+0 = -\frac{35.35}{EI}$$

$$f_{22} = \int \frac{h_{12}\, h_{12}}{EI}\, dy_3$$

$$= \frac{2}{EI}\left[10\times\frac{5\sqrt{2}}{2}\left(2\times\frac{10}{3}+\frac{5}{3}\right)+\frac{1}{2}\times 5\sqrt{2}\times 5\left(\frac{2}{3}\times 5+\frac{10}{3}\right)+\left(5\times\frac{5}{2}+5\times\frac{2}{3}\right)\right]$$

$$= \frac{908.3}{EI}$$

$$f_{23} = \int \frac{h_{12}\, h_{13}}{EI}\, dy_3 = 0 = f_{32}$$

$$f_{33} = \int \frac{h_{13}\, h_{13}}{EI}\, dy_3 = \frac{2}{EI}\left[\frac{5}{2}\times 5\sqrt{2}\times\frac{2}{3}\times 5\right]+0 = \frac{117.8}{EI}$$

Compatibility condition gives, $\Delta_0 + F\,p = 0$

or,
$$\begin{Bmatrix} -88.4 \\ 736.5 \\ 294.6 \end{Bmatrix} + \begin{bmatrix} 24.14 & 0 & -35.35 \\ 0 & 908.3 & 0 \\ -35.35 & 0 & 117.8 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} = 0$$

or,
$$p = \begin{Bmatrix} 0 \\ -0.811 \\ -2.50 \end{Bmatrix}$$

The stress components can be evaluated at different sections in the frame.

The bending moment x_1 at A in the member AB is given by

$$(x_1)_A^{AB} = (x_{10})_A^{AB} + \mathbf{H}^T \mathbf{p}$$

The row vector $\mathbf{H}^T$ gives the h_{ij} values at the sections under considerations due to the unit bi-actions.

$$(x_1)_A^{AB} = -25 + \{1 - 10 - 5\} \begin{Bmatrix} 0 \\ -0.811 \\ -2.50 \end{Bmatrix} = -4.40 \text{ kNm}$$

Similarly,

$$(x_1)_B^{AB} = 0 + \{1 - 5 \quad 0\} \begin{Bmatrix} 0 \\ -0.811 \\ -2.50 \end{Bmatrix} = 4.05 \text{ kNm}$$

$$(x_1)_B^{BC} = 0 + \{1 - 5 \quad 0\} \begin{Bmatrix} 0 \\ -0.811 \\ -2.50 \end{Bmatrix} = 4.05 \text{ kNm}$$

$$(x_1)_C^{BC} = 0 + \{1 \quad -5 \quad 0\} \begin{Bmatrix} 0 \\ -0.811 \\ -2.50 \end{Bmatrix} = -4.05 \text{ kNm}$$

$$(x_1)_C^{CD} = 0 + \{-1 \ -5 \quad 0\} \begin{Bmatrix} 0 \\ -0.811 \\ -2.50 \end{Bmatrix} = 4.05 \text{ kNm}$$

$$(x_1)_D^{CD} = 0 + \{-1 - 10 - 5\} \begin{Bmatrix} 0 \\ -0.811 \\ -2.50 \end{Bmatrix} = -4.40 \text{ kNm}$$

The resulting bending moment diagram is shown in Fig. 7.16 k.

PROBLEMS

7.1 Develop the influence coefficient matrix for the beam shown in Fig. P7.1 by
(i) introducing a thrust release at B and C, and
(ii) introducing a hinge at A and a thrust release at B.
Ignore axial and shear deformations.

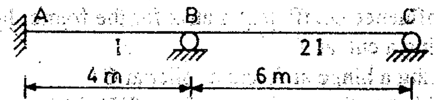


Fig. P7.1

7.2 Redo problem 7.1 by including the axial and shear deformations. Take $A_eG = 75$ EI, and AE = 100 EI.

7.3 Develop the influence coefficient matrix for the beam shown in Fig. P7.2 by
(i) introducing a thrust release each at B and C
(ii) introducing a thrust release each at B and D
(iii) introducing a thrust release each at C and D
Ignore axial and shear deformations.

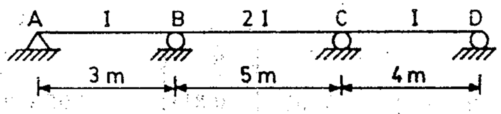


Fig. P7.2

7.4 Develop the influence coefficient matrix for the beam shown in Fig. P7.3 by
(i) introducing a hinge at D and a thrust release at B
(ii) introducing a hinge at D and a thrust release at C

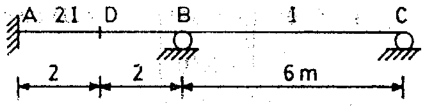


Fig. P.7.3

7.5 Develop the influence coefficient matrix for the frame shown in Fig. P7.4 by
(i) introducing a roller at D.
(ii) introducing a hinge at E

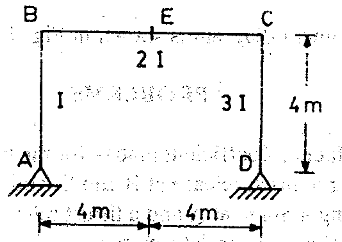


Fig. P7.4

7.6 Develop the influence coefficient matrix for the frame shown in Fig. P7.5 by
(i) introducing a cut at C
(ii) introducing a hinge at A and a roller at C
and considering the axial and shear deformations.
Take $A_eG = 25\,EI$, $AE = 50\,EI$.

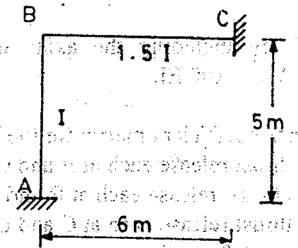


Fig. P7.5

7.7 Develop the influence coefficient matrix for the frame shown in Fig. P7.6 by considering any two different released structures. Take $A_eG = 25\,EI$, $AE = 50\,EI$.

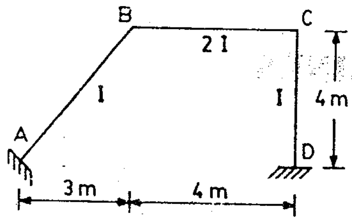


Fig. P7.6

7.8 Develop the influence coefficient matrix for the truss shown in Fig. P7.7 by
(i) introducing a thrust release each at 1-5 and 3-5.
(ii) introducing a thrust release each at 1-5 and 2-6.
Take AE of the vertical members and diagonal members = 0.5 AE of top and bottom chord members.

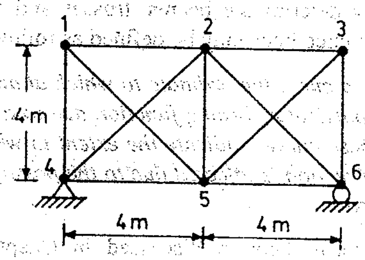


Fig. P7.7

7.9 Analyze the beams shown in Figs. P 3.2 and P 3.3 using the influence coefficient method and determine the redundant reactions.

7.10 Analyze the beams shown in Figs. P 3.7b and P 3.8c using the influence coefficient method and determine the redundant reactions.

7.11 Analyze the beams shown in Figs. P 6.1c and P 6.1d and determine the support reactions at A.

7.12 Analyze the frames shown in Figs. P 6.3c and P 6.3d by introducing a cut at section C, and determine the support reactions at A

CHAPTER
eight

INFLUENCE LINES

8.1 INTRODUCTION

Influence lines are required for the design of structures subjected to moving loads. Influence lines for statically determinate beams, trusses and arches were discussed in volume 1 of this book. Influence lines may be defined as follows:

> *An influence line is a curve the ordinate to which at any section equals the value of some particular forcing function due to a unit load acting at that section. These curves indicate the extent to which a particular function at a given section is affected due to the passage of a unit load across the entire span.*

The Maxwell's reciprocal theorem discussed in Chapter 5 is very helpful in developing the influence lines. In this chapter the influence lines are drawn for statically indeterminate beams and frames.

8.2 MULLER - BRESLAU PRINCIPLE

The Muller - Breslau principle proposed in 1886 - 87 is the most powerful tool for developing influence lines for both determinate and indeterminate structures, where the principle of superposition is valid. It states that :

> *The influence lines for any force quantity in a structure is represented to some scale by the deflected shape of the structure resulting from moving the force quantity under consideration through a small displacement.*

This principle is the basis for determining influence lines for various structures regardless of whether the method is analytical or experimental.

Consider a frame shown in Fig. 8.1a. It is desired to find the influence line ordinates

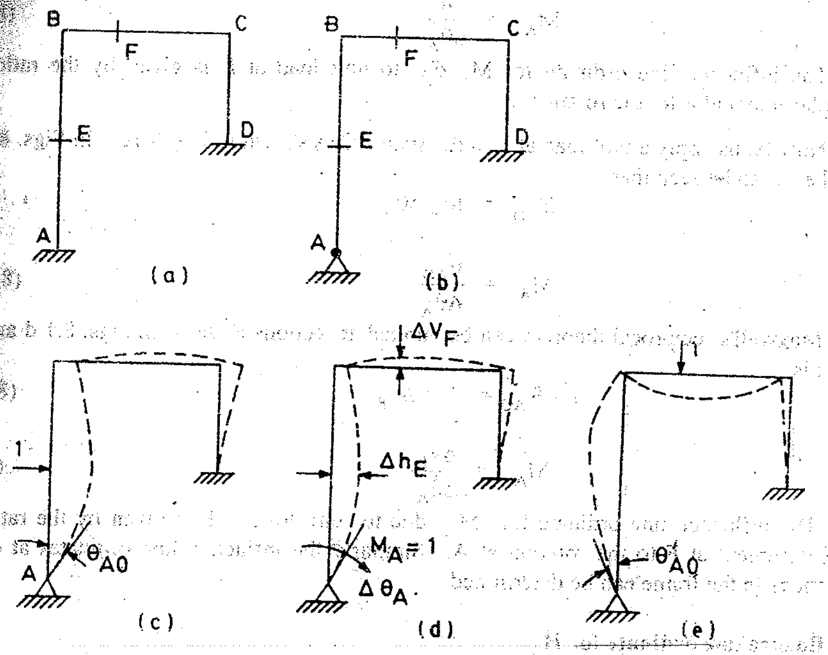


Fig. 8. 1 Influence line for moment at A

Influence line ordinates for M_A

Let us calculate the influence line ordinate for a unit load acting at points E and F, where E is any point in the span AB, and F is any point in the span BC. In accordance with the Muller - Breslau principle, the force component for which the influence line is desired is removed. In other words, a hinge is introduced at the support A as shown in Fig. 8.1b so that a specified rotation may be introduced. A unit load is applied at E and the rotation θ_{AO} at A as shown in Fig. 8.1c is computed by any method.

A unit moment is now applied at A and the frame will deflect as shown in Fig. 8.1 d. The rotation $\Delta\theta_A$, and deflection Δh_E and Δv_F can be computed by any method.

In Figs. 8.1c and d, it can be easily seen that

$$\theta_{AO} = M_A \Delta\theta_A$$

or,

$$M_A = \frac{\theta_{AO}}{\Delta\theta_A} \tag{8.1a}$$

Maxwell's reciprocal theorem can be applied to sections A and E in Figs. 8.1 c and d, that is,

$$\therefore \quad M_A = \frac{\Delta h_E}{\Delta\theta_A} \tag{8.2}$$

The influence line ordinate for M_A due to unit load at E is given by the ratio of displacement at E to the rotation at A.

Now let us apply a unit load at F in the span BC as shown in Fig. 8.1e. In Figs. 8.1 d and e, it can be seen that

$$\theta'_{AO} = M_A \Delta\theta_A$$

or,

$$M_A = \frac{\theta'_{AO}}{\Delta\theta_A} \tag{8.3a}$$

Maxwell's reciprocal theorem can be applied at sections F and A in Figs. 8.1 d and e, that is,

$$1 \times \theta'_{AO} = 1 \times \Delta v_F \tag{8.3b}$$

$$\therefore \quad M_A = \frac{\Delta v_F}{\Delta\theta_A} \tag{8.4}$$

The influence line ordinate for M_A due to unit load at F is given by the ratio of displacement at F to the rotation at A. Similarly the influence line ordinates at other sections in the frame can be determined.

Influence line ordinate for H_A

Let us now calculate the influence line ordinate for a unit load acting at points E and F of the frame shown in Fig. 8.2a. In accordance with the Muller - Breslau principle, a shear release is introduced at the support A so that a specified displacement may be introduced as shown in Fig. 8.2b. A unit load is applied at E and the horizontal displacement at A in the frame shown in Fig. 8.2c is computed by any method.

A unit horizontal force of 1 kN is next applied at A and the frame deflects as shown in Fig. 8.2 d. The deflections Δh_A, Δh_E and Δv_F can be computed by any method. In Figs. 8.2 c and d, it can be easily seen that

$$h_{AO} = H_A \Delta h_A$$

or,

$$H_A = \frac{h_{AO}}{\Delta h_A} \tag{8.5a}$$

Maxwell's reciprocal theorem can be applied to sections A and E in the above figures, that is,

$$1 \times h_{AO} = 1 \times \Delta h_E \tag{8.5b}$$

$$\therefore \quad H_A = \frac{\Delta h_E}{\Delta h_A} \tag{8.6}$$

Similarly, with respect to Figs. 8.2 c and e, it can be shown that

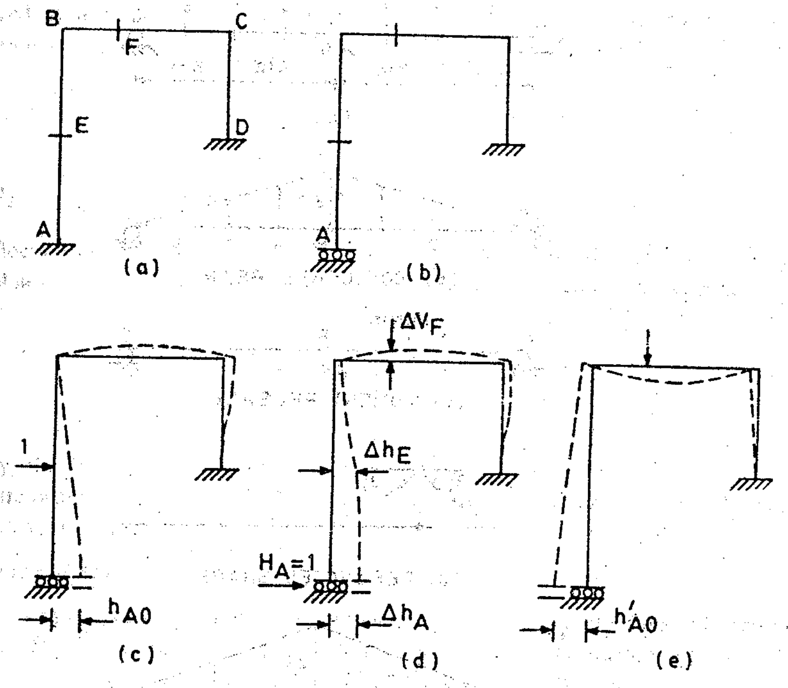


Fig. 8.2 Influence line for horizontal reaction at A

$$H_A = \frac{\Delta v_F}{\Delta h_A} \tag{8.7}$$

From Eq. 8.6 it is apparent that the deflection curve of the member AB is, to some scale, the influence line for H_A for a horizontal load on AB. Similarly, the deflection curve of the member BC, is to some scale, the influence line for H_A for a vertical load on BC. The slopes and deflections can be determined by any method.

8.3 ILLUSTRATIVE EXAMPLES

Example 8.1

Develop the influence lines for R_A, R_B, M_2, M_B and V_2 in a two span continuous beam shown in Fig. 8.3 a.

Solution

Influence Line For R_A

The vertical restraint at support A is removed and a unit vertical load is applied at A. The deflected shape of the beam is, to some scale, the influence line for the reaction at A.

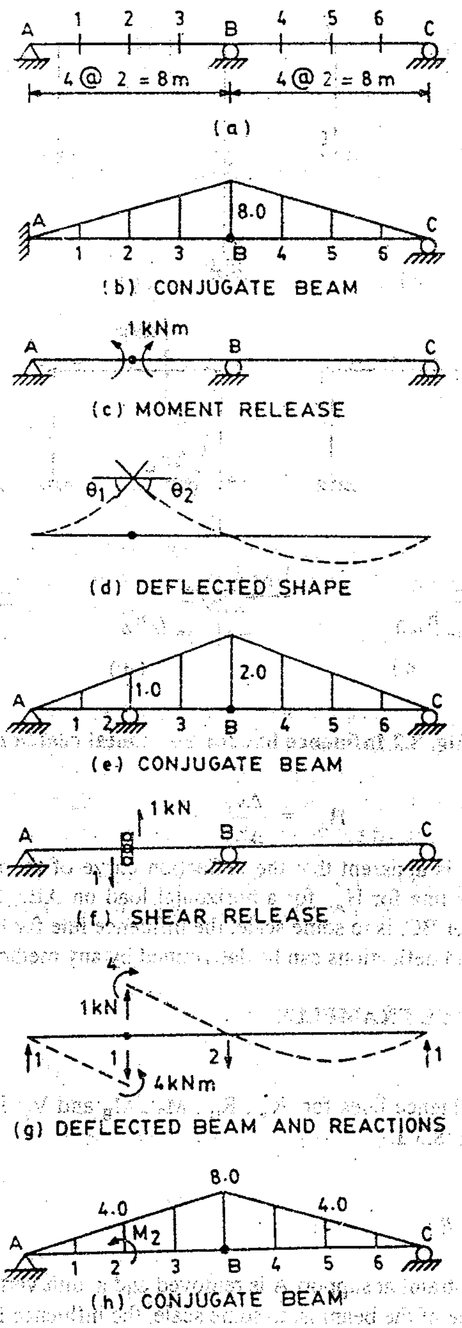


Fig. 8.3

The deflected shape can be evaluated using the conjugate beam method. The conjugate beam is shown in Fig. 8.3b along with the M/EI loading. For convenience 1/EI is taken as unity. The free end A is replaced by a fixed end at A, while the continuous support at B is replaced by a hinge at B in the conjugate beam ABC. Let us first calculate the bending moment, say at sections A and 2 in the conjugate beam of Fig. 8.3b.

$$R_A = \frac{160}{3}\uparrow, \quad R_C = \frac{32}{3}\uparrow, \quad M_A = \frac{1024}{3} \text{ anitclockwise}$$

and $$M_2 = \frac{1024}{3} - \frac{160}{3}\times 4 + \frac{1}{2}\times 4\times 4\times\frac{4}{3} = \frac{416}{3}$$

The deflection at A is given by the moment at A, and the deflection at 2 is given by the moment at 2. If the deflection at A is taken as unity, the deflection at 2 is equal to

$$\Delta_2 = \frac{416/3}{1024/3} = 0.406$$

Similarly, the influence line coefficient can be determined at other sections. The influence line ordinates are shown in Fig. 8.4b. Similarly, the influence line for R_B can be developed as shown in Fig. 8.4c.

Influence Line For Moment at 2

A hinge is introduced at section 2 and a pair of unit couples is introduced as shown in Fig. 8.3c. Its deflected shape is shown in Fig. 8.3 d. The net rotation at the hinge is $\theta_1 + \theta_2$. The conjugate beam and the M/EI loading is shown in Fig. 8.3e. Again 1/EI is taken as unity for convenience. The total rotation ($\theta_1 + \theta_2$) at the pin is given by sum of the shears on both sides of the pin in the conjugate beam.

$$(\theta_1 + \theta_2) \text{ at } 2 = \text{Reaction } R_2$$

$$\text{Shear at B in the span BC} = \frac{1}{2}\times 8\times 2\times\frac{2}{3} = 5.33\downarrow$$

Reaction R_A can be obtained by taking moment about the support 2 of all the loads acting on the span A - 2 - B.

$$R_A\times 4 - \left(\frac{1}{2}\times 4\times 1\right)\times\frac{4}{3} + \left(\frac{1+2}{2}\right)\times 4\times\left\{\frac{4}{3}\times\left(\frac{1+2\times 2}{1+2}\right)\right\} + 5.33\times 4 = 0$$

or, $$R_A = -8.00\downarrow$$

$$\therefore \quad \text{Reaction } R_C = \frac{1}{2}\times 8\times 2\times\frac{1}{3} = 2.67\uparrow$$

$$\therefore \quad \text{Reaction } R_2 = 16.0 + 8.00 - 2.67 = 21.33\uparrow$$

The moments at various sections can be easily computed.

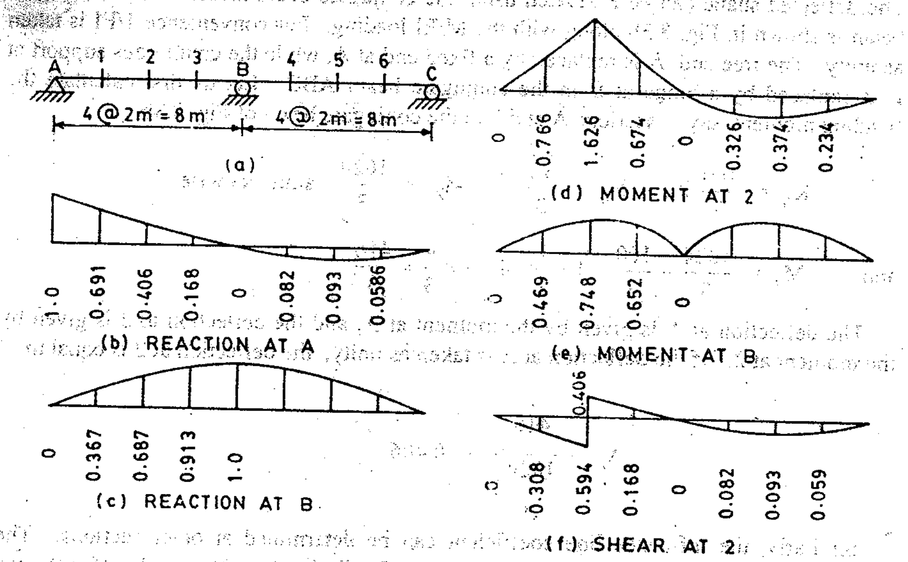


Fig. 8.4 Influence lines for 2 span beam

$$\text{Moment at 1} = -8\times2-\frac{1}{2}\times2\times0.5\times\frac{2}{3} = -16.333$$

$$\text{Moment at 5} = 2.67\times4-\frac{1}{2}\times4\times1\times\frac{4}{3} = 8.0$$

The influence line ordinate at any section is computed by dividing each moment by $(\theta_1 + \theta_2 = 21.33)$ as in Eq. 8.2 or 8.4.

$$\text{I. L. ordinate at 1} = -\frac{16.333}{21.33} = -0.7657$$

$$\text{I. L. ordinate at 5} = \frac{8.0}{21.33} = 0.375$$

Similarly the influence line ordinate at other sections can be obtained as shown in Fig. 8.4 d. The influence line for moment at B can also be obtained in the same manner and is shown in Fig. 8.4e.

Influence Line For Shear at 2

A shear release is introduced at section 2 and a pair of unit forces is applied as shown in Fig. 8.3f. Its deflected shape and reactions are shown in Fig. 8.3g. The corresponding conjugate beam is shown in Fig. 8.3h. A moment M_2 is imposed at section 2 corresponding to the relative displacement between the two cut portions at section 2 shown in Fig. 8.3g. Let us first calculate the reactions in the conjugate beam of Fig. 8.3h.

$$\text{Reaction } R_C = \frac{1}{2}\times8\times8\times\frac{1}{3} = 10.667\uparrow$$

$$\text{Reaction } R_A = \frac{1}{2}\times16\times8-10.667 = 53.333\uparrow$$

The moment M_2 can be determined by considering the equilibrium of the span AB.

$$53.333\times8-\frac{1}{2}\times8\times8\times\frac{8}{3}-M_2 = 0, \qquad \text{or,} \quad M_2 = 341.33$$

The moments at various sections can be computed as follows :

$$\text{Moment at 1} = 53.333\times2-\frac{1}{2}\times2\times2\times\frac{2}{3} = 105.33$$

$$\text{Moment just to the left of 2} = 53.333\times4-\frac{1}{2}\times4\times4\times\frac{4}{3} = 202.66$$

$$\text{Moment just to the right of 2} = -341.33+202.66 = -138.66$$

$$\text{Moment at 5} = 10.667\times4-\frac{1}{2}\times4\times4\times\frac{4}{3} = 32.0$$

$$\text{I. L. ordinate at 1} = \frac{105.33}{341.33} = 0.308$$

$$\text{at left of 2} = \frac{202.66}{341.33} = 0.594$$

$$\text{at right of 2} = -\frac{138.66}{341.33} = -0.406$$

$$\text{at 5} = \frac{32}{341.33} = 0.093$$

The influence line for shear at 2 is shown in Fig. 8.4 f.

Example 8.2

Develop the influence lines for reactions at A, B and C, moments at 2, B and C, and shear at 2 for the continuous beam shown in Fig. 8.5a. Take EI = constant.

Solution

The influence lines for the desired force functions for the fixed ended continuous beam can be drawn from the basic principles discussed earlier. For instance the qualitative influence line for M_B can be drawn using the Muller-Breslau principle. A hinge is introduced at B and the deflected shape is drawn due to a pair of unit couples. The ordinates at various sections can be evaluated using the conjugate beam method as shown in the previous example.

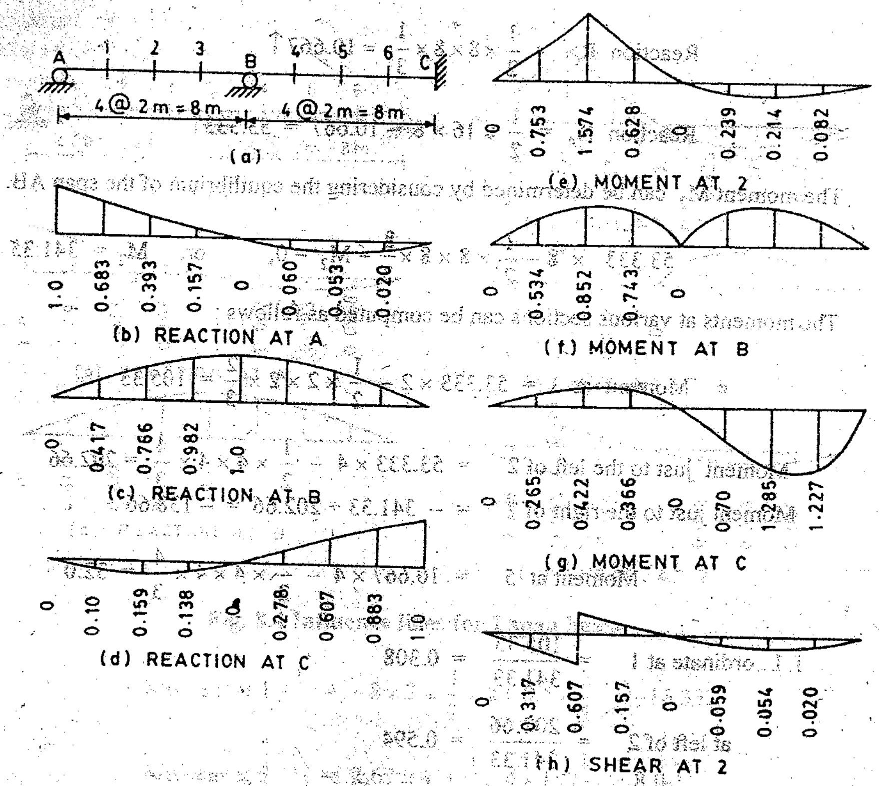


Fig. 8.5

Alternatively, the influence line ordinates can be calculated analytically. A unit load is placed at different sections from 1 to 6, one at a time, on the original beam shown in Fig. 8.5a. This is a statically indeterminate beam having a degree of static indeterminacy equal to 2. This can be analyzed using any of the methods such as the consistent deformation method, strain energy method or influence coefficient method. The values of R_A, R_B, R_C, M_2, M_B, M_C and V_2 can be easily evaluated for each loading case. In the present example, this beam will need to be analyzed completely six times for each of the six different positions of the unit load.

This beam can also be analyzed using a computer program based on the stiffness matrix method discussed in chapters 12 and 13. The details of the computer program are discussed in chapter 14 and its listing is given in Appendix D. The program gives the displacements and rotations directly. The results are shown in Figs 8.5b to 8.5h.

PROBLEMS

8.1 Sketch the influence lines for the moment at A, moment at C and shear at C for the beam shown in Fig. P8.1 and compute the ordinates at 0.5 m interval.

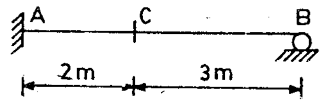


Fig. P8.1

8.2 Sketch the influence lines for the moment and shear at the midspan of BC for the beam shown in Fig. P8.2. Hence, determine their values if there is a uniform load of intensity 15 kN/m on the entire beam.

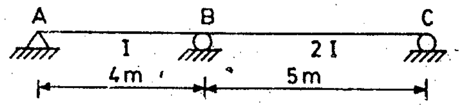


Fig. P8.2

8.3 Develop the influence line for the reaction at B @ 2 m interval for the beam shown in Fig. P8.3.

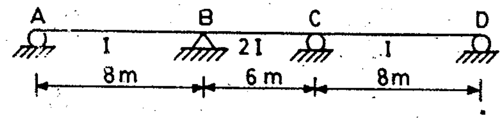


Fig. P8.3

8.4 Sketch the influence lines for the horizontal reaction and moment at A for the frames shown in Figs. P8.4 a and b. Compute the influence line ordinates at 1.5 m interval.

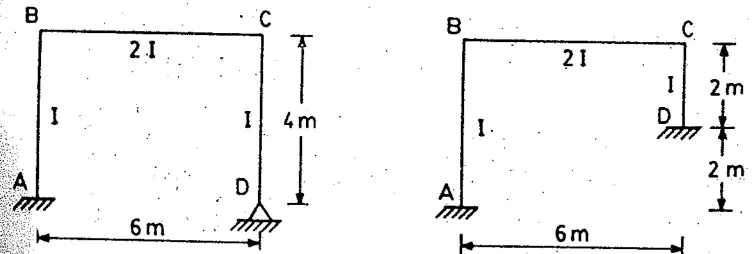


Fig. P8.4

8.5 Using the computer program given in Chapter 14, develop the influence lines for the vertical reaction at A, moment at A and bending moment at B in the span BE of the two storey frame shown in Fig. P8.5.

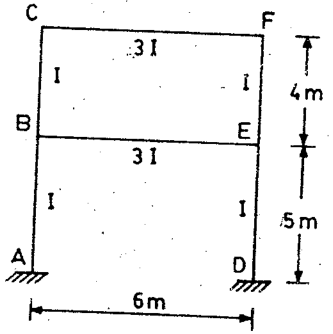


Fig. P8.5

CHAPTER
nine

ARCHES

9.1 INTRODUCTION

An arch may be defined as a curved structure which depends for its ability to resist applied vertical loads on the development of horizontal reaction components acting towards the center of the span of the arch. An arch may be classified as a three-hinged, two hinged, or fixed or hingeless. A three-hinged arch is statically determinate, whereas, the other arches are statically indeterminate. An arch may also be classified as to the shape and structural arrangement of the rib. Typical arches are shown in Fig. 9.1. Over the years with the improvement in the material, methods of analysis, design and construction techniques, the spans of new arch bridges have constantly increased. The economy of any arch depends on the rise to span ratio. Of course, this ratio will be influenced by the site conditions.

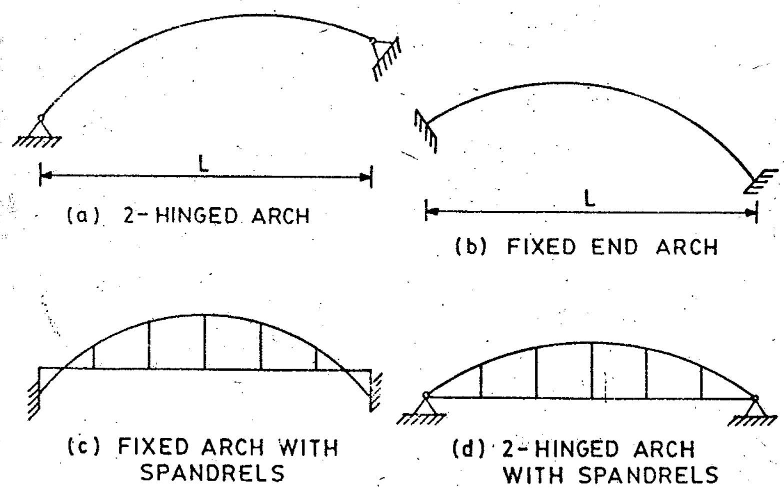


Fig. 9.1 Typical arches

9.2 TWO - HINGED ARCH

A two-hinged arch consists of two vertical support reactions and two horizontal support reactions. Thus, it is a statically indeterminate structure by degree 1. A two-hinged arch may be analyzed using any of the flexibility methods developed so far :

(i) method of consistent deformations,
(ii) strain energy method, and
(iii) column analogy method.

Column Analogy Method

Let us use the column analogy method to determine the horizontal thrust H. Consider a two hinged arch shown in Fig. 9.2a. It can be made statically determinate by removing one hinge support and replacing it with a roller support. Analogous column section is shown in Fig. 9.2 b. The bending moment at any point in the arch will be μ_x. The loading on the analogous column section at a point will be $\mu_x\, ds/EI$.

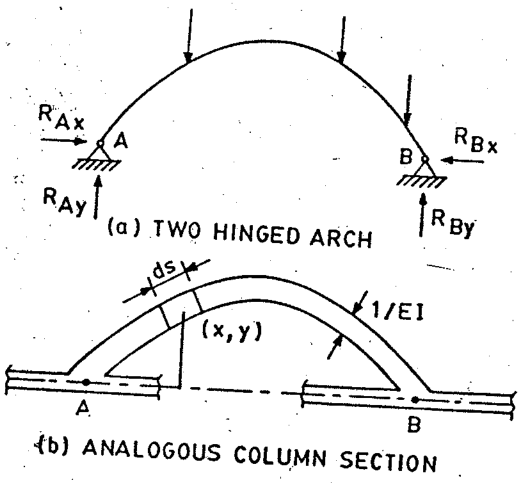


Fig. 9.2

$$\text{Total load on the column} = \int_A^B \frac{\mu_x\, ds}{EI}$$

$$\text{Moment of load about A – B axis} = \int_A^B \frac{\mu_x\, y\, ds}{EI}$$

$$\text{Area of analogous column} = \infty$$

$$\text{Moment of inertia of analogous column} = \int_A^B \frac{y^2 ds}{EI}$$

The stress at any point is given by Eq. 6

$$\sigma = \frac{P}{A} + \frac{My}{I}$$

or,

$$\sigma = \frac{\int_A^B \frac{\mu_x\, y\, ds}{EI}}{\infty} + \frac{\int_A^B \mu_x\, y \frac{ds}{EI}}{\int_A^B y^2 \frac{ds}{EI}}\, y = \frac{\int_A^B \mu_x\, y \frac{ds}{EI}}{\int_A^B y^2 \frac{ds}{EI}}\, y$$

The net sagging moment at any point (x,y) in the arch is equal to

$$M_x = \mu_x - \sigma = \mu_x - Hy$$

$$\therefore \quad \text{Horizontal thrust } H = \frac{\int_A^B \mu_x\, y \frac{ds}{EI}}{\int_A^B y^2 \frac{ds}{EI}} \tag{9.1}$$

For vertical loads only on the arch, $H = R_{Ax} = R_{By}$

Knowing the horizontal thrust, the vertical support reactions R_{Ay} and R_{By} can be easily evaluated.

In large span arches, change in temperature and elastic axial deformations in the arch rib may cause considerable change in the value of horizontal thrust. A detailed derivation can be carried out using the method of consistent deformations. This method will be used to analyze a fixed end arch in the next section. The expression for the horizontal thrust is given by :

$$H = \frac{\int_A^B \mu_x\, y \frac{ds}{EI} + \alpha L T}{\int_A^B y^2 \frac{ds}{EI} + \int_A^B \frac{ds \cos\theta}{AE}} \tag{9.2}$$

where,
α = coefficient of thermal expansion
T = change in temperature
A = area of cross-section of the rib

θ = inclination of the arch axis with the horizontal
E = modulus of elasticity of the arch material

The effect of rib shortening is to reduce the value of horizontal thrust by about 3%. It is generally important for very flat arches or for arches having a large depth of rib.

Strain Energy Method

Let us reanalyze the arch using the strain energy method.

$$\text{Strain energy } U = \int_A^B \frac{1}{2}\frac{M^2 ds}{EI}$$

For minimum strain energy

$$\frac{\partial U}{\partial H} = \frac{\partial U}{\partial M}\frac{\partial M}{\partial H} = 0$$

In a statically determinate arch, $M = \mu_x - Hy$

$$\frac{\partial M}{\partial H} = -y, \text{ and } \frac{\partial U}{\partial M} = \int_A^B \frac{M}{EI} ds$$

$$\therefore \quad \frac{\partial U}{\partial H} = \int_A^B \frac{Mds}{EI}\frac{\partial M}{\partial H} = \int_A^B (\mu_x - Hy)\frac{ds}{EI}(-y) = 0$$

or,
$$H\int_A^B y^2 \frac{ds}{EI} = \int_A^B \mu_x y \frac{ds}{EI}$$

or,
$$H = \frac{\int_A^B \mu_x y \frac{ds}{EI}}{\int_A^B y^2 \frac{ds}{EI}}$$

This is same as Eq. 9.1.

In the case of parabolic arches, the rib is thickened near the edges. Thus, its moment of inertia varies along the span. It is therefore difficult to integrate Eqs. 9.1 or 9.2 directly. A simplification is introduced. It is assumed that moment of inertia I at any section is equal to $I_o \sec\theta$.

where, I_o = value of moment of inertia at the crown
θ = inclination of the tangent at the section with horizontal.

In the case of circular arches, direct integration is possible if the moment of inertia of the arch rib is constant throughout. The analysis of arches is very sensitive to numerical round-off errors. Hence, it is recommended that arches should be analyzed using at least 5 or 6 significant digits on a digital calculator.

9.3 ILLUSTRATIVE EXAMPLES

Example 9.1

A 2-hinged parabolic arch has a span of 100 m and a rise of 25 m. It carries a uniformly distributed load of 20 kN/m intensity of the horizontal span over its left half span. Determine the reactions and draw bending moment diagram. Take $I = I_o \sec\theta$.

Solution

The horizontal thrust in a 2-hinged arch is given by Eq. 9.1

but $ds\cos\theta = dx$ or, $ds = dx\sec\theta$ and $I = I_o\sec\theta$

$\therefore$ Equation for H reduces to

$$H = \frac{\int_A^B \mu_x y\, dx}{\int_A^B y^2 dx}$$

Vertical reaction $R_A = 750$ kN, $R_B = 250$ kN

Equation of a parabola with A as origin is given by

$$y = \frac{4hx}{L^2}(L-x) = \frac{4\times 25x}{100^2}(100-x) = \frac{x}{100}(100-x)$$

Simple beam moment μ_x in the arch is given by

$$\mu_x = 750x - 20\frac{x^2}{2} \qquad \text{for } x \le 50 \text{ m} \qquad \text{(i)}$$

and
$$= 750x - 1000(x-25) \qquad \text{for } x > 50 \text{ m} \qquad \text{(ii)}$$

or,
$$\mu_x = 25000 - 250x$$

$$\therefore \int_A^B \mu_x y\, dx = \int_0^{50}\left[750x - 10x^2\right]\left[\frac{x}{100}(100-x)\right]dx + \int_{50}^{100}(25000-250x)\frac{x}{100}(100-x)dx$$

$$= 166.66 \times 10^5$$

$$\int_A^B y^2\,dx = \frac{1}{10^4}\int_0^{100} x^2(100-x)^2\,dx = \frac{1}{10^4}\int_0^{100}\left(10^4x^2 - 200x^3 + x^4\right)dx$$

$$= 3.33 \times 10^4$$

$$\therefore \quad H = \frac{166.66\times 10^5}{3.33\times 10^4} = 500 \text{ kN}$$

Net bending moment

$$M_x = \mu_x - Hy$$
$$= 750\,x - 10x^2 - 500\,y$$
$$= 750\,x - 10\,x^2 - 5x\,(100 - x) = 250x - 5x^2 \quad \text{for } x \le 50 \text{ m} \qquad \text{(iii)}$$

and
$$M_x = 25000 - 250x - 500\,y$$
$$= 25000 - 250x - 5x\,(100 - x)$$
$$= 25000 - 750x + 5x^2 \qquad \text{for } x > 50 \text{ m} \qquad \text{(iv)}$$

The point of inflection can be determined by equating Eq. (iii) or Eq. (iv) to zero, that is,

$$250x - 5x^2 = 0 \quad \text{or,} \quad x = 50 \text{ m}$$

Maximum sagging moment in segment AC can be determined by differentiating Eq. (iii) and equating it to zero, that is,

$$250x - 5x^2 = 0 \quad \text{or,} \quad x = 50 \text{ m}$$

$\therefore$ Maximum negative B.M. $= 25000 - 750 \times 75 + 5 \times 75^2$
$= -\,3125$ kNm

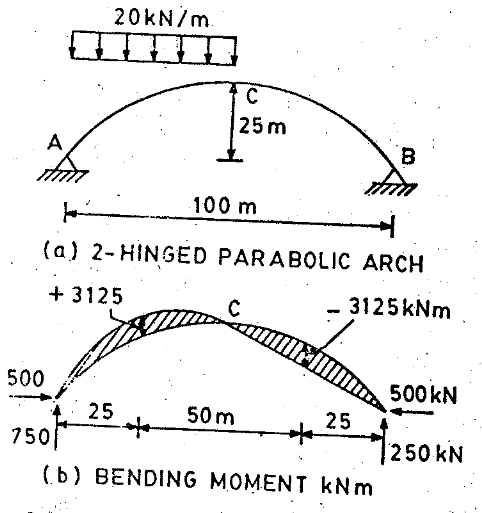


Fig. 9.3

It is of interest to note that the variation of moment in the segment AC is parabolic as given by Eq. (i), and that in the segment CB is linear as given by Eq. (ii). The bending moment diagram is drawn by first drawing the μ_x diagram and subtracting from it the Hy diagram. The μ_x diagram will be its line of thrust. The Hy diagram will be the arch axis itself if the vertical scale is selected accordingly. The B.M. diagram is shown in Fig. 9.3b.

Example 9.2

(a) A 2–hinged circular arch has a span of 150 m and rise of 25 m. It carries a uniform load of 30 kN/m of horizontal span from 50 m to 100 m from its left support as shown in Fig. 9.4 a. Draw the bending moment diagram. Also, determine the effect of rib shortening. The cross-section of the arch is 1m × 2m deep throughout. Take E = 2×10^4 kN/cm^2.

(b) If temperature rises by 40°C, determine the increase in horizontal thrust and the change in bending moment. Take coefficient of thermal expansion $\alpha = 12 \times 10^{-6}$/°C.

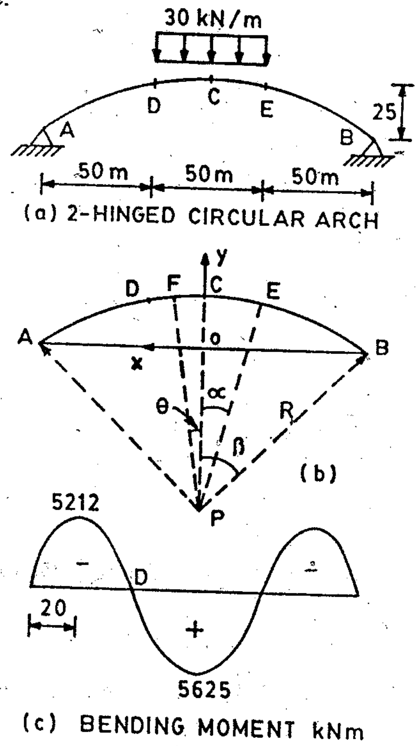


Fig. 9.4

Solution

(a) The horizontal thrust in a 2-hinged arch is given by

$$H = \frac{\int_A^B \mu_x y \frac{ds}{EI} + \alpha LT}{\int_A^B y^2 \frac{ds}{EI} + \int_A^B \frac{ds \cos\theta}{AE}} \qquad (9.2)$$

Let O be the origin. (Fig. 9.4b)

$$y = R\cos\theta - (R-h)$$

and $R = \left(\frac{L^2}{4}+h^2\right)\frac{1}{2h} = \left(\frac{150^2}{4}+25^2\right)\frac{1}{2\times 25} = 125 \text{ m}$

$\therefore \quad y = (125\cos\theta - 100)$

$\sin\alpha = \frac{25}{125} = 0.2, \quad \sin\beta = \frac{75}{125} = 0.6$

$\alpha = 11.54° = 0.20 \text{ rad.} \quad \beta = 36.87° = 0.643 \text{ rad.}$

Simple span moment μ_x at point G

$\mu_x = 750\,(75 - R\sin\theta) \qquad \beta > \theta > \alpha$

Simple span moment μ_x at point F

$\mu_x = 750\,(75 - R\sin\theta) - 30\,(R\sin\alpha - R\sin\theta)^2/2 \qquad \alpha > \theta > 0$

$ds = R\,d\theta$

$$\therefore \int_A^B \mu_x y\,dx = 2\int_0^\alpha\left[750(75-125\sin\theta) - 15\times125^2\times(\sin\alpha-\sin\theta)^2\right](125\cos\theta-100)$$

$$\times\ 125\,d\theta + 2\int_\alpha^\beta 750(75-125\sin\theta)(125\cos\theta-100)\times125\,d\theta$$

$$= 2\int_0^\alpha 125^4\left[(3.6-6\sin\theta) - 1.5\left(\sin^2\alpha - 2\sin\alpha\sin\theta + \sin^2\theta\right)\right](\cos\theta-0.8)\,d\theta$$

$$+\ 2\int_\alpha^\beta 125^4\left[(3.6-6\sin\theta)(\cos\theta-0.8)\right]d\theta$$

or, $\int_A^B \mu_x y\frac{ds}{EI} = \frac{2\times125^4}{EI}[0.105+0.067] = 0.344\times\frac{125^4}{EI}$

$$\int_A^B y^2\frac{ds}{EI} = \frac{2}{EI}\int_0^\beta (125\cos\theta-100)^2 R\,d\theta$$

$$= \frac{2\times125^3}{EI}\int_0^\beta\left(\cos^2\theta - 1.6\cos\theta + 0.64\right)d\theta = \frac{0.026\times125^3}{EI}$$

$$\int_A^B \frac{ds\cos\theta}{AE} = 2\int_0^\beta \frac{R\cos\theta\,d\theta}{AE} = \frac{2}{AE}\int_0^\beta 125\cos\theta\,d\theta = \frac{250}{AE}\int_0^\beta\cos\theta\,d\theta = \frac{150}{AE}$$

Area of cross - section of rib $A = b \times D = 1\times 2 = 2 \text{ m}^2$

Moment of inertia of rib $I = \frac{bD^3}{12} = \frac{1\times2^3}{12} = 0.667 \text{ m}^4$

$E = 2\times10^4 \text{ kN/cm}^2 = 2\times10^8 \text{ kN/m}^2$

Horizontal thrust is given by,

$$H = \frac{0.344\times125^4/EI}{0.0260\times125^3/EI + 150/AE} = \frac{0.6295}{38.12\times10^{-5}+3\times10^{-7}}$$

$H = 1651.3$ kN ignoring axial shortening

$H = 1650$ kN including axial shortening

The effect of axial shortening is to reduce H by 0.079%.

$V_A = 30\times50/2 = 750 \text{ kN}$

Moment at any section is given by

$M = 750x - 1650\,y \qquad 0 \le x \le 50 \text{ m}$

$\quad = 750x - 1650y - 30\,(x-50)^2/2 \qquad 50 \le x \le 75 \text{ m}$

where, x is measured from support A.

at $x = 20$ m, $y = 12.25$ m, $\therefore M = -5212$ kNm

at D, $x = 50$ m, $y = 22.47$ m, $\therefore M = 424.5$ kNm

at C, $x = 75$ m, $y = 25$ m, $\therefore M = 5625$ kNm

The bending moment diagram is shown in Fig. 9.4c.

(b) If temperature rises by 40°C,

$$H = \frac{0.6295 + 12\times10^{-6}\times150\times40}{0.0003812} = \frac{0.7015}{0.0003812} = 1840\ \text{kN}$$

The thrust H increases by 11.50%.

Moment at crown $= 750\times75 - 1840\times25 - 30\times25^2/2$

$M_C = 875$ kNm

Thus maximum bending moment at the crown decreases by 84.4%.

9.4 FIXED ARCH

A fixed arch is shown in Fig. 9.5a. There are six support reactions. Hence the fixed arch is statically indeterminate to degree 3. It can be analyzed using any flexibility method described earlier. Let us use the consistent deformation method which gives a better physical understanding of the structure. Let us first derive some basic relationships for deformations due to moment, thrust, shear and temperature. Let us remove the support B and the fixed arch is reduced to a curved cantilever beam fixed at support A.

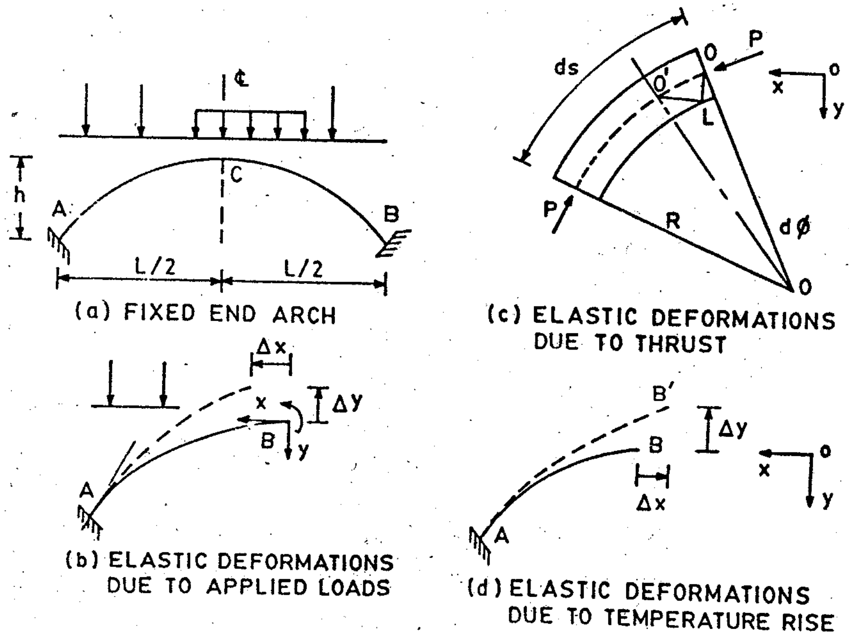


Fig. 9.5

Deformations due to the applied loads

Consider the released arch as shown in Fig. 9.5b. Let net moment at any section be sagging. If B is taken as the origin of coordinates with the positive direction of axes shown in the same figure, the deformations Δx, Δy and $\Delta\phi$ of a small element of the arch can be evaluated using the moment area theorem as follows:

horizontal displacement of B with respect to A, $$\Delta x = \int_B^A \frac{My\,ds}{EI} \qquad (9.3a)$$

vertical displacement of B with respect to A, $$\Delta y = -\int_B^A \frac{Mx\,ds}{EI} \qquad (9.3b)$$

rotation of B with respect to tangent at A, $$\Delta\phi = \int_A^B \frac{M\,ds}{EI} \qquad (9.3c)$$

$$= \text{area of } \frac{M}{EI} \text{ diagram}$$

Deformations due to thrust

Thrust due to the applied loads may be quite significant in certain conditions. Let us consider a small segment ds as shown in Fig. 9.5c. If the thrust is P,

shortening in element ds $= OO' = \dfrac{P\,ds}{AE}$

horizontal component of $OO' = O'L = OO'\cos\theta$

$$\text{or, } \Delta x = \frac{P\,ds}{AE}\cos\theta \qquad (9.4a)$$

vertical component of $OO' = OL = OO'\sin\theta$

$$\Delta y = \frac{P\,ds}{AE}\sin\theta \qquad (9.4b)$$

rotation $d\phi$ of point $O = \dfrac{OO'}{R}$

$$\text{or, } \Delta\phi = \frac{P\,ds}{AER} \qquad (9.4c)$$

since $R >> D$ (thickness of arch), $\Delta\phi \approx 0$.

Deformations due to temperature

Consider the released arch shown in Fig. 9.5 d. If the temperature rises by t° and α is the coefficient of thermal expansion, length of the element ds will increase by

α ds t

$\therefore$ horizontal elongation of element ds $= -\alpha$ t ds cos θ (9.5a)

and vertical elongation of element ds $= -\alpha$ t ds sin θ (9.5b)

Deformations due to shear

These may be neglected being very small. The net deformations in the arch are obtained by superposition of different effects and integrating along the arch:

$$\Delta x = \int_A^B \frac{My\,ds}{EI} + \int_A^B \frac{P\,ds}{AE}\cos\theta - \int_A^B \alpha\, t\, ds \cos\theta \qquad (9.6)$$

$$\Delta y = -\int_A^B \frac{Mx\,ds}{EI} + \int_A^B \frac{P\,ds}{AE}\sin\theta - \int_A^B \alpha\, t\, ds \sin\theta \qquad (9.7)$$

$$\Delta\phi = \int_A^B \frac{M\,ds}{EI} \qquad (9.8)$$

For compatibility,

$$\Delta x = 0, \quad \Delta y = 0, \quad \text{and} \quad \Delta\phi = 0 \qquad (9.9)$$

There are three unknowns M_A, H_A and R_A and three simultaneous equations. The moment M and thrust P can be expressed in terms of the above three unknowns. Solution of these three simultaneous equations gives the values of the unknowns M_A, H_A and R_A. Once the reactions are known the forces at any section of the arch can be easily calculated.

9.5 SYMMETRICAL FIXED ARCH

The solution of Eqs. 9.6, 9.7 and 9.8 becomes simpler in the case of symmetrical arches having their supports at the same level. Consider an arch shown in Fig. 9.6 which is symmetrical about a vertical axis through the crown C. The symmetry is in terms of geometry and not the loading. Let us introduce a cut at the crown C and the resulting arch is statically determinate. The redundants at C are H_C, R_C and M_C. These will act on either end at C such that the section C is in static equilibrium as shown in Fig. 9.6. Due to the external loads as well as these redundants, the point C in each half arch will undergo certain displacements. For compatibility, the displacements in either half must be consistent, that is, continuity of the arch axis must be maintained.

Let C be the origin for each half arch and positive directions of the axes are shown in the same figure. If Δx_A, Δy_A and $\Delta\phi_A$ represent the displacements of C in the portion AC and Δx_B, Δy_B and $\Delta\phi_B$ in the other half arch CB, then

for compatibility at C,

$$\Delta x_A = -\Delta x_B \qquad (9.10a)$$

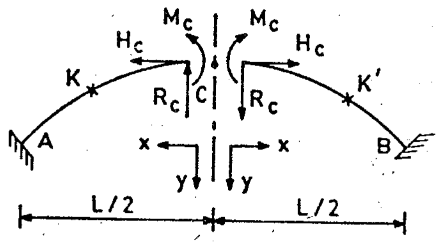


Fig. 9.6 Symmetrical fixed arch

$$\Delta y_A = \Delta y_B \qquad (9.10b)$$

$$\Delta\phi_A = -\Delta\phi_B \qquad (9.10c)$$

Net bending moment at any point K in the portion AC is given as

$$M = M_C + H_C y + R_C x - m_A \qquad (9.11a)$$

where m_A = cantilever B.M. about K of all loads between K and C

The effect of loads between C and B at K is already included through the values of three redundants at C. For vertical loads only, the thrust P at any section can be taken equal to the horizontal thrust.

that is, $P \approx H_C$ (9.11b)

This approximation is acceptable because the effect of rib shortening is itself very small and it simplifies the calculations.

The values of M and P given by Eq. 9.11 can be substituted in Eqs. 9.6, 9.7 and 9.8. Thus,

$$\Delta x_A = \int_C^A (M_C + H_C y + R_C x - m_A)\, y \frac{ds}{EI} + \int_C^A \frac{H_C \cos\theta\, ds}{AE} - \int_C^A \alpha\, t \cos\theta\, ds \qquad (9.12a)$$

$$\Delta y_A = -\int_C^A (M_C + H_C y + R_C x - m_A)\, x \frac{ds}{EI} + \int_C^A \frac{H_C \sin\theta\, ds}{AE} - \int_C^A \alpha\, t \sin\theta\, ds \qquad (9.12b)$$

$$\Delta\phi_A = \int_C^A (M_C + H_C y + R_C x - m_A) \frac{ds}{EI} \qquad (9.12c)$$

Similarly in the other half arch, the net moment at any point K' is given as

$$M = M_C + H_C y + R_C x - m_B \tag{9.13a}$$

and $$P = H_C \tag{9.13b}$$

where m_B = cantilever moment about K' of all loads between C and K'.

The values of M and P can be substituted in Eqs. 9.6, 9.7 and 9.8 to get Δx_B, Δy_B and $\Delta\phi_B$. Substituting these values of displacements in the compatibility conditions given by Eq. 9.10 yield the following expressions:

$$\Delta x_A = -\Delta x_B \text{ yields}$$

$$\int_C^A (M_C + H_C y + R_C x - m_A) y \frac{ds}{EI} + \int_C^A \frac{H_C \cos\theta\, ds}{AE} - \int_C^A \alpha t \cos\theta\, ds$$

$$= -\int_C^B (M_C + H_C y - R_C x - m_B) y \frac{ds}{EI} - \int_C^B \frac{H_C \cos\theta\, ds}{AE} + \int_C^B \alpha t \cos\theta\, ds$$

Since the arch is symmetrical about C, the integrals of similar functions from C to A or C to B have the same values. For a given loading, H_C, R_C and M_C are constant and can be taken outside the sign of integration. The above equations can be simplified as :

$$2M_C \int_C^A y \frac{ds}{EI} + 2H_C \int_C^A \frac{y^2 ds}{EI} - \int_C^A (m_A + m_B) y \frac{ds}{EI} + 2H_C \int_C^A \frac{\cos\theta\, ds}{AE}$$

$$= 2\alpha t \int_C^A ds \cos\theta = \alpha t L \tag{9.14}$$

$\Delta y_A = -$ yields,

$$-\int_C^A (M_C + H_C y + R_C x - m_A) x \frac{ds}{EI} + \int_C^A \frac{H_C \sin\theta\, ds}{AE} - \int_C^A \alpha t \sin\theta\, ds$$

$$= -\int_C^B (M_C + H_C y - R_C x - m_B) x \frac{ds}{EI} + \int_C^B \frac{H_C\, ds \sin\theta}{AE} - \int_C^B \alpha t \sin\theta\, ds$$

or, $$2R_C \int_C^A \frac{x^2 ds}{EI} = \int_C^A (m_A - m_B) x \frac{ds}{EI}$$

or, $$R_C = \frac{\int_C^A (m_A - m_B) x \frac{ds}{EI}}{2\int_C^A \frac{x^2 ds}{EI}} \tag{9.15}$$

Similarly,

$$\Delta\phi_A = -\Delta\phi_B \text{ gives}$$

$$\int_C^A (M_C + H_C y + R_C x - m_A) \frac{ds}{EI} = -\int_C^B (M_C + H_C y - R_C x - m_B) \frac{ds}{EI}$$

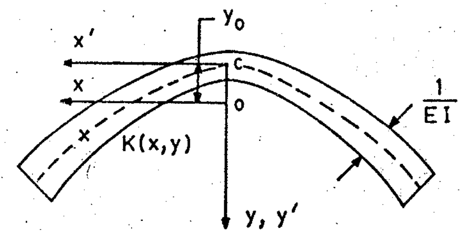
or, $$2M_C \int_C^A \frac{ds}{EI} + 2H_C \int_C^A \frac{y\, ds}{EI} = \int_C^A (m_A + m_B) \frac{ds}{EI} \tag{9.16}$$

Shear R_C is given by Eq. 9.15, the values of M_C and H_C can be evaluated by the solution of Eqs. 9.14 and 9.16. The solution of these equations can be further simplified by using the concept of elastic centre.

9.6 ELASTIC CENTRE

We know that displacements of a real beam could be found by computing shears and moment in a conjugate beam. A similar observation may be made in arches. An analogous column section of the fixed arch is shown in Fig. 9.7. Width of the analogous column at any point on the axis is equal to 1/EI where I is the moment of inertia of the arch rib at the same point. Centre of gravity of the analogous column is O. The coordinates of O are chosen so that

$$\int x \frac{ds}{EI} = 0 = \int y \frac{ds}{EI}$$



ANALOGOUS COLUMN SECTION

Fig. 9.7

Due to symmetry of the arch, the c.g. should be on the axis of symmetry. Let the point O be at a distance y_o below the crown C.

$$\therefore \qquad y_1 = y - y_0 \tag{9.17}$$

and
$$\int_C^A \frac{y_1 ds}{EI} = 0 \tag{9.18}$$

or,
$$\int_C^A (y - y_0)\frac{ds}{EI} = 0$$

or,
$$y_0 = \frac{\int_C^A y\frac{ds}{EI}}{\int_C^A \frac{ds}{EI}} \tag{9.19}$$

The point O is called the *Elastic Centre*. The concept of elastic centre is also useful in the analysis of frames. The origin of coordinates may be transferred from C to O, keeping positive direction of the axes same. Some useful relations can be derived first by making use of Eqs. 9.17 and 9.18.

$$\int_C^A y\frac{ds}{EI} = \int_C^A (y_1 + y_0)\frac{ds}{EI} = \int_C^A \frac{y_1 ds}{EI} + y_0\int_C^A \frac{ds}{EI} = y_0\int_C^A \frac{ds}{EI} \tag{9.20a}$$

$$\int_C^A y^2\frac{ds}{EI} = \int_C^A (y_1 + y_0)^2\frac{ds}{EI} = \int_C^A y_1^2\frac{ds}{EI} + y_0^2\int_C^A \frac{ds}{EI} + 2y_0\int_C^A y_1\frac{ds}{EI}$$

$$= \int_C^A y_1^2\frac{ds}{EI} + y_0^2\int_C^A \frac{ds}{EI} \tag{9.20b}$$

$$\int_C^A (m_A + m_B)y\frac{ds}{EI} = \int_C^A (m_A + m_B)(y_1 + y_0)\frac{ds}{EI}$$

$$= \int_C^A (m_A + m_B)y_1\frac{ds}{EI} + y_0\int_C^A (m_A + m_B)\frac{ds}{EI} \tag{9.20c}$$

Substituting the three integrals given by Eqs. 9.20 in Eqs. 9.14 and 9.16, we get

$$2M_C\, y_0\int_C^A \frac{ds}{EI} + 2H_C\left[\int_C^A y_1^2\frac{ds}{EI} + y_0^2\int_C^A \frac{ds}{EI}\right] + 2H_C\int_C^A \frac{\cos\theta\, ds}{AE} - \int_C^A (m_A + m_B)y_1\frac{ds}{EI} - y_0\int_C^A (m_A + m_B)\frac{ds}{EI} = \alpha t L \tag{9.21a}$$

and

$$2M_C\int_C^A \frac{ds}{EI} + 2H_C\, y_0\int_C^A \frac{ds}{EI} = \int_C^A (m_A + m_B)\frac{ds}{EI} \tag{9.21b}$$

Eq. 9.21b gives,
$$M_C = \frac{\int_C^A (m_A + m_B)\frac{ds}{EI}}{2\int_C^A \frac{ds}{EI}} - H_C\, y_0 \tag{9.22}$$

Substituting the value of M_C in Eq. 9.21a gives

$$H_C = \frac{\int_C^A (m_A + m_B)y_1\frac{ds}{EI} + \alpha t L}{2\left[\int_C^A y_1^2\frac{ds}{EI} + \int_C^A \frac{\cos\theta\, ds}{AE}\right]} \tag{9.23}$$

Thus a fixed symmetrical arch can be analyzed using Eqs. 9.15, 9.19, 9.22 and 9.23.

9.7 Illustrative Examples

Example 9.3

Analyze the arch of Example 9.1, if the two ends are fixed as shown in Fig. 9.8a.

Solution

The elastic centre of the arch is given by

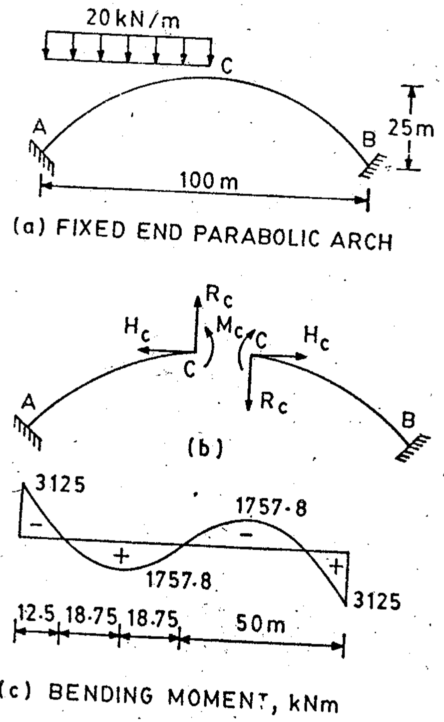


Fig. 9.8

$$y_0 = \frac{\int_C^A y\dfrac{ds}{EI}}{\int_C^A \dfrac{ds}{EI}} \tag{9.19}$$

The stress components at the crown R_C, H_C and M_C are given by

$$R_C = \frac{\int_C^A (m_A - m_B)x\dfrac{ds}{EI}}{2\int_C^A \dfrac{x^2 ds}{EI}} \tag{9.15}$$

$$H_C = \frac{\int_C^A (m_A + m_B)y_1\dfrac{ds}{EI} + \alpha t L}{2\left[\int_C^A y_1^2\dfrac{ds}{EI} + \int_C^A \dfrac{\cos\theta\, ds}{AE}\right]} \tag{9.23}$$

$$M_C = \frac{\int_C^A (m_A + m_B)\dfrac{ds}{EI}}{2\int_C^A \dfrac{ds}{EI}} - H_C y_0 \tag{9.22}$$

Equation of the arch with C as origin is given by

$$y = \frac{1}{100}x^2$$

$$\frac{ds}{I} = \frac{dx \sec\theta}{I_0 \sec\theta}$$

$$y_0 = \frac{\int_0^{50} y\dfrac{ds}{EI}}{\int_0^{50}\dfrac{ds}{EI}} = \frac{\int_0^{50}\dfrac{x^2}{100}\dfrac{dx}{EI_0}}{\int_0^{50}\dfrac{dx}{EI_0}} = \frac{\dfrac{1}{100}\int_0^{50} x^2\,dx}{\int_0^{50} dx} = 8.334 \text{ m}$$

$$y_1 = y - y_0 = \left(\frac{x^2}{100} - 8.334\right), \quad m_A = 20\frac{x^2}{2}, \quad m_B = 0$$

Various integrals can be evaluated as follows (Fig. 9.8b) :

$$\int_C^A (m_A - m_B)x\frac{dx}{EI_0} = \int_0^{50} 10x^3\frac{dx}{EI_0} = \frac{156.25\times 10^5}{EI_0}$$

$$\int_C^A (m_A + m_B)y_1\frac{ds}{EI_0} = \int_0^{50}(10x^2)\left(\frac{x^2}{100} - 8.334\right)\frac{dx}{EI_0} = 27.77\times\frac{10^5}{EI_0}$$

$$\int_C^A (m_A + m_B)\frac{ds}{EI_0} = \int_0^{50} 10x^2 \frac{dx}{EI_0} = \frac{4.167\times10^5}{EI_0}$$

$$\int_C^A x^2 \frac{ds}{EI_0} = \int_0^{50} x^2 \frac{dx}{EI_0} = \frac{0.416\times10^5}{EI_0}$$

$$\int_C^A {y_1}^2 \frac{ds}{EI} = \int_0^{50}\left(\frac{x^2}{100} - 8.334\right)^2 \frac{dx}{EI_0} = \frac{2780}{EI_0}$$

The reactions R_C, H_C and M_C at the crown can now be evaluated.

$$R_C = \frac{156.25\times10^5/EI_0}{2\times0.416\times10^5/EI_0} = 187.5 \text{ kN}$$

$$H_C = \frac{27.77\times10^5/EI_0}{2\times2780/EI_0} = 500 \text{ kN}$$

$$M_C = \frac{4.167\times10^5/EI_0}{2\times50/EI_0} - 500\times 8.334 = \text{zero}$$

Now the support reactions can be easily calculated.

$$R_A = 20\times50 - R_C = 812.5 \text{ kN}\uparrow$$

$$H_A = H_C = 500 \text{ kN}\rightarrow$$

$$M_x = \frac{wx^2}{2} - R_C\,x - H_C\,y - M_C$$

or, $$M_A = 20\times\frac{50^2}{2} - 187.5\times50 - 500\times25 = 3125 \text{ kNm hogging}$$

$$R_B = R_C = 187.5 \text{ kN}\uparrow$$

$$H_B = H_C = 500 \text{ kN}\leftarrow$$

$$M_x = R_C\,x - H_C\,y - M_C$$

or, $$M_B = 187.5\times50 - 500\times25 = -3125 \text{ kNm sagging}$$

The points of inflection can be determined by equating M_x equal to zero, in the left half arch,

$$\frac{wx^2}{2} - R_C\,x - H_C\,y - M_C = 0$$

or, $$10x^2 - 187.5x - 500\,\frac{x^2}{100} = 0 \quad \text{or,} \quad (5x - 187.5)x = 0$$

or, $x = 0$ or, $x = 37.5$ m.

The maxima can be located by setting $\frac{dM}{dx} = 0$,

$$\frac{dM_x}{dx} = wx - R_C - \frac{2x}{100}H_C = 0$$

or, $$20\,x - 187.5 - 10x = 0 \quad \text{or,} \quad x = 18.75 \text{ m}$$

$$M = 10\times18.75^2 - 187.5\times18.75 - 500\times\frac{18.75^2}{100} = -1757.8 \text{ kNm sagging}$$

Similar calculations can be done for the right half arch. The bending moment diagram can now be drawn as shown in Fig. 9.8c. The various integrals should be evaluated using 5 or 6 significant digits on a calculator to avoid numerical errors.

Example 9.4

Determine the reactions in a symmetrical fixed ended arch shown in Fig. 9.9a due to a load of 50 kN acting at 20 m from the left support using the column analogy method and elastic centre method. The width of the arch rib is 0.90 m constant, and depth varies as given by expression (i), where x is measured from the crown.

$$t = \left(1 + \frac{|x|}{50}\right)\text{m} \qquad \text{(i)}$$

Solution

(i) Column Analogy Method

Eq. of parabola with C as origin $y' = \dfrac{x'^2}{100}$

$$\frac{dy'}{dx'} = \frac{x'}{50} = \tan\theta, \quad \text{and} \quad ds = 10\sec\theta = 10\sqrt{1+\tan^2\theta}$$

$$y_0 = \frac{\int y'\frac{ds}{EI}}{\int \frac{ds}{EI}} = 5.16 \text{ m}$$

Moment of inertia of the arch section at the crown

$$I_C = \frac{0.9\times1^3}{12} = 0.075 \text{ m}^4$$

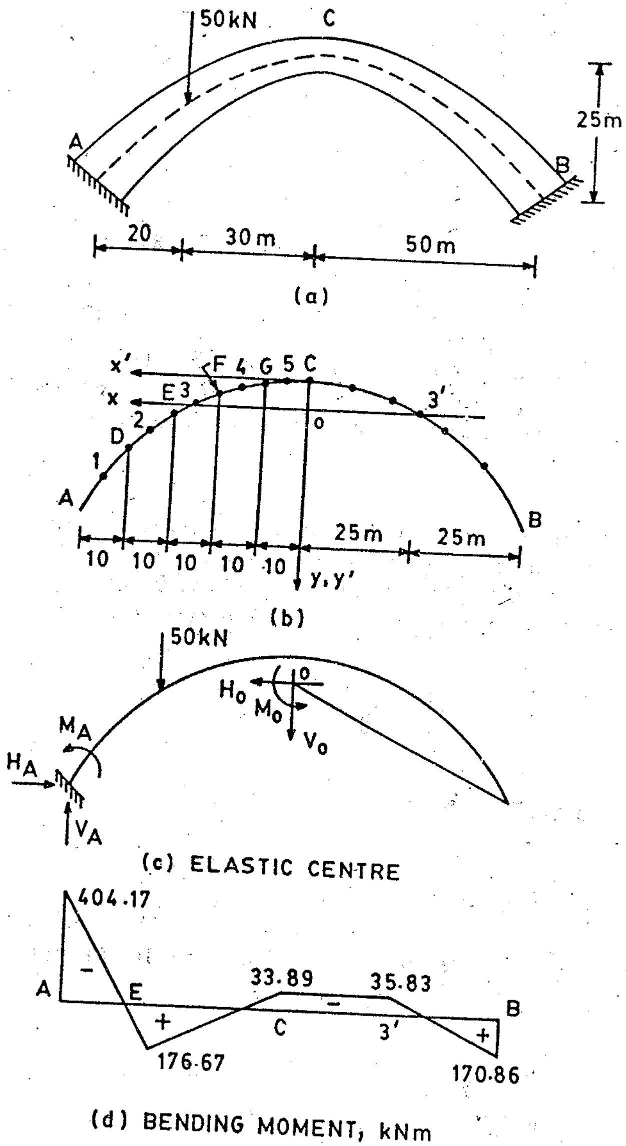


Fig. 9.9

The arch is symmetrical about the crown. Let half the arch be divided in five segments AD, DE, EF, FG and GC. Their centroids are indicated by points 1, 2, 3, 4, and 5, respectively. The computations of the elastic centre are done in a tabular form for convenience, as shown in Table 9.1. The computations of elastic loads and properties of the analogous column section are shown in Table 9.2. Thus, values of the reactions H_0, M_0 and V_0 can be also computed at the elastic centre. However, the stresses in an analogous column section can be determined directly without making use of the values at the elastic centre as shown in Table 9.3.

Table 9.1 Computation of elastic centre

Segment	Point	x' m	y' m	$\frac{dy}{dx}$	ds m	t m	I m^4	$\frac{I}{I_C}$	$\frac{ds}{EI}$	$y'\frac{ds}{EI}$
AD	1	45	20.25	0.90	13.454	1.9	0.514	6.853	26.175	-530.040
DE	2	35	12.25	0.70	12.207	1.7	0.368	4.907	33.171	406.345
EF	3	25	6.25	0.50	11.180	1.5	0.253	3.373	44.189	276.181
FG	4	15	2.25	0.30	10.440	1.3	0.165	2.200	63.273	142.364
GC	5	5	0.25	0.10	10.050	1.1	0.100	1.334	100.50	25.125

$\Sigma 267.308$ $\Sigma 1380.060$

$y_0 = 5.16$ m

Table 9.2 Computation of loads, moment and inertia

Segment	Point	x m	y m	$\frac{ds}{EI}$	$x^2\frac{ds}{EI}$	$y^2\frac{ds}{EI}$	M_s	$M_s\frac{ds}{EI}$	$M_s.x\frac{ds}{EI}$	$M_s y\frac{ds}{EI}$
AD	1	45	15.09	26.175	53004	5960	–750	–19631	– 883432	– 296232
DE	2	35	7.09	33.171	40635	1667	–250	– 8292.7	–290244	– 58795
EF	3	25	1.09	44.189	27618	52.50	0	0	0	0
FG	4	15	– 2.91	63.273	14236	535.80	0	0	0	0
GC	5	5	– 4.91	100.500	2512	2422.9	0	0	0	0
				Σ= 267.3	Σ= 138005	Σ = 10638	Σ = – 1000	P = – 27923.7	My = –1173676	Mx = – 355027
				A= 534.6	Iy = 276010	Ix = 21276				

$$M_0 = -\frac{P}{A} = \frac{27923.7}{534.6} = 52.23$$

$$H_0 = -\frac{M_x}{I_x} = \frac{355027}{21276} = 16.69$$

$$V_0 = \frac{M_y}{I_y} = -\frac{1173676}{276010} = -4.25$$

Table 9.3 Computation of stress in analogous column

Point	x	y	M_s	$\frac{P}{A}$	$\frac{M_x}{I_x}y$	$\frac{M_y}{I_y}x$	M_i	$M = M_s - M_i$
A	50	19.84	−1000	−52.23	−331.13	−212.5	−595.83	−404.17
3	25	1.09	0	−52.23	−18.19	−106.25	−176.67	176.67
C	0	−5.16	0	−52.23	86.12	0	33.89	−33.89
3'	−25	1.09	0	−52.23	−18.19	106.25	35.83	−35.83
B	−50	19.84	0	−52.23	−331.13	+212.5	−170.86	170.86

Reactions at support A are given by

$$H_A = H_0 = 16.69 \text{ kN} \rightarrow$$
$$V_A = 50 + V_0 = 50 - 4.25 = 45.75 \text{ kN} \uparrow$$
$$M_A = -404.17 \text{ kNm hogging}$$

Reactions at support B are given by

$$H_B = H_0 = 16.69 \text{ kN} \leftarrow$$
$$V_B = V_0 = 4.25 \text{ kN} \uparrow$$
$$M_B = 170.8 \text{ kNm sagging}$$

The bending moment diagram is shown in Fig.9.9d.

(ii) Elastic Centre Method

The moment at any section in the arch can be determined knowing the values of M_0, H_0 and V_0 at the elastic centre as follows:

$$M = M_s + M_0 + H_0 y - V_0 x$$

Sagging moment is taken as positive and hogging moment is taken as negative.

$$M_A = -1000 + 52.23 + 16.69 \times 19.84 + 4.25 \times 50 = -404.17 \text{ kNm hogging}$$

$$M_3 = 0 + 52.23 + 16.69 \times 1.09 + 4.25 \times 25 = 176.67 \text{ kNm sagging}$$

$$M_C = 0 + 52.23 + 16.69 \times (-5.16) + 4.25 \times 0 = -33.89 \text{ kNm hogging}$$

Example 9.5

Reanalyze the circular arch of Example 9.2 if its two ends are fixed.

Solution

$$y_0 = \frac{\int_C^A y \frac{ds}{EI}}{\int_C^A \frac{ds}{EI}} \qquad (9.19)$$

$$R_C = \frac{\int_C^A (m_A - m_B)x \frac{ds}{EI}}{2\int_C^A \frac{x^2 ds}{EI}} \qquad (9.15)$$

$$H_C = \frac{\int_C^A (m_A + m_B)y_1 \frac{ds}{EI}}{2\int_C^A y_1^2 \frac{ds}{EI}} \quad \text{(Ignoring axial deformations)} \qquad (9.23)$$

$$M_C = \frac{\int_C^A (m_A + m_B) \frac{ds}{EI}}{2\int_C^A \frac{ds}{EI}} - H_C y_0 \qquad (9.22)$$

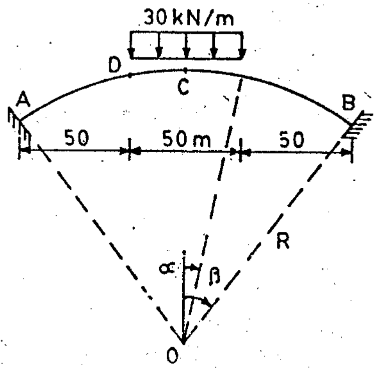


Fig. 9.10

Let C be the origin, $y = R - R\cos\theta$, $R = 125$ m

$\therefore \quad y = 125 - 125\cos\theta$

EI = constant, $\alpha = 0.2$ radian $= 11.54°$, $\sin\alpha = 0.2$, $\cos\alpha = 0.98$

$\beta = 0.6435 = 36.87°$, $\sin\beta = 0.6$, $\cos\beta = 0.8$

Let us first evaluate the various integrals used in these equations.

$$\int_C^A \frac{ds}{EI} = \int_0^\beta \frac{R\,d\theta}{EI} = \frac{125\,\beta}{EI} = \frac{80.37}{EI}$$

$$\int_C^A y\frac{ds}{EI} = \int_0^\beta (R - R\cos\theta)\frac{R\,d\theta}{EI} = \frac{125^2}{EI}\int_0^\beta (1-\cos\theta)\,d\theta = \frac{671.9}{EI}$$

$$\text{Elastic centre } y_0 = \frac{671.9/EI}{80.37/EI} = 8.36 \text{ m from C}$$

$$m_A = 30\frac{x^2}{2}, \qquad 0 < \theta \le \alpha$$
$$= 30 \times 25\,(R\sin\theta - 12.5), \quad \alpha \le \theta \le \beta$$

$$m_B = 30\frac{x^2}{2}, \qquad 0 < \theta \le \alpha$$
$$= 750\,(R\sin\theta - 12.5), \quad \alpha < \theta \le \beta$$

Since $m_A = m_B$, $R_C = 0$, $y_1 = (116.64 - R\cos\theta)$

y_1 when measured downward from the elastic centre is taken as positive

$$\int_C^A (m_A + m_B)\,y_1\frac{ds}{EI} = \int_0^\alpha 2\times 15x^2(116.64 - R\cos\theta)\frac{Rd\theta}{EI} +$$

$$\int_\alpha^\beta 2\times 750(R\sin\theta - 12.5)(116.64 - R\cos\theta)\frac{Rd\theta}{EI}$$

$$= \frac{174\times 10^5}{EI}$$

$$\int_C^A y_1^2\frac{ds}{EI} = \int_0^\beta (116.64 - R\cos\theta)^2\frac{Rd\theta}{EI} = \frac{R^3}{EI}\int_0^\beta (0.933 - \cos\theta)^2\,d\theta = \frac{4405}{EI}$$

$$\therefore \quad H_C = \frac{174\times 10^5}{2\times 4405} = 1975 \text{ kN}$$

$$\int_C^A (m_A + m_B)\frac{ds}{EI} = \int_0^\alpha 30R^2\sin^2\theta\frac{R\,d\theta}{EI} + \int_\alpha^\beta 2\times 750\,(R\sin\theta - 12.5)\frac{R\,d\theta}{EI}$$

$$= \frac{R^3}{EI}\int_0^\alpha 30\sin^2\theta\,d\theta + \int_\alpha^\beta (12\sin\theta - 1.2)\,d\theta = \frac{32.98\times 10^5}{EI}$$

$$M_C = \frac{32.98\times 10^5}{2\times 80.37} - 1975\times 8.36 = 4007 \text{ kNm (sagging)}$$

Now the support reactions can be easily calculated as the arch becomes statically determinate.

R_A = 750 kN ↑, H_A = 1975 kN →,

$M_x = 750\,(R\sin\theta - 12.5) - H_C\,y - M_C$

or, $M_A = 750\,(75 - 12.5) - 1975\times 25 - 4007 = -6507$ kNm sagging

Similarly, R_B = 750 kN ↑, H_B = 1975 kN ←, $M_B = -6507$ kNm sagging

The evaluation of integrals involve addition/ subtraction of quantities of nearly the same magnitude. Hence, there is a very large fluctuation in the value of the integrand depending upon the number of significant digits considered. It is recommended that all values be computed up to at least 5 or 6 significant digits on a calculator in the analysis of a circular arch with fixed ends.

Example 9.6

Reanalyze the circular arch of Example 9.5 using the column analogy method, and elastic centre method.

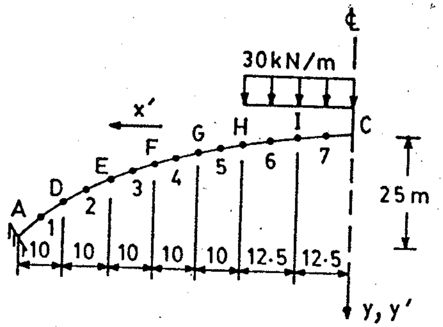
Solution

(i) Column Analogy Method

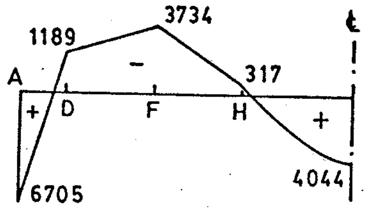
The circular arch is symmetric about the crown. It is divided in 7 segments as shown in Fig. 9.11a. The elastic centre is computed as shown in Table 9.5.

$$\text{The distance of elastic centre from C; } y_0 = \frac{674.329}{80.395} = 8.387 \text{ m}$$

The load on the analogous column and its properties are computed as shown in Table 9.6. The stresses in the analogous column are computed as shown in Table 9.7. Thus, net bending moment at different sections in the arch can be determined as shown in the same table.



(a) FIXED CIRCULAR ARCH



(b) MOMENT kNm

Fig. 9.11

Table 9.5 Computation of elastic centre

Segment	Point	x' m	y' m	$\frac{dy}{dx} = \tan\theta$	ds = dx sec θ	y' ds
AD	1	70	21.438	0.676	12.070	258.756
DE	2	60	15.341	0.547	11.400	174.887
EF	3	50	10.435	0.436	10.910	113.845
FG	4	40	6.573	0.338	10.556	69.384
GH	5	30	3.653	0.247	10.300	37.626
HI	6	18.75	1.414	0.152	12.643	17.877
IC	7	6.25	0.156	0.050	12.516	1.953
					Σ80.395	Σ674.329

Table 9.6 Computations of loads and inertia

Segment	Point	x m	y m	ds m	x^2 ds	y^2 ds	M_s	M_s ds	$M_s x$ ds $\times 10^5$	$M_s y$ ds $\times 10^5$
AD	1	70	13.051	12.070	59143.0	2055.8	− 43125	− 520518	− 364.36	− 67.93
DE	2	60	6.954	11.400	41040.0	551.3	− 35625	− 406125	− 243.67	− 28.24
EF	3	50	2.048	10.910	27275.0	45.7	− 28125	− 306840	− 153.42	− 6.28
FG	4	40	− 1.814	10.556	16889.6	34.7	− 20625	− 217718	− 87.08	+ 3.95
GH	5	30	− 4.734	10.300	9270.0	230.8	− 13125	− 135190	− 40.56	6.40
HI	6	18.75	− 6.973	12.643	4444.8	614.7	− 5273	− 66667	− 12.50	4.65
IC	7	6.25	− 8.231	12.516	488.9	847.9	− 585.9	− 7333	− 0.46	0.60
				Σ80.395	Σ158545	Σ4381	Σ146484	Σ−1660391	Σ−902.05	Σ− 86.85

$$A = 2 \times 80.395 = 160.79, \quad I_y = 2 \times 158545 = 317090$$

$$I_x = 2 \times 4381 = 8762, \quad P = -33.20 \times 10^5$$

$$M_x = -173.7 \times 10^5 \quad M_y = (-902.05 + 902.05) \times 10^5 = 0$$

$$\frac{P}{A} = \frac{-33.2 \times 10^5}{160.79} = -20653$$

$$\frac{M_x}{I_x} = \frac{-173.7 \times 10^5}{8762} = -1982$$

$$\frac{M_y}{I_y} = 0$$

Table 9.7 Stresses in analogous column

Point	x	y	Ms	$\frac{P}{A}$	$\frac{M_x}{I_x} y$	M_i	$M = M_s - M_i$
A	75	16.613	−46875	−20653	−32927	−53580	6705 sagging
F	45	− 0.006	−24375	−20653	11.89	−20640	−3734 hogging
H	25	− 5.850	− 9375	−20653	11595	− 9058	− 317 hogging
C	0	− 8.380	0	−20653	16609	− 4044	4044 sagging

(ii) Elastic Centre Method

Alternatively, the arch can be analyzed using the concept of elastic centre. Bending moment at any section in the arch is given by :

$$M = M_s + M_0 + H_0 y - V_0 x$$

Sagging moment is taken as positive and hogging moment is taken as negative.

$$M_0 = -\frac{P}{A} = 20653 \text{ kNm}$$

$$H_0 = -\frac{M_x}{I_x} = 1982 \text{ kN}$$

$$V_0 = 0$$

$$\therefore \quad M_A = -46875 + 20653 + 1982 \times 16.613 = -6705 \text{ kNm}$$

$$M_D = -35625 + 20653 + 1982 \times 6.954 = -1189 \text{ kNm}$$

$$M_F = -24375 + 20653 + 1982 \times (-0.006) = -3734 \text{ kNm}$$

$$M_C = 0 + 20653 + 1982 \times (-8.38) = 4044 \text{ kNm} \qquad \text{O.K.}$$

The bending moment diagram is shown in Fig. 9.11b.

9.8 INFLUENCE LINES FOR A HINGED ARCH

Influence lines are very useful in the design of arch bridges. An arch is economical because moments are reduced due to the presence of horizontal thrust. Influence lines for three hinged arch were drawn in section 13.9 of volume 1 of this book. Influence lines for horizontal thrust, radial shear and moment in a 2-hinged arch are drawn in the same manner. Let us draw influence lines for a parabolic arch. The following simplifications are introduced :

1. Moment of inertia at any section $I = I_O \sec\theta$
2. Effect of rib shortening is ignored.

Horizontal Thrust

$$H = \frac{\int_A^B \frac{\mu_x y\, ds}{EI}}{\int_A^B \frac{y^2\, ds}{EI}} = \frac{\int_A^B \mu_x y\, dx}{\int_A^B y^2\, dx} \tag{9.1}$$

since $I = I_O \sec\theta$ and $ds \cos\theta = dx$

Equation of the parabola with A as origin is (Fig. 9.2a)

$$y = \frac{4hx(L-x)}{L^2}$$

$$\therefore \quad \int_A^B y^2 dx = \int_0^L \frac{16h^2x^2(L-x)^2\, dx}{L^4} = \frac{8h^2L}{15}$$

If the unit load is at a distance z from A

$$\therefore \quad \mu_x = \frac{L-z}{L}x \qquad \text{for } x < z$$

and

$$\mu_x = \frac{z}{L}(L-x) \qquad \text{for } x > z$$

$$\int_A^B \mu_x y\, dx = \int_0^z \left(\frac{L-z}{L}\right) xy\, dx + \int_z^L \frac{z}{L}(L-x)\, y\, dx$$

$$= \frac{4h}{L^3}\int_0^z (L-z)x^2(L-x)\, dx + \frac{4h}{L^3}\int_z^L (L-x)^2 xz\, dx$$

$$= \frac{hz(L-z)(L^2+Lz-z^2)}{3L^2}$$

$\therefore$ Eq. 9.1 gives

$$H = \frac{hz(L-z)(L^2+Lz-z^2)}{3L^2 \times (8/15)h^2L}$$

or,

$$H = \frac{5}{8}\frac{L}{h}\left(\frac{z}{L} - \frac{2z^3}{L^3} + \frac{z^4}{L^4}\right) \tag{9.24}$$

Influence line for H can be plotted for different values of z from 0 to L as shown in Fig. 9.12 a. The maximum value of H occurs when z = L/2 and is equal to 25L/128h.

Normal Thrust Normal thrust at any section d is given as

$$T = H\cos\theta_x - V_x \sin\theta_x \tag{9.25}$$

where, θ = slope of the arch axis at D
V_x = vertical shear at D

The influence line for normal thrust is given by the influence line for horizontal thrust times cos θ minus the influence line for vertical shear times sin θ.

Radial Shear Radial shear at any section D is given by

$$Q = V_x \cos\theta_x + H \sin\theta_x \tag{9.26}$$

The influence line for radial shear is given by the sum of influence line for vertical shear times cos θ and the influence line for H times sin θ.

At crown $\theta = 0$, $\therefore \quad Q = V_x$

The resulting influence line is shown in Fig. 9.12b.

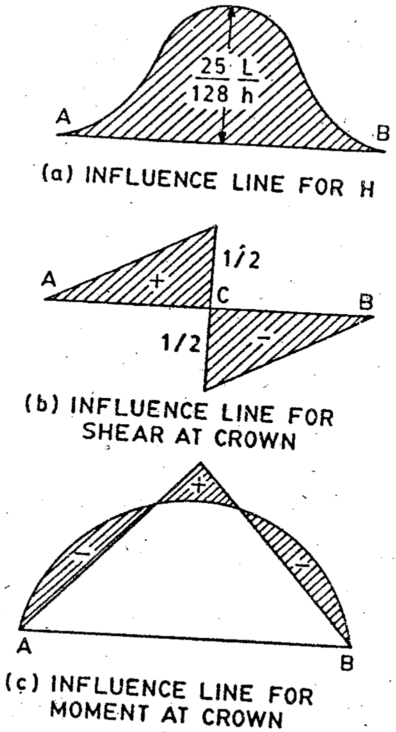


Fig. 9.12 Influence lines for a two-hinged arch - at crown

Moment Moment at any section is given as

$$M_x = \mu_x - Hy \tag{9.27}$$

Hence by plotting the influence line for μ_x, and subtracting from it y times the ordinate of the influence line for H, the influence line for M_x is obtained. Fig. 9.12c shows the influence line for M_x at the crown.

The influence lines at any other section can be plotted using Eqs. 9.24 to 9.26. The influence lines at left quarter span of parabolic arch for moment, normal thrust and radial shear are shown in Fig. 9.13 a, b and c. It is usual to check the arch rib at three sections : springing, quarter span and the crown. If the arch is safe at these three sections, it is assumed to be safe at all other points.

9.9 INFLUENCE LINES FOR A FIXED ARCH

The influence lines for moment, thrust and shear force at any section of a fixed arch can be easily calculated knowing the forces R_C, H_C and M_C. Thus, it is obvious that influence lines for these forces will have to be drawn first at the crown. The values of R_C, H_C and M_C will be calculated for different positions of a unit load on the arch from 0

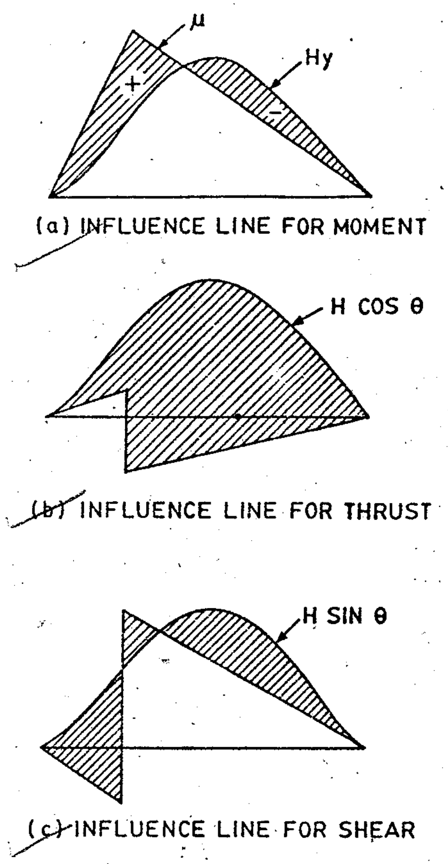


Fig. 9.13 Influence lines for a two-hinged arch - at quarter span

to L. Consider an arch shown in Fig. 9.14 where a unit load is placed at a distance z from the crown C.

Cantilever moment m_A at a point K $= (x - z)$ for $x > z$

and $= 0$ for $x < 0$

Cantilever moment m_B at a point $K' = 0$

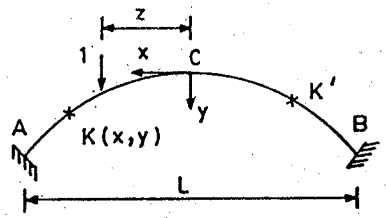


Fig. 9.14

$$\text{Eq. 9.15 gives,} \quad R_C = \frac{\int_C^A m_A x \frac{ds}{EI}}{2\int_C^A x^2 \frac{ds}{EI}} = \frac{\int_z^{L/2} (x-z)x \frac{ds}{EI}}{2\int_0^{L/2} x^2 \frac{ds}{EI}}$$

By putting different values of z between 0 and L/2, R_C can be evaluated. Due to symmetry, the same ordinates will apply when the load is on the right half arch but their sign will be opposite. Similarly, the values of H_C and M_C can be evaluated. The influence lines for Hc and Mc will be symmetrical about the crown. These influence lines are shown in Fig. 9.15 a, b and c. Once the influence line ordinates for R_C, H_C and M_C are known, the influence line ordinates for any force at any other section can be obtained by simple statics.

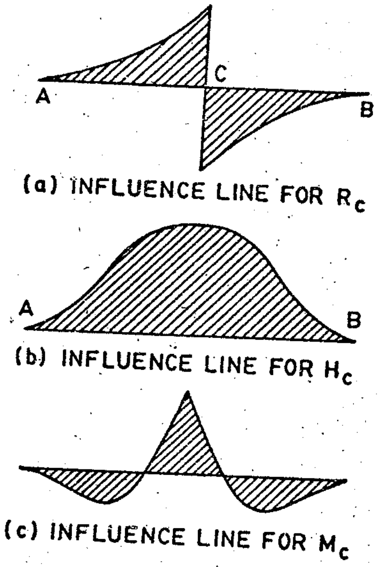


Fig. 9.15

PROBLEMS

9.1(a) A 2-hinged circular arch is shown in Fig. P9.1. Determine the support reactions due to a uniform load of 25 kN/m on the left half span and draw bending moment diagram. The cross-section of the arch is 0.75m × 1.5m deep throughout, Take $E = 2.5 \times 10^4$ kN/cm².
[H_A = 176.85 kN, R_A = 325.3 kN, R_B = 107.7 kN, H_C = 96.8 kNm sagging]

(ii) Reanalyze the circular arch if the uniform load of 25 kN/m is spread on the full horizontal span.
[H_A = 353.64 kN, R_A = 433.0 kN, M_C = 194.1 kNm sagging]

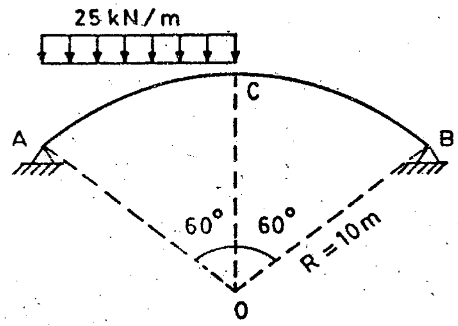


Fig. P9.1

9.2 If temperature of the arch of problem 9.1(a) drops by 50° C, determine the change in horizontal thrust (i) including axial deformation, (ii) excluding axial deformation. Take coefficient of thermal expansion $\alpha = 11 \times 10^{-6}$/°C

9.3 A two-hinged parabolic arch has a span of 150 m and a rise of 20 m. It carries a uniform load of 50 kN/m of horizontal length over its right half span. Draw the bending moment diagram assuming that the moment of inertia of the arch sections I varies as I_O sec θ.
[H_A = 3515 kN, R_A = 937.5 kN, Maximum moments = ∓ 17578 kNm at quarter spans].

9.4 Reanalyze problem 9.3 using the column analogy method if the ends of the arch are fully restrained.

9.5 For the arch in problem 9.3, draw influence lines for horizontal thrust H in the arch and the bending moment at a section at horizontal distance of 30 m from the left springing. Determine their maximum values if
(i) a concentrated load of 150 kN crosses the span.
(ii) a pair of concentrated loads of 75 kN and 125 kN spaced 10 m apart crosses the span.

9.6 Reanalyze problem 9.5 if the ends of the arch are fully restrained.

9.7 Determine the support reactions in a fixed parabolic arch carrying a vertical load of 100 kN at left quarter span if its span is 200 m and rise is 25 m. Take $I = I_O$ sec θ
[H_A = 105.63 kN, R_A = 84.15 kN, M_A = (–)1034 kNm hogging at left support, M_C = 1193 kNm sagging at left quarter span, M_B = – 795.5 kNm, hogging at right support]

9.8 Reanalyze problem 9.7 if the axis of the arch is circular.
[H_A = 106.80 kN, R_A = 84.12 kN, M_A = – 991.5 kNm hogging at left support, M = 1182 kNm sagging at left quarter span, M_B = – 832 kNm hogging at right support]

9.9(a) Determine the support reactions for a 2-hinged circular arch shown in Fig. P9.2. Also draw bending moment diagram. Take A = 1 m^2, I = 0.25 m^4 and E = 2 × 10^6 kN/m^2.
[H_A = 11.0 kN, R_A = 38.25 kN, M_C = – 13 kNm hogging, M_D = 12.34 kNm sagging, M_E = 15.11 kNm sagging, H_B = 11.0 kN, R_B = 1.75 kN]

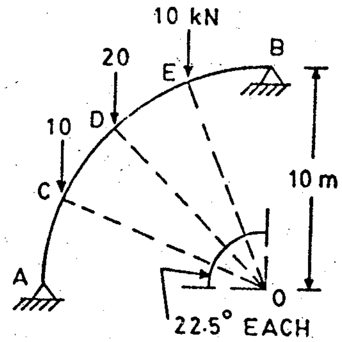


Fig. P9.2

(b) Reanalyze the circular arch if both its ends are fully restraint.
[H_A = 10.5 kN, R_A = 38.25 kN, M_A = 6.54 kNm sagging, M_C = 6.86 kNm hogging, M_D = 13.6 kNm sagging, M_E = 7.73 kNm sagging, M_B = 18.54 kNm hogging]

9.10(a) A 2-hinged parabolic arch segment shown in Fig. P9.3 is loaded with a concentrated load of 50 kN. Determine the support reactions and draw bending moment diagram. Take I = I_o sec θ, I_o = 0.25 m^4, E = 2.5 × 10^6 kN/m^2, A = 1.0 m^2.
[H_A = 77.38 kN, R_A = 59.1 kN ↑, R_B = – 9.1 kN ↓, H_B = 77.38 kNm , M_C = 80.5 kNm, sagging]

(b) Reanalyze the parabolic arch of Fig P9.3. if both its ends are fully restraint.
[H_A = 35.60 kN, R_A = 48.35 kN ↑, R_B = 1.65 kN ↑]

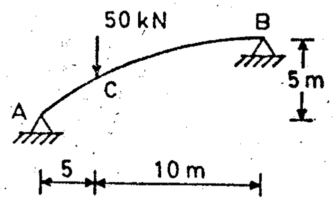


Fig. P9.3

PART 2

STIFFNESS METHODS

CHAPTER
ten

SLOPE-DEFLECTION METHOD

10.1 INTRODUCTION

The slope-deflection method can be used to analyze any statically indeterminate beam or rigid frame. It is based on the assumption that flexural deformations are predominant over axial and shear deformations. The rotational and translational deformations of the rigid joint are treated as unknowns, and their values are determined using the compatibility and equilibrium equations. A joint is said to be rigid when the angles between the members meeting at the joint remain unchanged after the application of the load. Thus, tangents to the various members meeting at a joint should undergo the same rotation, that is, the joint rotates as a whole. The end moments and shears of a member can be expressed in terms of the loads on the span, slopes and relative deflections of the two ends of the member. It will be shown that for each unknown joint rotation or deflection, there is a corresponding condition of joint equilibrium. Consequently, there are as many equations as there are unknown joint slopes and deflections. Thus, knowing the slopes and deflections, the end moments and shears can be determined.

The slope-deflection method was the forerunner of the modern stiffness matrix analysis. Unfortunately, it is not amenable to computer programming, hence it is now used only to understand the physical behaviour of simple problems.

10.2 DEVELOPMENT OF SLOPE-DEFLECTION EQUATIONS

Consider a member AB of a continuous beam or a rigid frame as shown in Fig.10.1. The member gets deformed as shown in the same figure under the application of the loads and deformations of the continuous beam or the frame. Let the end rotations be θ_A and θ_B both in the clockwise directions. The end B sinks by Δ relative to the end A such that the member rotates clockwise. Let the net end moments be M_A and M_B, again both clockwise. Thus, *clockwise rotations* and *clockwise moments* are considered to be *positive*. It is possible to express the two end moments acting on the member in terms of the two end rotations relative deflection Δ and the loads on the member.

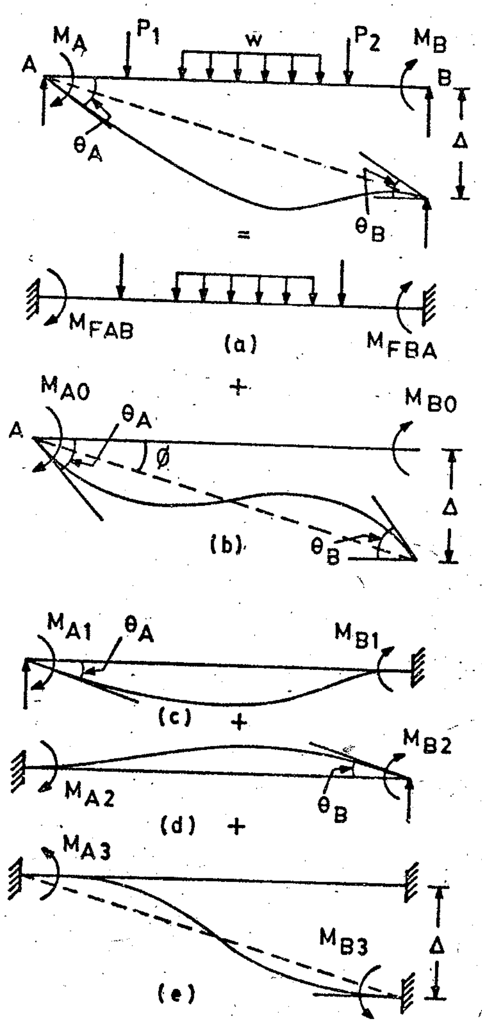


Fig. 10.1 Development of slope-deflection equation

The total deformations of this member may be separated into two parts shown in Figs.10.1a and b. The condition shown in Fig.10.1a is called the fixed end condition in which the moments M_{FAB} and M_{FBA} are capable of maintaining zero slopes at A and B with loads acting on the span AB. The condition shown in Fig.10.1b is called the free end condition in which the moments M_{AO} and M_{BO} are capable of maintaining the joint deformations θ_A, θ_B and Δ without any loads acting on the span AB. Thus,

$$M_A = M_{FAB} + M_{AO} \tag{10.1a}$$

$$M_B = M_{FBA} + M_{BO} \tag{10.1b}$$

The free end condition shown in Fig.10.1b can be further split in three parts as shown in Figs.10.1c, d and e. In Fig.10.1c, the end A is given a slope θ_A by applying a clockwise moment at end A while the end B is kept fixed. In Fig.10.1d, the end B is given a slope θ_B by applying a clockwise moment at end B while the end A is kept fixed. In Fig.10.1e, the end B is made to sink by Δ relative to A while the two ends are kept fixed. Thus, the end moments of Fig.10.1b are equal to the algebraic sum of the end moments shown in Figs.10.1c, d and e:

$$M_A = M_{FAB} + M_{A1} + M_{A2} - M_{A3} \tag{10.1c}$$

$$M_B = M_{FBA} + M_{B1} + M_{B2} - M_{B3} \tag{10.1d}$$

These end moments can be easily determined using the moment-area theorem, conjugate beam theorem or the unit load method. The fixed end moments M_{FAB} and M_{FBA} can be determined using any of the flexibility methods discussed earlier.

Using the conjugate beam method on the member shown in Fig.10.1c,

$$\frac{1}{EI}\left[M_{A1}\frac{L}{2}\times\frac{2}{3} - M_{B1}\frac{L}{2}\times\frac{1}{3}\right] = \theta_A \tag{10.2a}$$

and

$$\frac{1}{EI}\left[M_{A1}\frac{L}{2}\times\frac{1}{3} - M_{B1}\frac{L}{2}\times\frac{2}{3}\right] = 0 \tag{10.2b}$$

Solution of Eqs.10.2a and 10.2b gives,

$$M_{A1} = \frac{4EI}{L}\theta_A, \qquad M_{B1} = \frac{2EI}{L}\theta_A \tag{10.3}$$

Similarly, for the member shown in Fig.10.1d,

$$M_{A2} = \frac{2EI}{L}\theta_B, \qquad M_{B2} = \frac{4EI}{L}\theta_B \tag{10.4}$$

Using the moment-area theorem on the member shown in Fig.10.1e,

(i) net area of the moment diagram is equal to zero, because the total change of slope from A to B is zero, that is,

$$M_{A3} = M_{B3}$$

(ii) the moment of M/EI diagram about B should be equal to Δ, that is,

$$\frac{1}{EI}\left[\frac{1}{2}M_{A3}\frac{L}{2}\times\frac{5L}{6} - \frac{1}{2}M_{B3}\frac{L}{2}\times\frac{L}{6}\right] = \Delta$$

$$\text{or,}\quad M_{A3} = M_{B3} = \frac{6EI\Delta}{L^2} \tag{10.5}$$

Substituting the values of end moments in Eq.10.1c and d,

$$M_A = M_{FAB} + \frac{4EI}{L}\theta_A + \frac{2EI}{L}\theta_B - \frac{6EI\Delta}{L^2}$$

$$\text{or,}\quad M_A = M_{FAB} + \frac{2EI}{L}\left[2\theta_A + \theta_B - \frac{3\Delta}{L}\right] \tag{10.6a}$$

$$\text{and}\quad M_B = M_{FBA} + \frac{2EI}{L}\theta_A + \frac{4EI}{L}\theta_B - \frac{6EI\Delta}{L^2}$$

$$\text{or,}\quad M_B = M_{FBA} + \frac{2EI}{L}\left[\theta_A + 2\theta_B - \frac{3\Delta}{L}\right] \tag{10.6b}$$

Thus, if θ_A, θ_B and Δ are known, the end moments in any member AB can be determined using Eqs. 10.6. Typical values of the fixed end moments due to different loadings are shown in Appendix B.

10.3 EQUATIONS OF EQUILIBRIUM

At any joint of a statically indeterminate structure, two conditions must be satisfied :

(a) compatibility condition, and
(b) moment equilibrium.

The compatibility condition requires that angle between the members meeting at a joint before the application of the load must remain unchanged after the application of the load. This condition is satisfied by assuming the same value of θ for all members meeting at a joint. Thus there will be as many unknown θ's as the number of joints that can undergo a rotation. The condition of moment equilibrium is satisfied by making the sum of the end moments of all members meeting at any joint equal to zero. This provides as many linear simultaneous equations as the number of unknown θ's and Δ's.

The step-by-step procedure for the solution of any continuous beam or non-sway frame is as follows :

Step 1 - Determine the fixed end moments due to the applied loads on any member.

Step 2 - Write the slope-deflection equation for both ends of each member.

Step 3 - Write the moment equilibrium equations. For the end moments, substitute the slope-deflection equations developed in step 2 in terms of θ' and Δ's.

Step 4 - Solve the linear simultaneous equations for θ's and Δ's.

Step 5 - Substitute the values of θ's and Δ's in the slope-deflection equations to get the member end moments.

Step 6 - Draw free body diagram of each member and draw shear force diagrams.

Step 7 - Plot shear force, bending moment diagrams and the deflected shape of the structure.

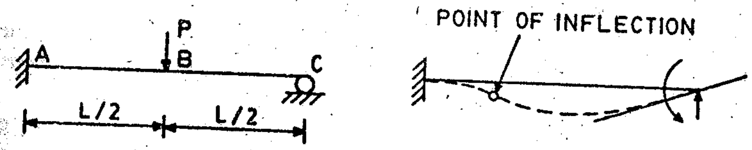
Deflected Shape

The deflected shape of the centre line of the structure helps in understanding its behaviour. The deflected shape follows the bending moment diagram. The slopes and deflection as calculated are plotted at the appropriate supports or joints by drawing tangents in the proper directions. The points of inflections are projected from the bending moment diagram. The elastic curve or the deflected shape can be interpolated through the supports or member joints and points of inflections.

10.4 BEAMS

Example 10.1

Analyze the propped cantilever beam shown in Fig.10.2a using the slope-deflection method. Draw bending moment diagram and deflected shape.



(a) (b) DEFLECTED SHAPE

Fig. 10.2

Solution

(a) *Fixed End Moments*

$$M_{FAC} = -\frac{Pab^2}{L^2} = -\frac{P\left(\frac{L}{2}\right)\left(\frac{L}{2}\right)^2}{L^2} = -\frac{PL}{8}$$

$$M_{FCA} = \frac{PL}{8}$$

(b) *Slope-Deflection Equations*

$$M_{AC} = M_{FAC} + \frac{2EI}{L}(2\theta_A + \theta_C)$$

$$= -\frac{PL}{8} + \frac{2EI}{L}\theta_C, \qquad \theta_A = 0 \text{ being a fixed end}$$

$$M_{CA} = \frac{PL}{8} + \frac{4EI}{L}\theta_C$$

(c) *Joint-Moment Equilibrium Equations*

$$\Sigma M_C = 0 \quad \text{or} \quad M_{CA} = 0 \quad \text{or} \quad \boxed{\theta_C = -\frac{PL^2}{32EI}}$$

(d) *Back- Substitution*

$$M_{AC} = -\frac{PL}{8} - \frac{PL}{16} = -\frac{3PL}{16}$$

$$M_{CA} = 0$$

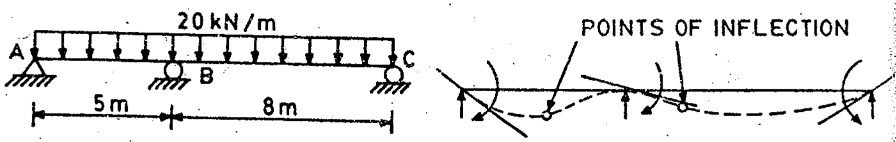
Free span moment at $L/2 = \frac{PL}{4}$, and Moment at $L/2$ due to continuity $= -\frac{3PL}{32}$

$\therefore$ Net moment at $L/2 = \frac{PL}{4} - \frac{3PL}{32} = \frac{5PL}{32}$ sagging

The bending moment diagram is shown in Fig. 3.1b. The deflected shape is shown in Fig. 10.2b.

Example 10.2

Analyze a two span continuous beam shown in Fig.10.3 using the slope-deflection method. Draw bending moment diagram and deflected shape. Take EI = constant.



(a) (b) DEFLECTED SHAPE

Fig. 10.3

Solution

(a) *Fixed End Moments*

$$M_{FAB} = -\frac{WL^2}{12} = -20 \times \frac{5^2}{12} = -41.67 \text{ kNm} \quad \text{and} \quad M_{FBA} = 41.67 \text{ kNm}$$

$$M_{FBC} = -\frac{WL^2}{12} = -20 \times \frac{8^2}{12} = -106.67 \text{kNm} \quad \text{and} \quad M_{FCB} = 106.67 \text{ kNm}$$

(b) *Slope-Deflection Equations*

$$M_{AB} = M_{FAB} + \frac{2EI}{5}(2\theta_A + \theta_B)$$

$$M_{AB} = -41.67 + 0.8\, EI\, \theta_A + 0.4\, EI\, \theta_B$$

$$M_{BA} = 41.67 + 0.4\, EI\, \theta_A + 0.8\, EI\, \theta_B$$

$$M_{BC} = -106.67 + \frac{2EI}{8}(2\theta_B + \theta_C)$$

$$= -106.67 + 0.5\, EI\, \theta_B + 0.25\, EI\, \theta_C$$

$$M_{CB} = 106.67 + 0.25\, EI\, \theta_B + 0.5\, EI\, \theta_C$$

(c) Joint-Moment Equilibrium Equations

$M_{AB} = 0$, or, $0.8\, EI\, \theta_A + 0.4\, EI\, \theta_B - 41.67 = 0$ (i)

$\Sigma M_B = 0$, or, $M_{BA} + M_{BC} = 0$

or, $0.4\, EI\, \theta_A + 0.8\, EI\, \theta_B + 41.67 + 0.5\, EI\, \theta_B + 0.25\, EI\, \theta_C - 106.67 = 0$

or, $0.4\, EI\, \theta_A + 1.3\, EI\, \theta_B + 0.25\, EI\, \theta_C - 65 = 0$ (ii)

$M_{CB} = 0$, or, $0.25\, EI\, \theta_B + 0.5\, EI\, \theta_C + 106.67 = 0$ (iii)

(d) *Matrix Solution*

Let us rearrange Eqs. (i) to (iii) in the matrix form :

$$EI \begin{bmatrix} 0.8 & 0.4 & 0 \\ 0.4 & 1.3 & 0.25 \\ 0 & 0.25 & 0.5 \end{bmatrix} \begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \end{Bmatrix} = \begin{Bmatrix} 41.67 \\ 65 \\ -106.67 \end{Bmatrix}$$

or,

$$\begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \end{Bmatrix} = \begin{Bmatrix} 2.087 \\ 100 \\ -263.3 \end{Bmatrix} \frac{1}{EI}$$

(e) *Back - Substitution*

On substituting the values of θ's in the slope-deflection equations :

$$M_{AB} = 0, \quad M_{BA} = -122.5 \text{ kNm}, \quad M_{BC} = 122.5 \text{ kNm} \quad \text{and} \quad M_{CB} = 0$$

The bending moment diagram is shown in Fig.3.2d and deflected shape is shown in Fig. 10.3b.

Example 10.3

Analyze the continuous beam shown in Fig.10.4a using the slope-deflection method and draw shear force and bending moment diagrams.

Solution

(a) *Fixed End Moments*

$$M_{FAC} = -P\frac{ab^2}{L^2} = -100 \times \frac{2.5^3}{5^2} = -62.5 \text{ kNm} = -M_{FCA}$$

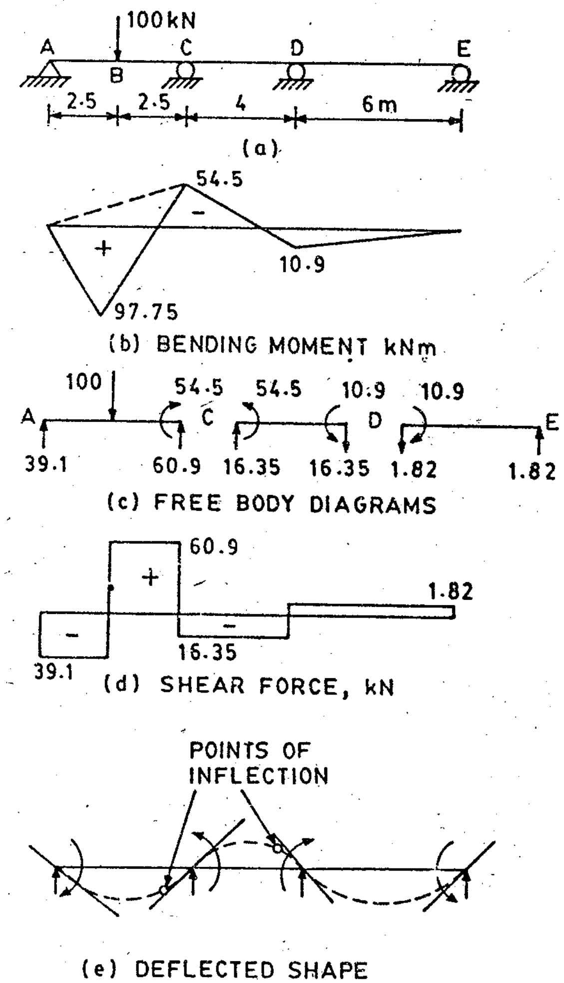


Fig. 10.4

(b) ***Slope-Deflection Equations***

$$M_{AC} = M_{FAC} + \frac{2EI}{L}(2\theta_A + \theta_C)$$

$$= -62.5 + \frac{2EI}{5}(2\theta_A + \theta_C) = 0.8\,EI\,\theta_A + 0.4\,EI\,\theta_B - 62.5$$

$$M_{CA} = 62.5 + \frac{2EI}{5}(\theta_A + 2\theta_C) = 0.4\,EI\,\theta_A + 0.8\,EI\,\theta_B + 62.5$$

$$M_{CD} = 0 + \frac{2EI}{4}(2\theta_C + \theta_D) = EI\,\theta_C + 0.5\,EI\,\theta_D$$

$$M_{DC} = 0 + \frac{2EI}{4}(\theta_C + 2\theta_D) = 0.5EI\,\theta_C + EI\,\theta_D$$

$$M_{DE} = 0 + \frac{2EI}{6}(2\theta_D + \theta_E) = 0.6667\,EI\,\theta_D + 0.3334\,EI\,\theta_E$$

$$M_{ED} = 0 + \frac{2EI}{6}(\theta_D + 2\theta_E) = 0.3334\,EI\,\theta_D + 0.6667\,EI\,\theta_E$$

(c) ***Joint-Moment Equilibrium Equations***

$M_{AC} = 0$, A being a simple support (i)

$M_{CA} + M_{CD} = 0$ (ii)

$M_{DC} + M_{DE} = 0$ (iii)

$M_{ED} = 0$, E being a simple support (iv)

$\Sigma M_A = 0$ gives, $0.8\,EI\theta_A + 0.4\,EI\theta_B - 62.5 = 0$

$\Sigma M_C = 0$ gives, $0.4\,EI\theta_A + 0.8\,EI\theta_C + 62.5 + EI\theta_C + 0.5\,EI\theta_D = 0$

or $0.4\,EI\theta_A + 1.8\,EI\theta_C + 0.5\,EI\theta_D + 62.5 = 0$

$\Sigma M_D = 0$ gives, $0.5\,EI\theta_C + EI\theta_D + 0.6667\,EI\theta_D + 0.3334\,EI\theta_E = 0$

$0.5\,EI\theta_C + 1.6667\,EI\theta_D + 0.3334\,EI\theta_E = 0$

$\Sigma M_E = 0$ gives, $0.3334\,EI\theta_D + 0.6667\,EI\theta_E = 0$

The four simultaneous equations may be rearranged in the matrix form :

$$EI\begin{bmatrix} 0.8 & 0.4 & 0 & 0 \\ 0.4 & 1.8 & 0.5 & 0 \\ 0 & 0.5 & 1.6667 & 0.3334 \\ 0 & 0 & 0.3334 & 0.6667 \end{bmatrix}\begin{Bmatrix}\theta_A \\ \theta_C \\ \theta_D \\ \theta_E\end{Bmatrix} = \begin{Bmatrix} 62.5 \\ -62.5 \\ 0 \\ 0 \end{Bmatrix}$$

or

$$\begin{Bmatrix}\theta_A \\ \theta_C \\ \theta_D \\ \theta_E\end{Bmatrix} = \begin{Bmatrix} 110.829 \\ -65.4 \\ 21.8 \\ -10.9 \end{Bmatrix}\frac{1}{EI}$$

Substituting the values of θ_A, θ_C, θ_D and θ_E in the slope-deflection equations,

$M_{AC} = 0$, $M_{CA} = 54.51$ kNm, $M_{CD} = -54.50$ kNm, $M_{DC} = -10.9$ kNm,
$M_{DE} = 10.9$ kNm, $M_{ED} = 0$

Free span moment in span AC at B $= \dfrac{PL}{4} = 125$ kNm. The bending moment diagram is shown in Fig. 10.4b. Shear force diagram can be drawn by first calculating the support reactions. The free body diagrams of each of the three spans are shown in Fig. 10.4c. The free body diagram gives, $R_C = 77.25$ kN and $R_D = -18.17$ kN.

The shear force diagram can now be easily drawn in Fig. 10.4d. The deflected shape is shown in Fig. 10.4e.

Example 10.4

Determine the reactions in the continuous beam due to a vertical settlement of 10 mm at the support B as shown in Fig. 10.5a. Draw shear force and bending moment diagrams. Take EI = constant, E = 200 GPa, I = 200 × 10^{-6} m^4.

Solution

(a) *Fixed End Moments*

There are no loads on the beam. Hence fixed end moments due to loads in each span are zero.

(b) *Slope-Deflection Equations*

$$M_{AB} = \frac{2EI}{5}\left(2\theta_A + \theta_B - \frac{3\times 0.01}{5}\right) = 0.8\,EI\,\theta_A + 0.4\,EI\,\theta_B - 0.0024EI$$

$$M_{BA} = \frac{2EI}{5}\left(\theta_A + 2\theta_B - \frac{3\times 0.01}{5}\right) = 0.4\,EI\,\theta_A + 0.8\,EI\,\theta_B - 0.0024EI$$

$$M_{BC} = \frac{2EI}{4}\left(2\theta_B + \theta_C + \frac{3\times 0.01}{4}\right) = EI\,\theta_B + 0.5\,EI\,\theta_C + 0.00375EI$$

$$M_{CB} = \frac{2EI}{4}\left(\theta_B + 2\theta_C + \frac{3\times 0.01}{4}\right) = 0.5\,EI\,\theta_B + EI\,\theta_C + 0.00375\,EI$$

$$M_{CD} = \frac{2EI}{6}(2\theta_C + \theta_D) = 0.6667\,EI\,\theta_C + 0.3334\,EI\,\theta_D$$

$$M_{DC} = \frac{2EI}{6}(\theta_C + 2\theta_D) = 0.3334\,EI\,\theta_C + 0.6667\,EI\,\theta_D$$

(c) *Joint-Moment Equilibrium Equations*

$M_{AB} = 0$, A being a simple support

$M_{BA} + M_{BC} = 0$

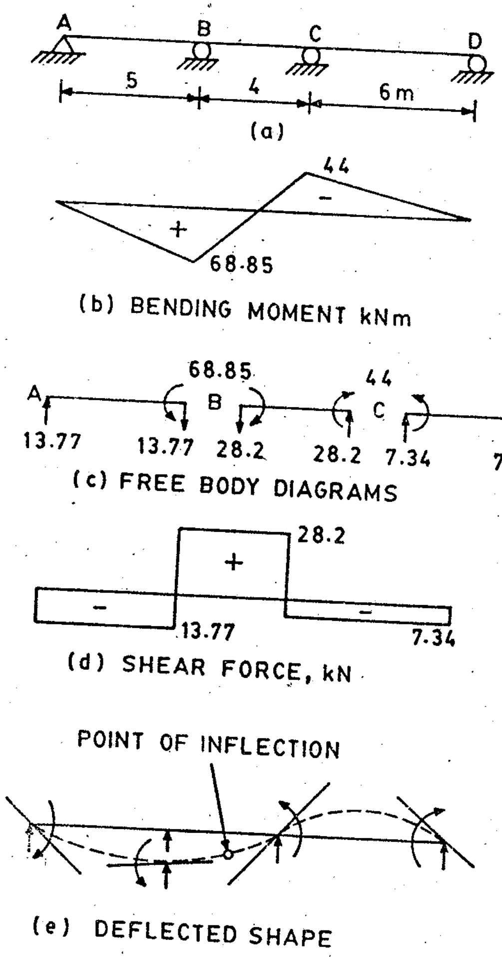


Fig. 10.5

$M_{CB} + M_{CD} = 0$

$M_{DC} = 0$, D being a simple support

$\Sigma M_A = 0$, gives, $0.8\,EI\,\theta_A + 0.4\,EI\,\theta_B - 0.0024\,EI = 0$

$\Sigma M_B = 0$, gives, $0.4\,EI\,\theta_A + 1.8\,EI\,\theta_B + 0.5\,EI\,\theta_C + 0.00135\,EI = 0$

$\Sigma M_C = 0$, gives,
$0.5\,EI\,\theta_B + 1.6667\,EI\,\theta_C + 0.3334\,EI\,\theta_D + 0.00375\,EI = 0$

$\Sigma M_D = 0$, gives, $0.3334\,EI\,\theta_C + 0.6667\,EI\,\theta_D = 0$

The four simultaneous equations may be rearranged in the matrix form :

$$\begin{bmatrix} 0.8 & 0.4 & 0 & 0 \\ 0.4 & 1.8 & 0.5 & 0 \\ 0 & 0.5 & 1.6667 & 0.3334 \\ 0 & 0 & 0.3334 & 0.6667 \end{bmatrix} \begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \theta_D \end{Bmatrix} = \begin{Bmatrix} 0.0024 \\ -0.00135 \\ -0.00375 \\ 0 \end{Bmatrix}$$

or,

$$\begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \theta_D \end{Bmatrix} = \begin{Bmatrix} 0.0035 \\ -0.0009 \\ -0.0022 \\ 0.0011 \end{Bmatrix}$$

Substituting the values of θ_A, θ_B, θ_C and θ_D in the slope-deflection equations gives :

$$M_{AB} = 0,\ M_{BA} = -68.8 \text{ kNm},\ M_{BC} = 68.9 \text{ kNm}$$

$$M_{CB} = 44 \text{ kNm},\ M_{CD} = -44 \text{ kNm, and } M_{DC} = 0$$

The free body diagram gives, $R_B = -41.97$ kN, and $R_C = 35.5$ kN

The bending moment diagram is shown in Fig. 10.5b. The shear force diagram can be drawn if the support reactions are known. The free body diagram is shown in Fig. 10.5c. The shear force diagram can be easily drawn as shown in Fig. 10.5d. The deflected shape is shown in Fig. 10.5e

10.5 FRAMES : NO SIDE SWAY

A frame tends to deflect horizontally under lateral loads or unsymmetrical vertical loads unless it is restrained to do so. Typical frames that are restrained to deflect horizontally, that is, with no side sway, are shown in Figs. 10.6a to d. In addition, if a frame is symmetrical with respect to its geometry, boundary conditions, and vertical loading, it will not undergo side sway as shown in Figs. 10.6e and f. The following examples illustrate the analysis of such frames.

Imp

Example 10.5

Analyze the frame shown in Fig. 10.7a using the slope-deflection method and draw bending moment diagram.

Solution

(a) *Fixed End Moments*

$$M_{FAB} = -\frac{wL^2}{12} = -50 \times \frac{5^2}{12} = -104.17 \text{ kNm}$$

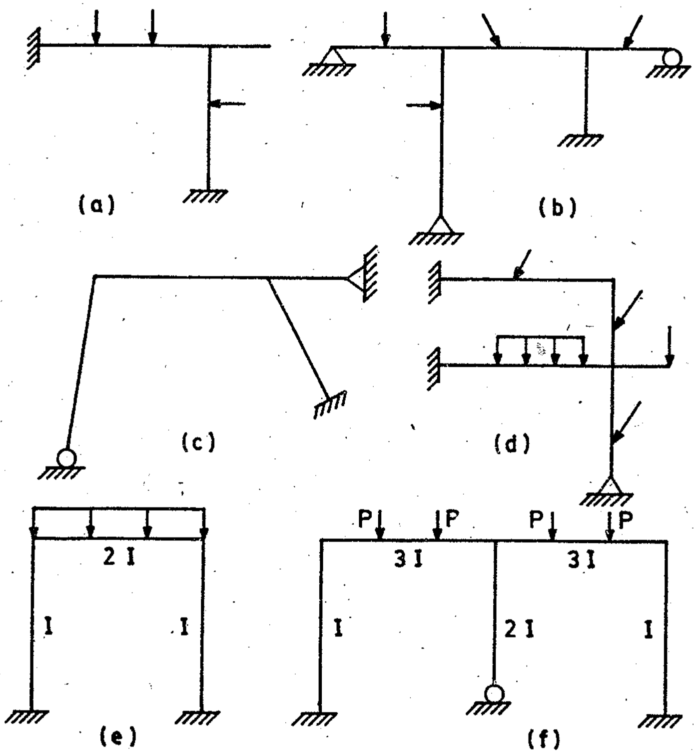


Fig. 10.6 Typical frames with no side sway

$$M_{FBA} = 104.17 \text{ kNm}$$

BC is a cantilever span. It is statically determinate. We do not write slope-deflection equations for such spans. The end moment at B in the span BC is 15 kNm, anticlockwise.

(b) *Slope-Deflection Equations*

$$M_{AB} = -M_{FAB} + \frac{2E(2I)}{5}(2\theta_A + \theta_B)$$

$$= -104.17 + \frac{4}{5} EI(\theta_B) \quad \text{since } \theta_A = 0$$

$$M_{BA} = 104.17 + \frac{4}{5} EI(2\theta_B),\quad M_{BC} = -15 \text{ kNm}$$

$$M_{BD} = 0 + \frac{2EI}{4}(2\theta_B + \theta_D) = EI\,\theta_B \qquad \text{since } \theta_D = 0$$

$$M_{DB} = \frac{2EI}{4}(\theta_B)$$

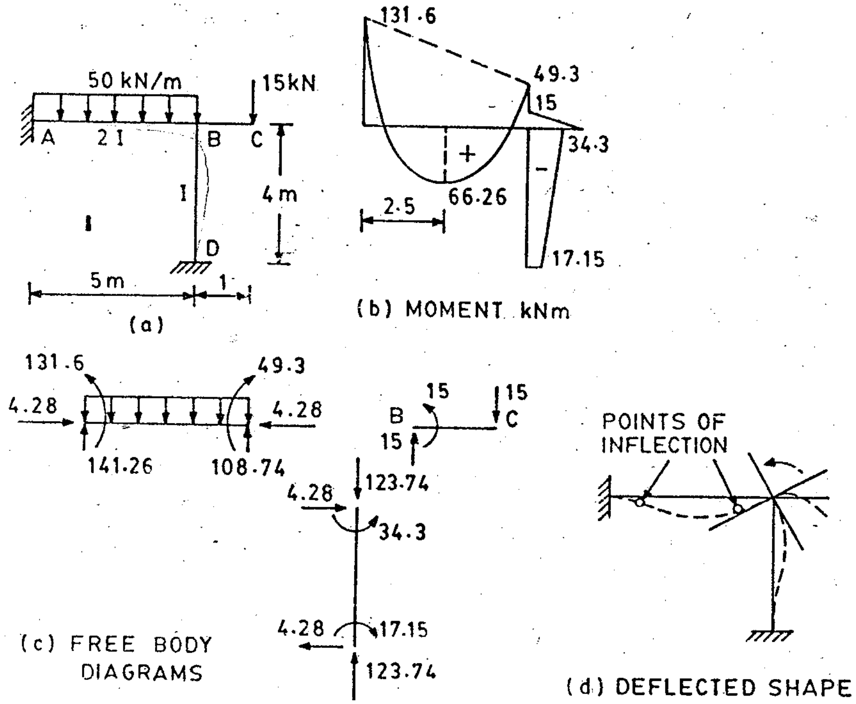


Fig. 10.7

(c) *Joint-Moment Equilibrium Equations*

$$\Sigma M_B = 0, \quad \text{or,} \quad M_{BA} + M_{BC} + M_{BD} = 0$$

or, $\quad 104.17 + 1.6\,E\,I\theta_B - 15 + EI\,\theta_B = 0$

or, $\quad 2.6\,E\,I\,\theta_B = -\,89.17$

or, $\quad \theta_B = -\dfrac{34.3}{EI}$

(d) *Back-Substitution*

$M_{AB} = -\ 131.60$ kNm, $M_{BA} = 49.30$ kNm, $M_{BC} = -\ 15$ kNm,

$M_{BD} = -\ 34.3$ kNm, $M_{DB} = 17.15$ kNm

Free span moment at L/2 in the span AB $= \dfrac{wL^2}{8} = 50 \times \dfrac{5^2}{8} = 156.26$ kNm

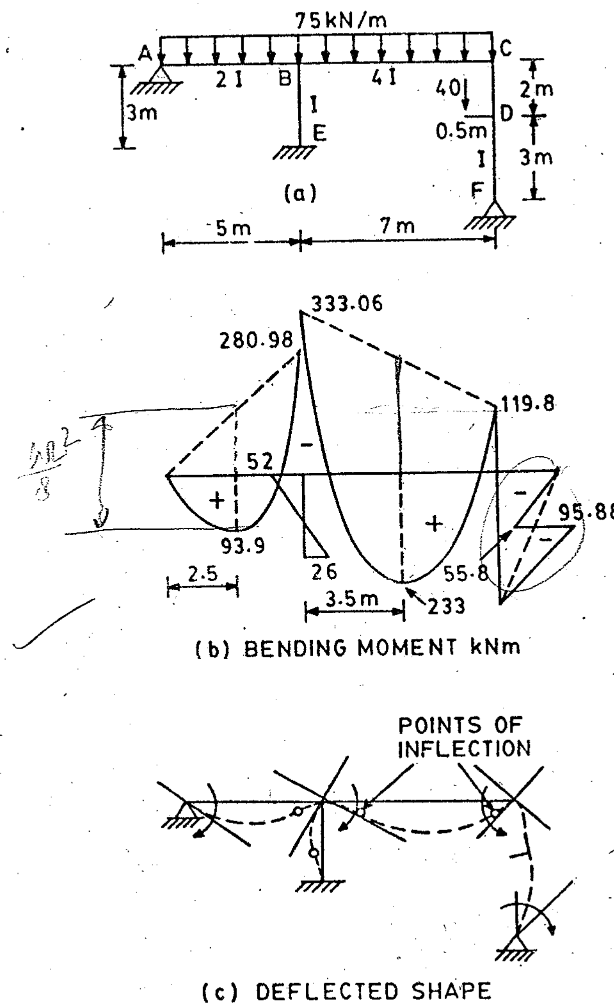
Hogging moment at L/2 $= 0.5\,(130.6 + 49.3) = 90$ kNm

$\therefore$ Net sagging moment at L/2 $= +\ 66.26$ kNm

The resulting bending moment and free body diagrams are shown in Fig. 10.7 b and c. The shear force diagram can be now easily drawn. The deflected shape is shown in Fig. 10.7d.

Example 10.6

Analyze the frame shown in Fig. 10.8a using the slope-deflection method and draw bending moment diagram.



449.43

140

234.375

280.98

Fig. 10.8

Solution

(a) *Fixed End Moments*

$$M_{FAB} = -\frac{wL^2}{12} = -75\times\frac{5^2}{12} = -156.25 \text{ kNm}$$

$$M_{FBA} = 156.25 \text{ kNm}$$

$$M_{FBC} = -\frac{wL^2}{12} = -75\times\frac{7^2}{12} = -306.25 \text{ kNm}$$

$$M_{FCB} = 306.25 \text{ kNm}$$

$$M_{FCF} = -\frac{Mb}{L^2}(2a-b) = -40\times\frac{3}{5^2}(2\times2-3) = -4.8 \text{ kNm}$$

$$M_{FFC} = -\frac{Ma}{L^2}(2b-a) = -40\times\frac{2}{5^2}(2\times3-2) = -12.8 \text{ kNm}$$

(b) *Slope-Deflection Equations*

$$M_{AB} = M_{FAB} + \frac{2E(2I)}{5}(2\theta_A + \theta_B)$$

$$M_{AB} = -156.25 + 1.6\,EI\,\theta_A + 0.8\,EI\,\theta_B$$

$$M_{BA} = 156.25 + 0.8\,EI\,\theta_A + 1.6\,EI\,\theta_B$$

$$M_{BC} = M_{FBC} + \frac{2E(4I)}{7}(2\theta_B + \theta_C)$$

$$= -306.25 + 2.28\,EI\,\theta_B + 1.14\,EI\,\theta_C$$

$$M_{CB} = 306.25 + 1.14\,EI\,\theta_B + 2.28\,EI\,\theta_C$$

$$M_{BE} = 0 + \frac{2EI}{3}(2\theta_B + \theta_E) = 1.34\,EI\,\theta_B + 0.67\,EI\,\theta_E$$

$$M_{EB} = 0.67\,EI\,\theta_B + 1.34\,EI\,\theta_E$$

$$M_{CF} = -4.8 + \frac{2E(2I)}{5}(2\theta_C + \theta_F) = -4.8 + 1.6\,EI\,\theta_C + 0.8\,EI\,\theta_F$$

$$M_{FC} = -12.8 + 0.8\,EI\,\theta_C + 1.6\,EI\,\theta_F$$

$\theta_E = 0$ being a fixed support

(c) *Joint-Moment Equilibrium Equations*

$\Sigma M_A = 0$, or, $1.6\,EI\,\theta_A + 0.8\,EI\,\theta_B - 156.25 = 0$ (i)

$\Sigma M_B = 0$, or, $M_{BA} + M_{BC} + M_{BE} = 0$

or, $0.8\,EI\,\theta_A + 5.22\,EI\,\theta_B + 1.14\,EI\,\theta_C - 150 = 0$ (ii)

$\Sigma M_C = 0$, or, $M_{CB} + M_{CF} = 0$

or, $1.14\,EI\,\theta_B + 3.88\,EI\,\theta_C + 0.8\,EI\,\theta_F + 301.45 = 0$ (iii)

$\Sigma M_{FC} = 0$, or, $0.8\,EI\,\theta_C + 1.6\,EI\,\theta_F - 12.8 = 0$ (iv)

(d) ***Matrix Solution***

$$EI\begin{bmatrix}1.6 & 0.8 & 0 & 0\\ 0.8 & 5.22 & 1.14 & 0\\ 0 & 1.14 & 3.88 & 0.8\\ 0 & 0 & 0.8 & 1.6\end{bmatrix}\begin{Bmatrix}\theta_A\\ \theta_B\\ \theta_C\\ \theta_F\end{Bmatrix} = \begin{Bmatrix}156.25\\ 150\\ -301.45\\ 12.8\end{Bmatrix}$$

or,

$$\begin{Bmatrix}\theta_A\\ \theta_B\\ \theta_C\\ \theta_F\end{Bmatrix} = \begin{Bmatrix}78.23\\ 38.84\\ -101.2\\ 58.59\end{Bmatrix}\frac{1}{EI}$$

(e) *Back-Substitution*

$M_{AB} = 0$, $M_{BA} = 280.98$ kNm, $M_{BC} = -333.06$ kNm, $M_{BE} = 52.04$ kNm,

$M_{CB} = 119.80$ kNm, $M_{CF} = -119.80$ kNm, $M_{FC} = 0$.

The bending moment diagram is shown in Fig.10.8b and deflected shape in Fig. 10.8c.

10.6 FRAMES : WITH SIDE SWAY

In practice there are many frames that undergo lateral displacements under vertical loads alone or under lateral loads. There will be additional joint translation at each storey of the frame in addition to joint rotations. Additional equilibrium equations are required to solve for each joint translation. Typical frames that can undergo lateral displacements are shown in Fig. 10.9. A frame can undergo lateral sway if there is unsymmetry with respect to geometry as in Fig. 10.9a or unsymmetry with respect to loading as in Fig. 10.9b. The frame shown in Fig. 10.9c can have two independent lateral joint translations at the top of the two columns. The additional equilibrium equations are obtained by considering the shear equilibrium in any storey of the frame. In the six storey frame of Fig. 10.9e, there will be six additional shear equilibrium equations.

In order to analyze a frame with side sway, it is essential to be able to draw its deflected shape. The following rules are very helpful in sketching the deflected shape to find the relation between the lateral displacements of various members meeting at a joint :

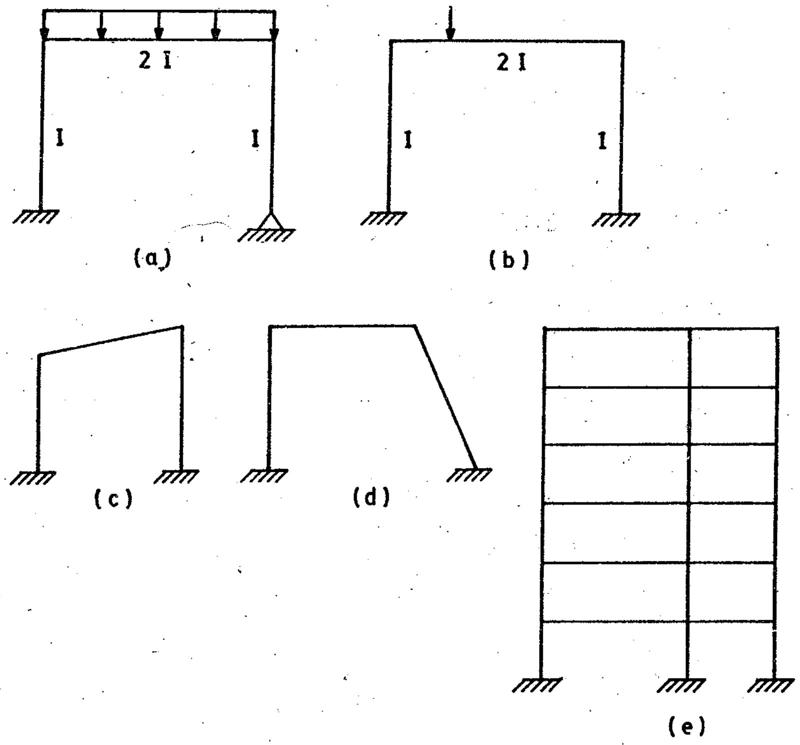


Fig. 10.9 Typical frames with side sway

Rule 1 : The frame undergoes small displacement.
Rule 2 : Axial deformations are not permitted.
Rule 3 : The joints are not allowed to rotate while angle between members at a rigid joint remains unchanged.
Rule 4 : The member ends can only displace in a direction perpendicular to itself.
Rule 5 : The distance between the member ends does not decrease due to curvature of the member.

Consider a single storey frame shown in Fig. 10.10a. The free body diagrams of the columns are shown in Fig. 10.10b and that of the girder in Fig. 10.10c with respect to shear. There are three columns with six joints and each column end will have certain shear. The shear equilibrium equation can be written as follows :

$$H_1 + H_3 + H_5 + P_1 + P_2 + P_3 = 0 \tag{10.7}$$

where, $H_1 = \dfrac{M_1 + M_2 - P_1 a - P_2 b}{L_1}$, $\qquad H_3 = \dfrac{M_4}{L_2}$

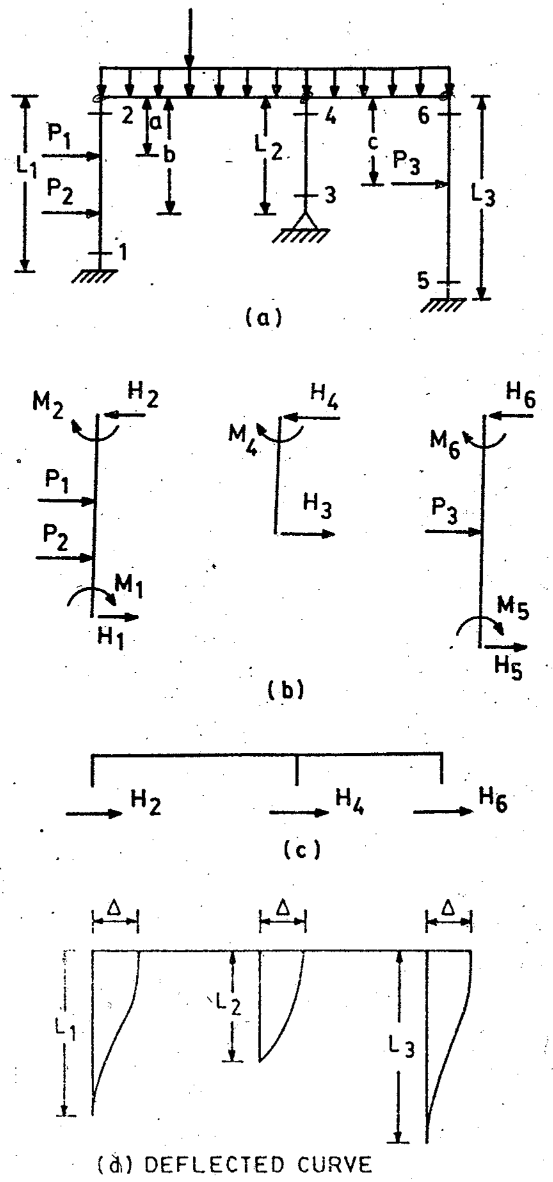


Fig. 10.10 Single storey frame with side sway- free body diagram

$$H_5 = \frac{M_5 + M_6 - P_3 c}{L_3}$$

Alternatively,

The shear equilibrium equation or simply the shear equation may be written as :

$$H_2 + H_4 + H_6 = 0 \tag{10.8}$$

where, $H_2 = \dfrac{M_1 + M_2 + P_1(L_1 - a) + P_2(L_1 - b)}{L_1}$, $H_4 = \dfrac{M_4}{L_2}$

$$H_6 = \frac{M_5 + M_6 + P_3(L_3 - c)}{L_3}$$

Upon simplification, Eqs. 10.7 and 10.8 turn out to be identical. Thus, the shear equation helps determine the lateral sway Δ at the level of the beam assuming there is no axial deformation in the frame.

Consider a two storey-two bay frame shown in Fig. 10.11 a. Its deflected shape is

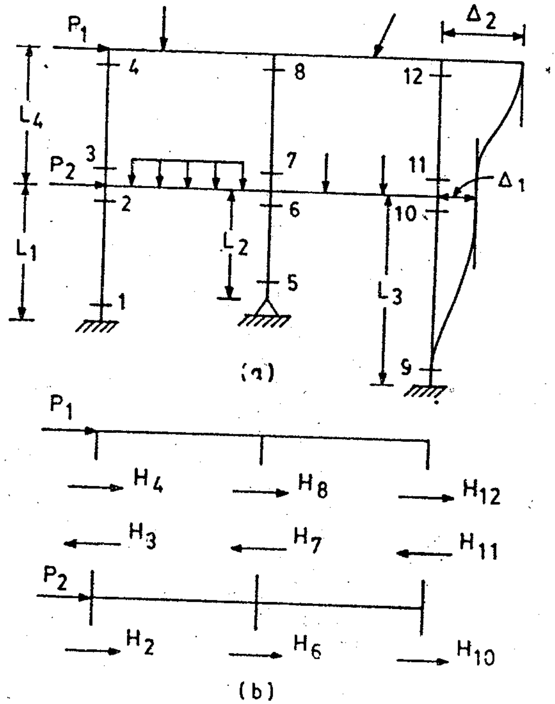


Fig. 10.11 Two storey frame with side sway- free body diagram

also shown in the same figure. At the first floor it deflects by Δ_1, while at the second floor, its total sway is Δ_2 and net sway is $\Delta_2 - \Delta_1$. It is assumed that displacements Δ_1 and Δ_2 are small and axial deformations are neglected. Since the top of the columns are in the same horizontal line, the points 2, 6 and 10 will have the same horizontal displacement Δ_1, and the points 4, 8 and 12 will have the same horizontal displacement Δ_2. Each storey of the frame is cut through an imaginary plane as shown in Fig. 10.11b. There are two unknown lateral floor displacements Δ_1 and Δ_2, and two additional shear equations are required. The shear equations are :

$$H_4 + H_8 + H_{12} + P_1 = 0 \tag{10.9a}$$

and
$$H_2 + H_6 + H_{10} + P_1 + P_2 = 0 \tag{10.9b}$$

where, $H_4 = \dfrac{M_3 + M_4}{L_4}$, $H_8 = \dfrac{M_7 + M_8}{L_4}$, $H_{12} = \dfrac{M_{11} + M_{12}}{L_4}$

$$H_2 = \frac{M_1 + M_2}{L_1},\quad H_6 = \frac{M_5 + M_6}{L_2},\quad H_{10} = \frac{M_9 + M_{10}}{L_3}$$

The following examples illustrate the application of the slope deflection method for the analysis of general frames.

Example 10.7

Analyze the frame shown in Fig. 10.12a using the slope-deflection method and draw bending moment diagram and its elastic curve.

Solution

(a) *Fixed End Moments*

$$M_{FBC} = -\frac{wL^2}{12} = -50 \times \frac{6^2}{12} = -150 \text{ kNm},\quad M_{FCB} = 150 \text{ kNm}$$

$$M_{FCE} = -150 \text{ kNm},\quad M_{FEC} = 150 \text{ kNm}$$

(b) *Slope-Deflection Equations*

The frame is symmetrical with respect to geometry, loading and boundary conditions, hence, the lateral sway may be taken as zero. Nevertheless, the sway is taken as non-zero which later turns out to be zero.

$$M_{AB} = 0 + \frac{2EI}{4.5}\left(2\theta_A + \theta_B - \frac{3\Delta}{4.5}\right)$$

$$= 0.89\,EI\,\theta_A + 0.45\,EI\,\theta_B - 0.29\,EI\,\Delta$$

$$M_{BA} = 0 + 0.45\,EI\,\theta_A + 0.89\,EI\,\theta_B - 0.29\,EI\,\Delta$$

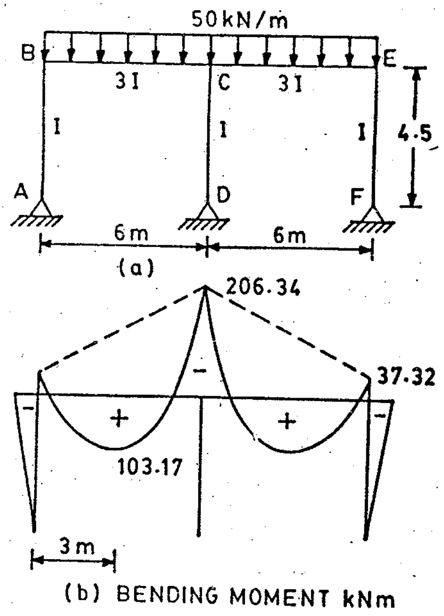


Fig. 10.12

$$M_{BC} = -150 + \frac{2E(3I)}{6}(2\theta_B + \theta_C)$$

$$= -150 + 2EI\theta_B + EI\theta_C$$

$$M_{CB} = 150 + EI\theta_B + 2EI\theta_C$$

$$M_{CE} = -150 + 2EI\theta_C + EI\theta_E$$

$$M_{EC} = 150 + EI\theta_C + 2EI\theta_E$$

$$M_{DC} = 0 + \frac{2EI}{4.5}\left(2\theta_D + \theta_C - \frac{3\Delta}{4.5}\right)$$

$$= 0.89\,EI\theta_D + 0.45\,EI\theta_C - 0.29\,EI\Delta$$

$$M_{CD} = 0 + 0.45\,EI\theta_D + 0.89\,EI\theta_C - 0.29\,EI\Delta$$

$$M_{FE} = 0.89\,EI\theta_F + 0.45\,EI\theta_E - 0.29\,EI\Delta$$

$$M_{EF} = 0.45\,EI\theta_F + 0.89\,EI\theta_E - 0.29\,EI\Delta$$

(c) *Joint-Moment Equilibrium Equations*

$$M_{AB} = 0, \quad \text{or,} \quad 0.89\,EI\theta_A + 0.45\,EI\theta_B - 0.29\,EI\Delta = 0 \qquad \text{(i)}$$

$$\Sigma M_B = 0, \quad \text{or,} \quad M_{BA} + M_{BC} = 0$$

$$0.45\,EI\theta_A + 2.89\,EI\theta_B + EI\theta_C - 0.29\,EI\Delta - 150 = 0 \qquad \text{(ii)}$$

$$\Sigma M_C = 0, \quad \text{or,} \quad M_{CB} + M_{CE} + M_{CD} = 0$$

$$EI\theta_B + 4.89\,EI\theta_C + 0.45EI\theta_D + EI\theta_E - 0.29\,EI\Delta = 0 \qquad \text{(iii)}$$

$$M_{DC} = 0, \quad \text{or,} \quad 0.45\,EI\theta_C + 0.89\,EI\theta_D - 0.29\,EI\Delta = 0 \qquad \text{(iv)}$$

$$\Sigma M_E = 0, \quad \text{or,} \quad M_{EC} + M_{EF} = 0$$

or, $$EI\theta_C + 2.89\,EI\theta_E + 0.45\,EI\theta_F - 0.29\,EI\Delta + 150 = 0 \qquad \text{(v)}$$

$$M_{FE} = 0, \quad \text{or,} \quad 0.45\,EI\theta_E + 0.89\,EI\theta_F - 0.29\,EI\Delta = 0 \qquad \text{(vi)}$$

Shear equation gives,

$$\boxed{H_A + H_D + H_F = 0}$$

or, $$\boxed{M_{AB} + M_{BA} + M_{DC} + M_{CD} + M_{FE} + M_{EF} = 0}$$

or, $$1.34\,EI\,[\theta_A + \theta_B + \theta_C + \theta_D + \theta_E + \theta_F] - 1.74\,EI\Delta = 0 \qquad \text{(vii)}$$

(d) *Matrix Solution*

$$EI\begin{bmatrix} 0.89 & 0.45 & 0 & 0 & 0 & 0 & -0.29 \\ 0.45 & 2.89 & 1 & 0 & 0 & 0 & -0.29 \\ 0 & 1 & 4.89 & 0.45 & 1 & 0 & -0.29 \\ 0 & 0 & 0.45 & 0.89 & 0 & 0 & -0.29 \\ 0 & 0 & 1 & 0 & 2.89 & 0.45 & -0.29 \\ 0 & 0 & 0 & 0 & 0.45 & 0.89 & -0.29 \\ 1.34 & 1.34 & 1.34 & 1.34 & 1.34 & 1.34 & -1.74 \end{bmatrix}\begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \theta_D \\ \theta_E \\ \theta_F \\ \Delta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 150 \\ 0 \\ 0 \\ -150 \\ 0 \\ 0 \end{Bmatrix}$$

or, $$\begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \theta_D \\ \theta_E \\ \theta_F \\ \Delta \end{Bmatrix} = \begin{Bmatrix} -28.49 \\ 56.34 \\ 0 \\ 0 \\ -56.34 \\ 28.49 \\ 0 \end{Bmatrix}\frac{1}{EI}$$

(e) *Back-Substitution*

$$M_{AB} = 0,\ M_{BA} = 37.32 \text{ kNm},\ M_{BC} = -37.32 \text{ kNm}$$

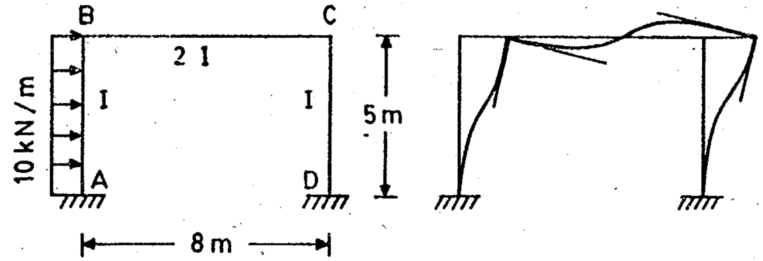
$$M_{CB} = 206.34 \text{ kNm},\ M_{CD} = 0,\ M_{CE} = -206.34 \text{ kNm}$$

$$M_{EC} = 37.32 \text{ kNm},\ M_{EF} = -37.32 \text{ kNm}$$

The free body diagrams and the bending moment diagram are shown in Figs. 10.12b and c.

Example 10.8

Analyze the frame shown in Fig. 10.13 using the slope-deflection method. Draw bending moment diagram and elastic curve.



(a) PORTAL FRAME (b) ELASTIC CURVE

Fig. 10.13

Solution

(a) *Fixed End Moments*

$$M_{FAB} = -\frac{wL^2}{12} = -10\times\frac{5^2}{12} = -20.84 \text{ kNm}$$

$$M_{FBA} = 20.84 \text{ kNm}$$

(b) *Slope-Deflection Equations*

$$M_{AB} = -20.84 + \frac{2EI}{5}\left(2\theta_A + \theta_B - \frac{3\Delta}{5}\right)$$

$$= -20.84 + 0.4\,EI\,\theta_B - 0.24\,EI\,\Delta, \qquad \theta_A = 0 \text{ being at a fixed end}$$

$$M_{BA} = 20.84 + 0.8\,EI\,\theta_B - 0.24\,EI\,\Delta$$

$$M_{BC} = 0 + \frac{2E(2I)}{8}(2\theta_B + \theta_C) = EI\,\theta_B + 0.5\,EI\,\theta_C$$

$$M_{CB} = 0.5\,EI\,\theta_B + EI\,\theta_C$$

$$M_{CD} = 0 + \frac{2EI}{5}\left(2\theta_C + \theta_D - \frac{3\Delta}{5}\right) \qquad \theta_D = 0 \text{ being at a fixed end}$$

$$\therefore \quad M_{CD} = 0.8\,EI\,\theta_C - 0.24\,EI\,\Delta,$$

$$M_{DC} = 0.4\,EI\,\theta_C - 0.24\,EI\,\Delta$$

(c) *Joint-Moment Equilibrium Equations*

$$\Sigma M_B = 0, \quad \text{or,} \quad M_{BA} + M_{BC} = 0$$

$$1.8\,EI\,\theta_B + 0.5\,EI\,\theta_C - 0.24\,EI\,\Delta + 20.84 = 0 \qquad \text{(i)}$$

$$\Sigma M_C = 0, \quad \text{or,} \quad M_{CB} + M_{CD} = 0$$

$$0.5\,EI\,\theta_B + 1.8\,EI\,\theta_C - 0.24\,EI\,\Delta = 0 \qquad \text{(ii)}$$

There are three unknowns, hence there should be three independent equations to determine them. The third equation can be obtained using the shear criteria. (Fig. not shown).

$$R_{Ax} + R_{Dx} + 10 \times 5 = 0,$$

or, $$\left[\frac{M_{AB} + M_{BA}}{5} - 25\right] + \frac{M_{DC} + M_{CD}}{5} + 50 = 0$$

or, $$1.2\,EI\,\theta_B + 1.2\,EI\,\theta_C - 0.96\,EI\,\Delta + 125 = 0 \qquad \text{(iii)}$$

(d) *Matrix Solution*

$$EI\begin{bmatrix} 1.8 & 0.5 & -0.24 \\ 0.5 & 1.8 & -0.24 \\ 1.2 & 1.2 & -0.96 \end{bmatrix}\begin{Bmatrix} \theta_B \\ \theta_C \\ \Delta \end{Bmatrix} = \begin{Bmatrix} -20.84 \\ 0 \\ -125 \end{Bmatrix}$$

or, $$\begin{Bmatrix} \theta_B \\ \theta_C \\ \Delta \end{Bmatrix} = \begin{Bmatrix} 4.24 \\ 20.27 \\ 160.84 \end{Bmatrix}\frac{1}{EI}$$

(e) *Back-Substitution*

Back substituting the values of slopes and deflection in the slope-deflection equations give the values of end moments.

$$M_{AB} = -57.75 \text{ kNm},\quad M_{BA} = -14.37 \text{ kNm},\quad M_{BC} = 14.38 \text{ kNm}$$

$$M_{CB} = 22.39 \text{ kNm},\quad M_{CD} = -22.38 \text{ kNm},\quad M_{DC} = -30.49 \text{ kNm},$$

Check $R_{Dx} = -10.58$ kN O. K.

This is the same result as obtained by the strain energy method in Ex. 5.7.

Example 10.9

Figure 10.14 shows a portal frame with hinged supports A and F. Draw the shear force and bending moment diagrams and elastic curve of the frame.

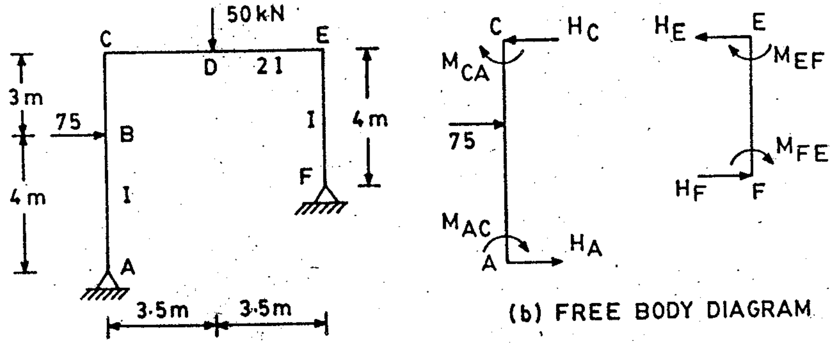


Fig. 10.14

Solution

(a) *Fixed End Moments*

$$M_{FAC} = -\frac{P\,ab^2}{L^2} = -75\times\frac{4\times 3^2}{7^2} = -55.10 \text{ kNm}$$

$$M_{FCA} = \frac{P\,a^2b}{L^2} = 75\times\frac{4^2\times 3}{7^2} = 73.47 \text{ kNm}$$

$$M_{FCE} = -50\times\frac{3.5^3}{7^2} = -43.75 \text{ kNm}$$

$$M_{FEC} = 43.75 \text{ kNm}$$

$$M_{FEF} = 0 = M_{FFE}$$

(b) *Slope-Deflection Equations*

Member AC

$$M_{AC} = M_{FAC} + \frac{2EI}{L}\left(2\theta_A + \theta_C - \frac{3\Delta}{L}\right)$$

$$= -55.10 + \frac{2EI}{7}\left(2\theta_A + \theta_C - \frac{3\Delta}{7}\right)$$

or, $$M_{AC} = \frac{4}{7}EI\theta_A + \frac{2}{7}EI\theta_C - \frac{6}{49}EI\Delta - 55.10$$

$$M_{CA} = 73.47 + \frac{2EI}{7}\left(\theta_A + 2\theta_C - \frac{3\Delta}{7}\right)$$

$$= \frac{2}{7}EI\theta_A + \frac{4}{7}EI\theta_C - \frac{6}{49}EI\Delta + 73.47$$

Member CE

$$M_{CE} = -43.75 + \frac{2E(2I)}{7}(2\theta_C + \theta_E)$$

$$= \frac{8}{7}EI\theta_C + \frac{4}{7}EI\theta_E - 43.75$$

$$M_{EC} = 43.75 + \frac{2E(2I)}{7}(\theta_C + 2\theta_E)$$

$$= \frac{4}{7}EI\theta_C + \frac{8}{7}EI\theta_E + 43.75$$

Member EF

$$M_{EF} = 0 + \frac{2EI}{4}\left(2\theta_E + \theta_F - \frac{3\Delta}{4}\right)$$

$$= EI\theta_E + \frac{EI}{2}\theta_F - \frac{3}{8}EI\Delta$$

$$M_{FE} = 0 + \frac{2EI}{4}\left(\theta_E + 2\theta_F - \frac{3\Delta}{4}\right)$$

$$= \frac{EI}{2}\theta_E + EI\theta_F - \frac{3}{8}EI\Delta$$

(c) *Joint -Moment Equilibrium Equations*

$M_{AC} = 0$ (i)

$M_{CA} + M_{CE} = 0$ (ii)

$M_{EC} + M_{EF} = 0$ (iii)

$M_{FE} = 0$ (iv)

The shear condition is :

$H_A + H_F + 75 = 0$ (v)

$M_{AC} = 0$, gives

$0.5714\,EI\,\theta_A + 0.2857\,EI\,\theta_C - 0.1224\,EI\,\Delta - 55.10 = 0$ (i)

$M_{CA} + M_{CE} = 0$, gives

$$0.2857EI\theta_A + 1.7142EI\theta_C + 0.5714EI\theta_E - 0.1224EI\Delta + 29.72 = 0 \quad \text{(ii)}$$

$M_{EC} + M_{EF} = 0$, gives

$$0.5714EI\theta_C + 2.1428EI\theta_E + 0.5000EI\theta_F - 0.375EI\Delta + 43.75 = 0 \quad \text{(iii)}$$

$M_{FE} = 0$, gives

$$0.5EI\theta_E + EI\theta_F - 0.3750EI\Delta = 0 \quad \text{(iv)}$$

The shear equation gives, (Fig.10.14b)

$$H_A = \frac{M_{AC}+M_{CA}}{7} - 75\times\frac{3}{7}, \text{ and } H_F = \frac{M_{EF}+M_{FE}}{4}$$

$$0.0408EI\theta_A + 0.0816EI\theta_C + 0.25EI\theta_E + 0.125EI\theta_F - 0.1112EI\Delta + 53.3547 = 0 \quad \text{(v)}$$

These five simultaneous equations may be arranged in the matrix form :

$$EI\begin{bmatrix} 0.5714 & 0.2857 & 0 & 0 & -0.1224 \\ 0.2857 & 1.7142 & 0.5714 & 0 & -0.1224 \\ 0 & 0.5714 & 2.1428 & 0.5 & -0.375 \\ 0 & 0 & 0.5 & 1 & -0.375 \\ 0.0408 & 0.0816 & 0.25 & 0.125 & -0.1112 \end{bmatrix}\begin{Bmatrix}\theta_A \\ \theta_C \\ \theta_E \\ \theta_F \\ \Delta\end{Bmatrix} = \begin{Bmatrix} 55.10 \\ -29.72 \\ -43.75 \\ 0 \\ -53.3547 \end{Bmatrix}$$

or,

$$\begin{Bmatrix}\theta_A \\ \theta_C \\ \theta_E \\ \theta_F \\ \Delta\end{Bmatrix} = \begin{Bmatrix} 415.581 \\ -26.808 \\ 126.368 \\ 472.06 \\ 1427.32 \end{Bmatrix}\frac{1}{EI}$$

(d) *Back-Substitution*

Substituting the values of θ_A, θ_C, θ_E, θ_F and Δ in the slope deflection equations give the values of end moments.

$M_{AC} = 0$, $M_{CA} = 2.175$ kNm, $M_{CE} = -2.17$ kNm

$M_{EC} = 172.85$ kNm, $M_{EF} = -172.85$ kNm, $M_{FE} = 0$

$H_A = -31.80$ kN←, $H_F = -43.20$ kN←, $R_A = 0.62$ kN ↑, $R_F = 49.38$ kN ↑

This is the same result as obtained by the consistent deformation method in Ex. 3.11.

Example 10.10

Figure 10.15a shows a portal frame with fixed supports A and F. Draw the shear force and bending moment diagrams and elastic curve of the frame.

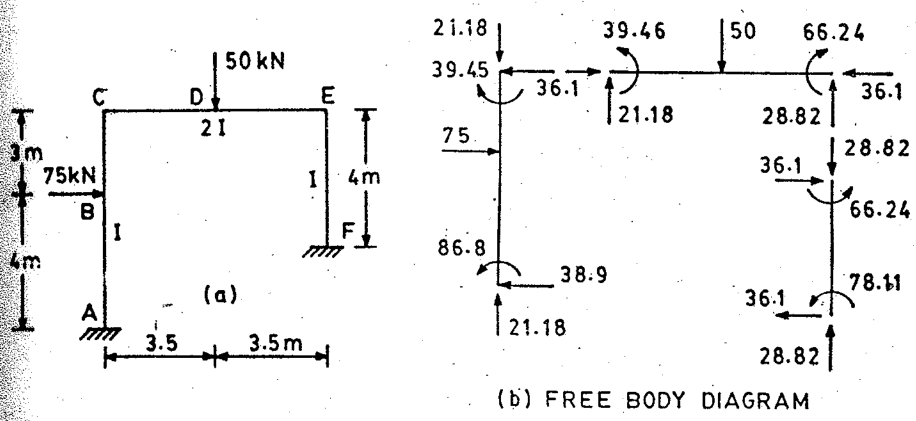


Fig. 10.15

Solution

(a) *Fixed End Moments*

These are same as given in Ex. 10.9.

$\theta_A = 0 = \theta_F$ due to fixity of support

(b) *Joint-Moment Equilibrium Equations*

$\Sigma M_C = 0$, or, $M_{CA} + M_{CE} = 0$ (i)

$\Sigma M_E = 0$, or, $M_{EC} + M_{EF} = 0$ (ii)

The shear condition is (as in Ex. 10.9) :

$H_A + H_F + 75 = 0$ (iii)

$\Sigma M_C = 0$ gives,

$$1.7142EI\theta_C + 0.5714EI\theta_E - 0.1224EI\Delta + 29.72 = 0 \quad \text{(i)}$$

$\Sigma M_E = 0$ gives,

$$0.5714EI\theta_C + 2.1428EI\theta_E - 0.375EI\Delta + 43.75 = 0 \quad \text{(ii)}$$

Shear equation gives,

$$0.1224EI\theta_C + 0.375EI\theta_E - 0.2225EI\Delta + 45.4812 = 0 \quad \text{(iii)}$$

The three simultaneous equations may be arranged in the matrix form :

$$EI\begin{bmatrix}1.7142 & 0.5714 & -0.1224\\ 0.5714 & 2.1428 & -0.375\\ 0.1224 & 0.375 & -0.2225\end{bmatrix}\begin{Bmatrix}\theta_C\\ \theta_E\\ \Delta\end{Bmatrix}=\begin{Bmatrix}-29.72\\ -43.75\\ -45.4812\end{Bmatrix}$$

or,

$$\begin{Bmatrix}\theta_C\\ \theta_E\\ \Delta\end{Bmatrix}=\frac{1}{EI}\begin{Bmatrix}-8.117\\ 23.741\\ 239.957\end{Bmatrix}$$

(c) *Back Substitution*

Substituting the values of θ_C, θ_E and Δ in the slope deflection equations gives

$M_{AC} = -86.8$ kNm, $M_{CA} = 39.45$ kNm, $M_{CE} = -39.46$ kNm

$M_{EC} = 66.24$ kNm, $M_{EF} = -66.24$ kNm, $M_{FE} = -78.11$ kNm

The values of reactions can be obtained from the free body diagrams shown in Figs. 10.15b.

$H_A = -38.90$ kN←, $H_F = -36.1$ kN←, $R_A = 21.18$ kN ↑, $R_F = 28.82$ kN ↑

This is the same result as obtained by the consistent deformation method in Ex. 3.12.

Imp

Example 10.11

Analyze all reactions induced in a rigid portal frame as shown in Fig. 10.16a due to the vertical settlement of support D by 10 mm and its clockwise rotation by 0.004 radian. Take E = 200 GPa, $I = 300 \times 10^{-6}$ m^4.

Solution

(a) *Slope-Deflection Equations*

The members AB and DC sway to right by an amount x (Fig. 10.16b).

$$M_{AB} = \frac{2EI}{7}\left(2\theta_A + \theta_B - \frac{3x}{7}\right) = 0.2857\,EI\,\theta_B - 0.1224\,EI\,x$$

$$M_{BA} = \frac{2EI}{7}\left(\theta_A + 2\theta_B - \frac{3x}{7}\right) = 0.5714\,EI\,\theta_B - 0.1224\,EI\,x$$

Due to the settlement of support D, member BC sways clockwise by the same amount, y = 10 mm.

$$M_{BC} = \frac{2E(2I)}{7}\left(2\theta_B + \theta_C - \frac{3y}{7}\right) = 1.1428\,EI\,\theta_B + 0.5714\,EI\,\theta_C - 0.2449\,EI\,y$$

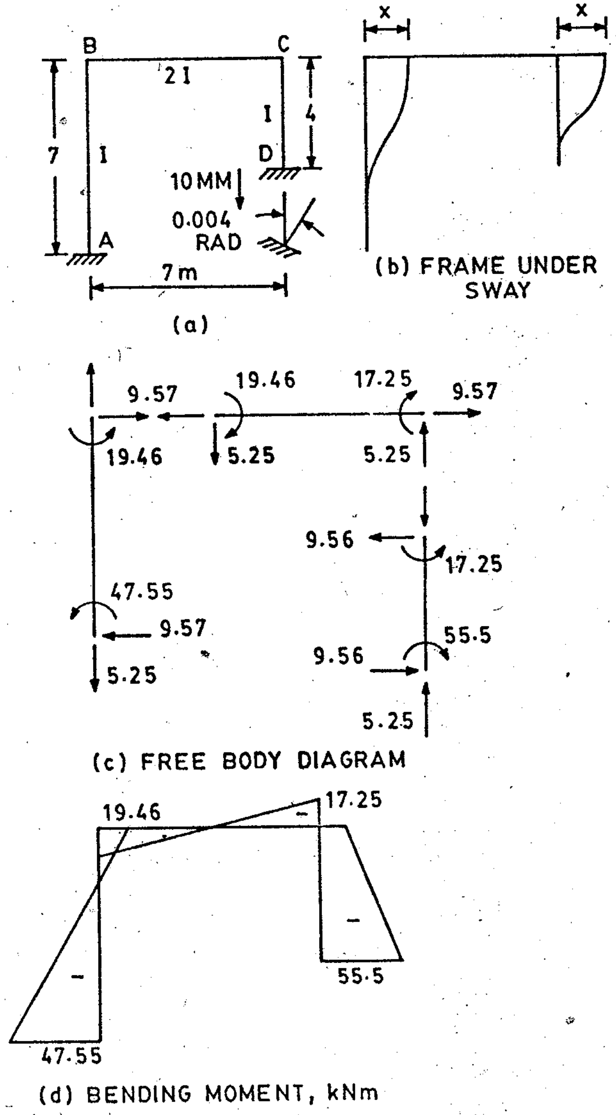


Fig. 10.16 Frame with support settlement

$$M_{CB} = \frac{4EI}{7}\left(\theta_B + 2\theta_C - \frac{3y}{7}\right) = 0.5714\,EI\,\theta_B + 1.1428\,EI\,\theta_C - 0.2449\,EI\,y$$

$$M_{CD} = \frac{2EI}{4}\left(2\theta_C + \theta_D - \frac{3x}{4}\right) = EI\,\theta_C + 0.5\,EI\,\theta_D - 0.375\,EI\,x$$

$$M_{DC} = \frac{2EI}{4}\left(\theta_C + 2\theta_D - \frac{3x}{4}\right) = 0.5\,EI\theta_C + EI\theta_D - 0.375\,EIx$$

Given : y = 10 mm = 0.01 m, and θ_D = 0.004

(b) *Joint-Moment Equilibrium Equations*

$$M_{BA} + M_{BC} = 0 \qquad \text{(i)}$$

$$M_{CB} + M_{CD} = 0 \qquad \text{(ii)}$$

$$H_A + H_D = 0$$

or,

$$\frac{M_{AB} + M_{BA}}{7} + \frac{M_{CD} + M_{DC}}{4} = 0 \qquad \text{(iii)}$$

$\Sigma M_B = 0$ gives,

$$1.7142\,EI\theta_B + 0.5714\,EI\theta_C - 0.1224\,EIx - 0.2449 \times 0.01 \times EI = 0$$

$\Sigma M_C = 0$ gives,

$$0.5714\,EI\theta_B + 2.1428\,EI\theta_C - 0.375\,EIx - 0.00045\,EI = 0$$

Shear equation gives,

or,

$$0.1224\,EI\theta_B + 0.357\,EI\theta_C - 0.2224\,EIx + 0.0015\,EI = 0$$

The three simultaneous equations can be arranged as :

$$\begin{bmatrix} 1.7142 & 0.5714 & -0.1224 \\ 0.5714 & 2.1428 & -0.375 \\ -0.1224 & -0.375 & 0.2224 \end{bmatrix} \begin{Bmatrix} \theta_B \\ \theta_C \\ x \end{Bmatrix} = \begin{Bmatrix} 0.002449 \\ 0.00045 \\ 0.0015 \end{Bmatrix}$$

or,

$$\begin{Bmatrix} \theta_B \\ \theta_C \\ x \end{Bmatrix} = \frac{1}{EI} \begin{Bmatrix} 0.001639 \\ 0.00157 \\ 0.0103 \end{Bmatrix}$$

Substituting the values of θ_B, θ_C and x in the slope deflection equations

M_{AB} = − 47.55 kNm, M_{BA} = − 19.45 kNm, M_{BC} = 19.47 kNm

M_{CB} = 17.30 kNm, M_{CD} = − 17.20 kNm, M_{DC} = 55.5 kNm

The free body and the bending moment diagrams are shown in Figs. 10.16 c and d.

Example 10.12

Analyze the two storey portal frame shown in Fig. 10.17a using the slope-deflection method and draw bending moment diagram and elastic curve.

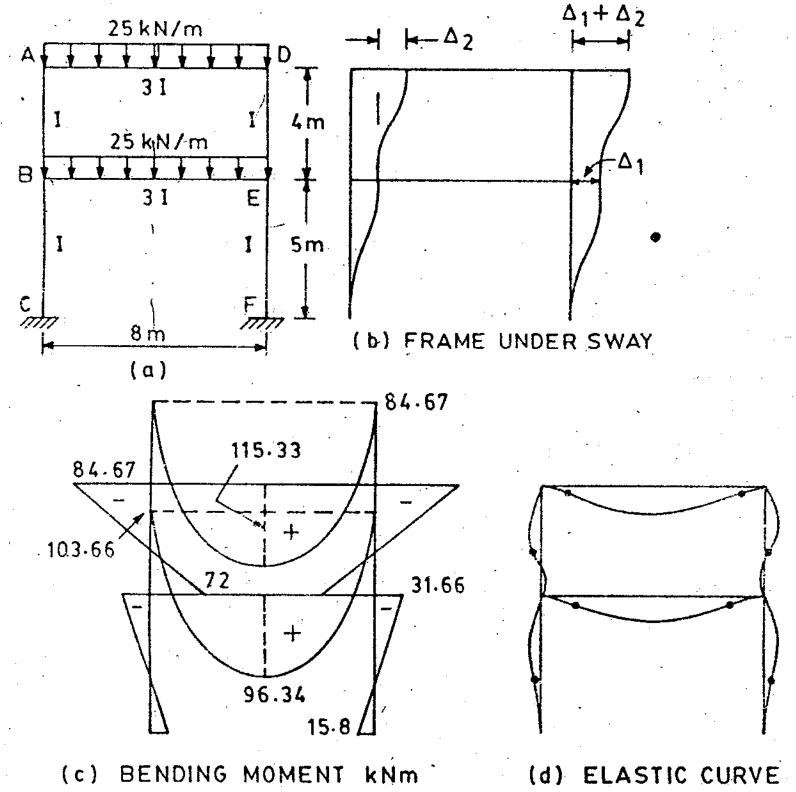


Fig. 10.17

Solution

(a) *Fixed End Moments*

$$M_{FAD} = -\frac{wL^2}{12} = -25 \times \frac{8^2}{12} = -133.4 \text{ kNm}$$

M_{FDA} = 133.4 kNm, M_{FBE} = − 133.4 kNm, M_{FEB} = 133.4 kNm

(b) *Slope-Deflection Equations*

The frame is symmetrical about its centre line with respect to geometry, vertical loads and boundary conditions. Hence, lateral sway will be zero. Nevertheless, it is taken as non-zero at each floor level to illustrate the point (Fig. 10.17b).

$$M_{CB} = 0 + \frac{2EI}{5}\left(2\theta_C + \theta_B - \frac{3\Delta_1}{5}\right), \ \theta_C = 0, \text{ C being a fixed end}$$

$$= 0.4\,EI\theta_B - 0.24\,EI\Delta_1$$

$$M_{BC} = 0.8\,EI\,\theta_B - 0.24\,EI\,\Delta_1$$

$$M_{BA} = 0 + \frac{2EI}{4}\left(2\theta_B + \theta_A - \frac{3\Delta_2}{4}\right) = EI\,\theta_B + 0.5EI\,\theta_A - 0.375\,EI\,\Delta_2$$

$$M_{AB} = 0.5\,EI\,\theta_B + EI\,\theta_A - 0.375\,EI\,\Delta_2$$

$$M_{AD} = -133.4 + \frac{2E(3I)}{8}(2\theta_A + \theta_D) = -133.4 + 1.5\,EI\,\theta_A + 0.75EI\,\theta_D$$

$$M_{DA} = 133.4 + 0.75EI\,\theta_A + 1.5\,EI\,\theta_D$$

$$M_{BE} = -133.4 + \frac{2E(3I)}{8}(2\theta_B + \theta_E) = -133.4 + 1.5\,EI\,\theta_B + 0.75EI\,\theta_E$$

$$M_{EB} = 133.4 + 0.75EI\,\theta_B + 1.5\,EI\,\theta_E$$

$$M_{DE} = 0 + \frac{2EI}{4}\left(2\theta_D + \theta_E - \frac{3\Delta_2}{4}\right) = EI\,\theta_D + 0.5\,EI\,\theta_E - 0.375\,EI\,\Delta_2$$

$$M_{ED} = 0.5\,EI\,\theta_D + EI\,\theta_E - 0.375\,EI\,\Delta_2$$

$$M_{EF} = 0 + \frac{2EI}{5}\left(2\theta_E + \theta_F - \frac{3\Delta_1}{5}\right) = 0.8\,EI\,\theta_E - 0.24\,EI\,\Delta_1$$

$$M_{FE} = 0.4\,EI\,\theta_E - 0.24\,EI\,\Delta_1$$

(c) *Joint-Moment Equilibrium Equations*

$\Sigma M_A = 0$, or, $M_{AB} + M_{AD} = 0$

or, $2.5\,EI\,\theta_A + 0.5\,EI\,\theta_B + 0.75\,EI\,\theta_D - 0.375\,EI\,\Delta_2 - 133.4 = 0$ (i)

$\Sigma M_B = 0$, or, $M_{BC} + M_{BE} + M_{BA} = 0$

$0.5\,EI\,\theta_A + 3.3\,EI\,\theta_B + 0.75\,EI\,\theta_E - 0.24\,EI\,\Delta_1 - 0.375\,EI\,\Delta_2 - 133.4 = 0$ (ii)

$\Sigma M_D = 0$, or, $M_{DA} + M_{DE} = 0$

or, $0.75\,EI\,\theta_A + 2.5\,EI\,\theta_D + 0.5\,EI\,\theta_E - 0.375\,EI\,\Delta_2 + 133.4 = 0$ (iii)

$\Sigma M_E = 0$, or, $M_{ED} + M_{EB} + M_{EF} = 0$

or, $0.75\,EI\,\theta_B + 0.5\,EI\,\theta_D + 3.3\,EI\,\theta_E - 0.24\,EI\,\Delta_1 - 0.375\,EI\,\Delta_2 + 133.4 = 0$ (iv)

There will be two shear equations : one corresponding to the upper storey and the other corresponding to the lower storey.

$\boxed{H_C + H_F = 0}$ (v)

and $\boxed{H_B + H_E = 0}$ (vi)

Eq. (v) can be rewritten as :

$$\frac{M_{CB} + M_{BC}}{5} + \frac{M_{EF} + M_{FE}}{5} = 0$$

$1.2\,EI\,(\theta_B + \theta_E) - 0.96\,EI\,\Delta_1 = 0$ (v)

Eq. (vi) can be rewritten as :

$$\frac{M_{AB} + M_{BA}}{4} + \frac{M_{DE} + M_{ED}}{4} = 0$$

or, $1.5\,EI\,\theta_A + 1.5\,EI\,\theta_B + 1.5\,EI\,\theta_D + 1.5\,EI\,\theta_E - 1.5\,EI\,\Delta_2 = 0$ (vi)

(c) *Matrix Solution*

$$EI\begin{bmatrix} 2.5 & 0.5 & 0.75 & 0 & 0 & -0.375 \\ 0.5 & 3.3 & 0 & 0.75 & -0.24 & -0.375 \\ 0.75 & 0 & 2.5 & 0.5 & 0 & -0.375 \\ 0 & 0.75 & 0.5 & 3.3 & -0.24 & -0.375 \\ 0 & 1.2 & 0 & 1.2 & -0.96 & 0 \\ 1.5 & 1.5 & 1.5 & 1.5 & 0 & -1.5 \end{bmatrix}\begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_D \\ \theta_E \\ \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} 133.4 \\ 133.4 \\ -133.4 \\ -133.4 \\ 0 \\ 0 \end{Bmatrix}$$

or,
$$\begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_D \\ \theta_E \\ \Delta_1 \\ \Delta_2 \end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix} 194.66 \\ 118.7 \\ -194.66 \\ -118.7 \\ 0 \\ 0 \end{Bmatrix}$$

(d) *Back-Substitution*

Back-substituting the values of slopes and deflections in the slope-deflection equations gives,

M_{CB} = 15.80 kNm, M_{BC} = 31.65 kNm, M_{BE} = − 103.66 kNm,
M_{BA} = 72 kNm, M_{AB} = 84.67 kNm, M_{AD} = − 84.67 kNm,
M_{DA} = 84.67 kNm, M_{DE} = − 84.67 kNm, M_{ED} = − 72 kNm,
M_{EB} = 103.66 kNm, M_{EF} = − 31.65 kNm, M_{FE} = − 15.80 kNm,

The bending moment diagram and the elastic curve are shown in Figs. 10.17c and d.

Example 10.13

Analyze the box frame shown in Fig.10.18 by the slope-deflection method.

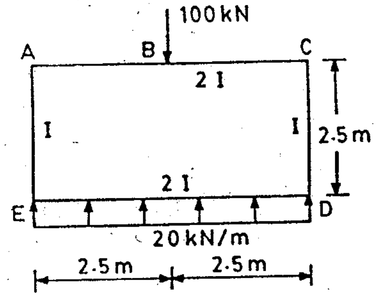


Fig. 10.18 Box section

Solution

(a) *Fixed End Moments*

$$M_{FAC} = -\frac{P\,ab^2}{L^2} = -\frac{PL}{8} = -100\times\frac{5}{8} = -62.5 \text{ kNm}$$

$$M_{FED} = \frac{wL^2}{12} = 20\times\frac{5^2}{12} = 41.67 \text{ kNm}$$

(b) *Slope-Deflection Equations*

$$M_{AC} = M_{FAC} + \frac{2EI}{L}\left(2\theta_A + \theta_C - \frac{3\Delta}{L}\right)$$

$$M_{AC} = -62.5 + \frac{2E(2I)}{5}\left(2\theta_A + \theta_C - \frac{3\Delta}{L}\right)$$

$$M_{AE} = 0 + \frac{2EI}{2.5}(2\theta_A + \theta_E)$$

$$M_{EA} = 0 + \frac{2EI}{2.5}(\theta_A + 2\theta_E)$$

$$M_{ED} = 41.67 + \frac{2E(2I)}{5}\left(2\theta_E + \theta_D - \frac{3\Delta}{L}\right)$$

Due to symmetry, $\theta_C = -\theta_A$
$\theta_D = -\theta_E$
$\Delta = 0$

(c) *Joint-Moment Equilibrium Equations*

Joint A, $M_{AC} + M_{AE} = 0$

or, $$-62.5 + \frac{4EI}{5}\theta_A + \frac{4EI}{2.5}\theta_A + \frac{2EI}{2.5}\theta_E = 0$$

or, $$2.4\,EI\,\theta_A + 0.8\,EI\,\theta_E - 62.5 = 0 \qquad \text{(i)}$$

Joint E, $M_{EA} + M_{ED} = 0$

or, $$\frac{2EI}{2.5}\theta_A + \frac{4EI}{2.5}\theta_E + 41.67 + \frac{4EI}{5}\theta_E = 0$$

or, $$0.8\,EI\,\theta_A + 2.4EI\,\theta_E + 41.67 = 0 \qquad \text{(ii)}$$

Rearranging Eqs. (i) and (ii) in matrix form :

$$EI\begin{bmatrix}2.4 & 0.8\\ 0.8 & 2.4\end{bmatrix}\begin{Bmatrix}\theta_A\\ \theta_E\end{Bmatrix} = \begin{Bmatrix}62.5\\ -41.67\end{Bmatrix}$$

or, $$\begin{Bmatrix}\theta_A\\ \theta_E\end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix}35.8\\ -29.3\end{Bmatrix}$$

(d) *Back - Substitution*

$M_{AC} = -33.86$ kNm, $M_{AE} = 33.84$ kNm, $M_{EA} = -18.24$ kNm,
$M_{ED} = 18.23$ kNm.

Free span moment at B $= \dfrac{PL}{4} = 100\times\dfrac{5}{4} = 125$ kNm

Free span moment at midspan of ED $= \dfrac{wL^2}{8} = 20\times\dfrac{5^2}{8} = 62.5$ kNm

The bending moment diagram is shown in Fig. 5.14 e.

10.7 FRAMES WITH SLOPING LEGS

Example 10.14

Analyze the frame shown in Fig. 10.19a using the slope-deflection method. Also draw its elastic curve.

Solution

In the slope-deflection method, axial deformations of the member are ignored. Thus, members AB, BC and CD can not deform axially. The joint B will deflect horizontally towards right. The member BC being horizontal and CD being inclined, the joint C will move perpendicular to both BC and CD so as to take the final position at C". C' C" is perpendicular to BC, and CC" is perpendicular to CD as shown in Fig. 10.19b. The various joint deflections may be written as follows :

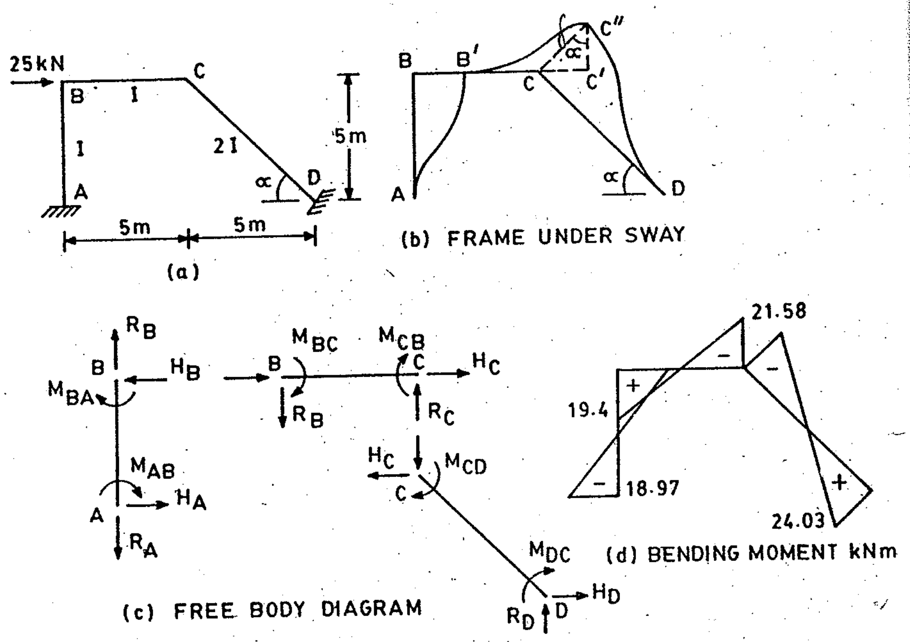


Fig. 10.19 Frame with an inclined leg

$\alpha = 45°, \qquad CC'' = \delta$

$$CC' = \delta \cos\alpha = BB' = \frac{\delta}{\sqrt{2}}$$

$$C'C'' = \delta \sin\alpha = \frac{\delta}{\sqrt{2}}$$

(a) *Slope-Deflection Equations*

$$M_{AB} = 0 + \frac{2EI}{5}\left(2\theta_A + \theta_B - \frac{3\frac{\delta}{\sqrt{2}}}{5}\right)$$

$$= 0.8\,EI\theta_A + 0.4\,EI\theta_B - 0.168\,EI\delta$$

$$M_{BA} = 0.4\,EI\theta_A + 0.8\,EI\theta_B - 0.168\,EI\delta$$

$$M_{BC} = 0 + \frac{2EI}{5}\left(2\theta_B + \theta_C + \frac{3\frac{\delta}{\sqrt{2}}}{5}\right)$$

$$= 0.8\,EI\theta_B + 0.4\,EI\theta_C + 0.168\,EI\delta$$

$$M_{CB} = 0.4\,EI\theta_B + 0.8\,EI\theta_C + 0.168\,EI\delta$$

$$M_{CD} = 0 + \frac{2E(2I)}{5\sqrt{2}}\left(2\theta_C + \theta_D - \frac{3\frac{\delta}{\sqrt{2}}}{5}\right)$$

$$= 1.12\,EI\theta_C - 0.24\,EI\delta, \quad \text{since } \theta_D = 0$$

$$M_{DC} = 0.56\,EI\theta_C - 0.24\,EI\delta$$

(b) *Joint-Moment Equilibrium Equations*

$\Sigma M_B = 0, \quad$ or, $\quad M_{BA} + M_{BC} = 0$

or, $\quad 1.6\,EI\theta_B + 0.4\,EI\theta_C = 0 \qquad$ (i)

$\Sigma M_C = 0, \quad$ or, $\quad M_{CB} + M_{CD} = 0$

$0.4\,EI\theta_B + 1.92\,EI\theta_C - 0.072\,EI\delta = 0 \qquad$ (ii)

The free body diagrams of the frame members are shown in Fig. 10.19c.

Leg AB

Moment about B gives, $H_A = \dfrac{M_{AB} + M_{BA}}{5}$

$\Sigma F_x = 0$ gives, $\quad H_A = H_B$

$\Sigma F_y = 0$ gives, $\quad R_A = R_B$

Leg BC

Moment about C gives, $R_B = \dfrac{M_{BC} + M_{CB}}{5}\downarrow = R_C\uparrow$

Leg CD

Moment about C gives, $H_D = \dfrac{M_{CD} + M_{DC} - R_D 5}{5}$

$R_C = R_D$

Shear equation

$$\boxed{H_A + H_D + 25 = 0}$$

or, $$\frac{M_{AB}+M_{BA}}{5} + \frac{M_{DC}+M_{CD}}{5} - \frac{M_{CB}+M_{BC}}{5} + 25 = 0$$

or, $$0.48\,EI\,\theta_C - 1.152\,EI\,\delta + 125 = 0 \qquad \text{(iii)}$$

(c) *Matrix Solution*

$$EI\begin{bmatrix}1.6 & 0.4 & 0\\ 0.4 & 1.92 & -0.072\\ 0 & 0.48 & -1.152\end{bmatrix}\begin{Bmatrix}\theta_B\\ \theta_C\\ \delta\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ -125\end{Bmatrix}$$

or, $$\begin{Bmatrix}\theta_B\\ \theta_C\\ \delta\end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix}-1.09\\ 4.36\\ 110.32\end{Bmatrix}$$

Back - Substitution

$M_{AB} = -\ 18.97$ kNm, $M_{BA} = -\ 19.4$ kNm, $M_{BC} = 19.4$ kNm

$M_{CB} = 21.58$ kNm, $M_{CD} = -\ 21.59$ kNm, $M_{DC} = -\ 24.03$ kNm

The bending moment diagram is shown in Figs. 10.19 d.

Example 10.15

Draw shear force and bending moment diagram for a portal frame shown in Fig. 10.20a.

Solution

Under the lateral load, the frame will sway to the right. Its deflected shape can be drawn using the rules cited earlier and is shown in Fig. 10.20b. The displacements of members AB and BC at joint B are as follows :

$BB'' = CC''$

$BB' = CC' = \delta$

$BB'' = \delta \cos 30$, $B'B'' = \delta \sin 30$

The displacement BB' is perpendicular to AB, while B'B" is perpendicular to BC. The displacement of joint C is identical to that of joint B.

(a) *Slope-Deflection Equations*

$$M_{AB} = \frac{2EI}{6}\left(2\theta_A + \theta_B - \frac{3\delta}{6}\right)$$

$$= 0.67\,EI\,\theta_A + 0.34\,EI\,\theta_B - 0.167\,EI\,\delta$$

$$M_{BA} = 0.34\,EI\,\theta_A + 0.67\,EI\,\theta_B - 0.167\,EI\,\delta$$

$$M_{BC} = \frac{2E(2I)}{6}\left(2\theta_B + \theta_C + \frac{3(B'B''+C'C'')}{6}\right)$$

$$= 1.34\,EI\,\theta_B + 0.67\,EI\,\theta_C + 0.333\,EI\,(2\delta \sin 30)$$

or, $$M_{BC} = 1.34\,EI\,\theta_B + 0.67\,EI\,\theta_C + 0.333\,EI\,\delta$$

$$M_{CB} = 0.67\,EI\,\theta_B + 1.34\,EI\,\theta_C + 0.333\,EI\,\delta$$

$$M_{CD} = 0.67\,EI\,\theta_C + 0.34\,EI\,\theta_D - 0.167\,EI\,\delta$$

$$M_{DC} = 0.34\,EI\,\theta_C + 0.67\,EI\,\theta_D - 0.167\,EI\,\delta$$

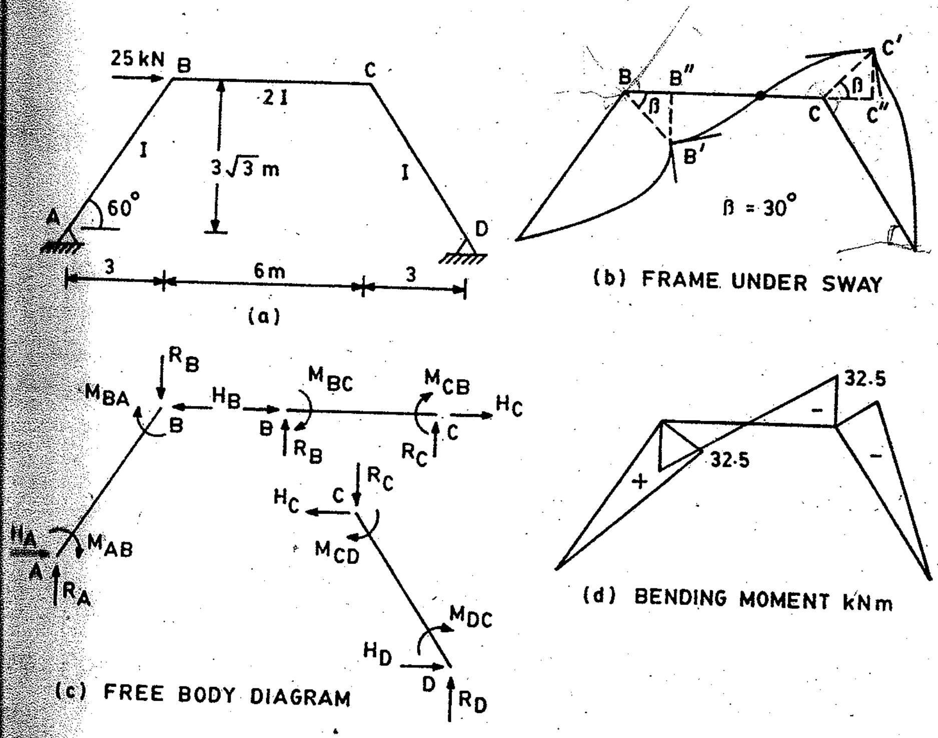


Fig. 10.20 Two hinged frame with inclined legs

(b) *Joint-Moment Equilibrium Equations*

$M_{AB} = 0$, $\quad 0.67\,EI\,\theta_A + 0.34\,EI\,\theta_B - 0.167\,EI\,\delta = 0 \qquad$ (i)

$\Sigma M_B = 0$, or, $\quad M_{BA} + M_{BC} = 0$

or, $\quad 0.34\,EI\,\theta_A + 2\,EI\,\theta_B + 0.67\,EI\,\theta_C + 0.167\,EI\,\delta = 0 \qquad$ (ii)

$$\Sigma M_C = 0, \quad \text{or,} \quad M_{CB} + M_{CD} = 0$$

or, $$0.67\,EI\theta_B + 2\,EI\theta_C + 0.34\,EI\theta_D - 0.167\,EI\delta = 0 \qquad \text{(iii)}$$

$$\Sigma M_D = 0, \quad 0.34\,EI\theta_C + 0.67\,EI\theta_D - 0.167\,EI\delta = 0 \qquad \text{(iv)}$$

The free body diagram of the frame is shown in Fig. 10.20c.

Member AB Taking moment about B

$$H_A = \frac{M_{AB} + M_{BA} + R_A 3}{3\sqrt{3}}$$

$$R_A = R_B, \quad H_A = H_B$$

Member BC Taking moment about C

$$R_B = -\frac{M_{BC} + M_{CB}}{6}$$

$$R_B + R_C = 0 \quad \text{or,} \quad R_C = \frac{M_{BC} + M_{CB}}{6}$$

Member CD Taking moment about C

$$H_D = \frac{M_{CD} + M_{DC} - R_D 3}{3\sqrt{3}}$$

$$R_D = R_C$$

Shear equation

$$H_A + H_D + 25 = 0$$

or, $$\frac{M_{AB} + M_{BA} + R_A 3}{3\sqrt{3}} + \frac{M_{CD} + M_{DC} - R_D 3}{3\sqrt{3}} + 25 = 0$$

$$(M_{AB} + M_{BA}) - 3\left(\frac{M_{BC} + M_{CB}}{6}\right) + (M_{CD} + M_{DC}) - 3\left(\frac{M_{BC} + M_{CB}}{6}\right) + 75\sqrt{3} = 0$$

or, $$(M_{AB} + M_{BA}) - (M_{BC} + M_{CB}) + M_{CB} + 75\sqrt{3} = 0$$

or, $$0.34\,EI\theta_A - 1.33\,EI\theta_B - 1.33\,EI\theta_C + 0.34\,EI\theta_D - EI\delta + 129.90 = 0 \qquad \text{(v)}$$

(c) *Matrix Solution*

$$EI\begin{bmatrix} 0.67 & 0.34 & 0 & 0 & -0.167 \\ 0.34 & 2 & 0.67 & 0 & 0.167 \\ 0 & 0.67 & 2 & 0.34 & 0.167 \\ 0 & 0 & 0.34 & 0.67 & -0.167 \\ 0.34 & -1.33 & -1.33 & 0.34 & -1 \end{bmatrix}\begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \theta_D \\ \delta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -129.9 \end{Bmatrix}$$

or, $$\begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \theta_D \\ \delta \end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix} 73.96 \\ -24.82 \\ -24.82 \\ 73.96 \\ 246.2 \end{Bmatrix}$$

(d) *Back - Substitution*

$$M_{AB} = 0, \; M_{BA} = -32.5\text{ kNm}, \; M_{BC} = 32.5\text{ kNm},$$

$$M_{CB} = 32.5\text{ kNm}, \; M_{CD} = -32.5\text{ kNm}, \; M_{DC} = 0$$

The bending moment diagram is shown in Fig. 10.20 d.

Example 10.16

Reanalyze the fixed end frame shown in Fig. 10.21a using the slope-deflection method. Draw the elastic curve.

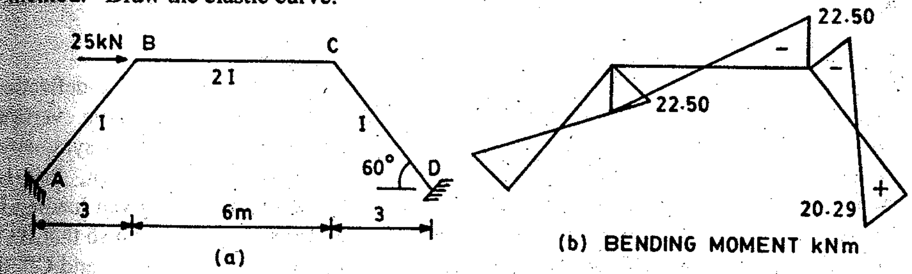


Fig. 10.21 Fixed end frame with inclined legs

Solution

Proceeding as in example 10.15,

$\theta_A = 0 = \theta_B$ being fixed ends, the joint-moment equilibrium Eqs. (ii) and (iii) become :

$$2\,EI\theta_B + 0.67\,EI\theta_C + 0.167\,EI\delta = 0 \qquad \text{(i)}$$

$$0.67\,EI\theta_B + 2\,EI\theta_C + 0.167\,EI\delta = 0$$

Shear equation is $H_A + H_D + 25 = 0,$

or, $\quad M_{AB} + M_{BA} - (M_{BC} + M_{CB}) + M_{CD} + M_{DC} + 75\sqrt{3} = 0$

or, $\quad -EI\theta_B - EI\theta_C - 1.33EI\delta + 129.90 = 0$

(c) *Matrix Solution*

$$EI\begin{bmatrix} 2 & 0.67 & 0.167 \\ 0.67 & 2 & 0.167 \\ -1 & -1 & -1.33 \end{bmatrix}\begin{Bmatrix} \theta_B \\ \theta_C \\ \delta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -129.9 \end{Bmatrix}$$

or,
$$\begin{Bmatrix} \theta_B \\ \theta_C \\ \delta \end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix} -6.74 \\ -6.74 \\ 107.8 \end{Bmatrix}$$

(d) *Back - Substitution*

$M_{AB} = -20.29$ kNm, $M_{BA} = -22.50$ kNm, $M_{BC} = 22.50$ kNm

$M_{CB} = 22.50$ kNm, $M_{CD} = -22.5$ kNm, $M_{DC} = -20.29$ kNm

The bending moment diagram is shown in Fig. 10.21 b.

Example 10.17

Analyze the gable frame shown in Fig. 10.22a using the slope-deflection metho[d]. Draw bending moment diagram and the elastic curve.

Solution

This gable frame has 5 rigid joints A,B, C, D and E. Since the slope-deflecti[on] method ignores the axial deformations, joints B and D cannot have verti[cal] displacements. The joint C cannot have a horizontal displacement due to symmetr[y]. Thus out of three joint translations, only one translation is independent, that [is] BB' = DD' = Δ. A possible elastic curve of the gable frame under the given loading [is] shown in Fig. 10.22b. The parallelograms B'BCC" and D'DCC"' are drawn so that B'[C"] and D'C"' are equal to BC and DC, respectively. Draw perpendiculars at C" and C"' [to] B'C" and D'C"' so that they intersect at C'.

$$\angle C\,C''\,C' = 90^\circ = \angle C\,C'''\,C'$$

$$C'C'' = C'C''' = \Delta_1$$

$$BB' = \Delta = \Delta_1 \sin\theta = CC''$$

(a) *Fixed End Moments*

The fixed end moment in a beam with a horizontal span of length L when subject[ed]

(ii) [Th]e total load wL is resolved into two components : along the member and at right [an]gles to the member (Fig. 10.22c). The perpendicular load per unit length is [w]L cos θ / L sec θ = w cos² θ: The fixed end moment is equal to

(iii)
$$\frac{w\cos^2\theta(L\sec\theta)^2}{12} = \frac{wL^2}{12}$$

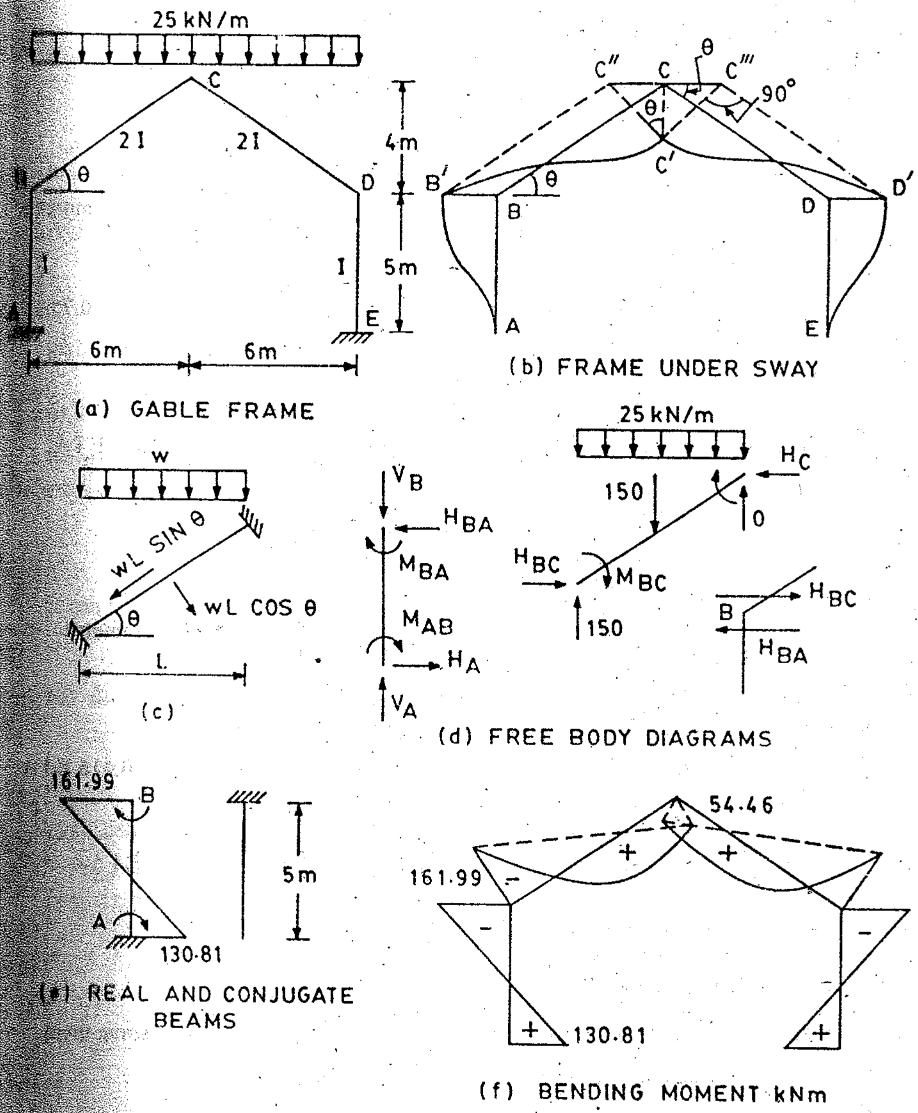


Fig. 10.22 Gable frame under symmetric loading

if $w = 25$ kN/m, $L = 6$ m, Fixed end moment $= 25 \times \frac{6^2}{12} = \pm 75$ kNm

(b) *Slope-Deflection Equations*

$$M_{AB} = 0 + \frac{2EI}{5}\left(2\theta_A + \theta_B + \frac{3\Delta}{5}\right)$$

$$= 0.4\,EI\theta_B + 0.24\,EI\Delta,\ \theta_A = 0,\ \text{A being a fixed end.}$$

$$M_{BA} = 0 + \frac{2EI}{5}\left(\theta_A + 2\theta_B + \frac{3\Delta}{5}\right) = 0.8\,EI\theta_B + 0.24\,EI\Delta$$

$$M_{BC} = -75 + \frac{2E(2I)}{6\sec\theta}\left(2\theta_B + \theta_C - \frac{3\Delta\,\text{cosec}\,\theta}{6\sec\theta}\right),\ \theta_C = 0,\ \text{due to symmetry}$$

$$= -75 + \frac{2EI}{3}\frac{3}{\sqrt{13}}\left(2\theta_B - \frac{3}{6}\times\frac{6}{4}\Delta\right) = -75 + \frac{2EI}{\sqrt{13}}(2\theta_B - 0.75\Delta)$$

or, $$M_{BC} = -75 + 1.10\,EI\theta_B - 0.416\,EI\Delta$$

$$M_{CB} = 75 + \frac{2EI}{\sqrt{13}}(\theta_B - 0.75\Delta) = 75 + 0.555\,EI\theta_B - 0.416\,EI\Delta$$

$$M_{CD} = -75 + \frac{2EI}{\sqrt{13}}(2\theta_C + \theta_D + 0.75\Delta)$$

$$= -75 + \frac{2EI}{\sqrt{13}}(\theta_D + 0.75\Delta),\ \text{but } \theta_D = -\theta_B \text{ due to symmetry}$$

or, $$M_{CD} = -75 - 0.555\,EI\theta_B + 0.416\,EI\Delta$$

$$M_{DC} = 75 + \frac{2EI}{\sqrt{13}}(\theta_C + 2\theta_D + 0.75\Delta) = 75 - 1.10\,EI\theta_B + 0.416\,EI\Delta$$

$$M_{DE} = 0 + \frac{2EI}{5}\left(2\theta_D + \theta_E - \frac{3\Delta}{5}\right) = -0.8\,EI\theta_B - 0.24\,EI\Delta$$

$$M_{ED} = -0.4\,EI\theta_B - 0.24\,EI\Delta$$

(c) *Joint-Moment Equilibrium Equations (Fig. 10.22d)*

There are two unknowns and two independent equations are required to solve them.

Joint B, $M_{BA} + M_{BC} = 0$

or, $0.8\,EI\theta_B + 0.24\,EI\Delta - 75 + 1.10\,EI\theta_B - 0.416\,EI\Delta = 0$

or, $1.9\,EI\theta_B - 0.176\,EI\Delta - 75 =$ (i)

Shear equation is given by :

$$H_A + H_E = 0,$$

$$H_A = \frac{M_{AB} + M_{BA}}{5} \quad \text{and} \quad H_E = \frac{M_{DE} + M_{ED}}{5}$$

$M_{AB} + M_{BA} + M_{DE} + M_{ED} = 0$, This leads to a trivial solution.

$\Sigma M_C = 0$ gives,

$$H_{BC} \times 4 + 150 \times 3 - M_{BC} - M_{CB} - 150 \times 6 = 0,$$

or, $$H_{BC} = \frac{M_{BC} + M_{CB} + 450}{4}$$

At joint B, the shear equation is :

$$H_{BC} - H_{BA} = 0$$

or, $$\frac{1}{4}[1.664\,EI\theta_B - 0.832EI\Delta + 450] - \left[\frac{1.2EI\theta_B + 0.48EI\Delta}{5}\right] = 0$$

or, $0.176\,EI\theta_B - 0.304\,EI\Delta + 112.5 = 0$ (ii)

Rewriting these equations in matrix form :

$$EI\begin{bmatrix} 1.9 & -0.176 \\ -0.176 & 0.304 \end{bmatrix}\begin{Bmatrix} \theta_B \\ \Delta \end{Bmatrix} = \begin{Bmatrix} 75 \\ 112.5 \end{Bmatrix}$$

It can be seen that the coefficient matrix is symmetric.

$\therefore$ $EI\theta_B = 77.933$ and $EI\Delta = 415.185$

(d) *Back - Substitution*

$M_{AB} = 130.81$ kNm, $M_{BA} = 161.99$ kNm, $M_{BC} = -161.99$ kNm

$M_{CB} = -54.46$ kNm

(e) *Compatibility Check*

The displacement at joint B can be checked using the conjugate beam method (Fig. 10.22e).

$$\theta_B = \frac{1}{2}\times 161.99 \times 5 - \frac{1}{2}\times 130.81 \times 5 = \frac{77.95}{EI} \text{ clockwise} \qquad \text{O. K.}$$

$$BB' = \frac{1}{2}\times 130.81 \times 5 \times \frac{2}{3}\times 5 - \frac{1}{2}\times 161.99 \times 5 \times \frac{5}{3} = \frac{415.186}{EI} \qquad \text{O. K.}$$

Similarly, it can be shown that $\theta_C = 0$. The bending moment diagram is shown in Fig. 10.22 f. The elastic curve was shown in Fig. 10.22b.

Example 10.18

Analyze the gable frame shown in Fig. 10.23a using the slope-deflection method. Draw bending moment diagram and the elastic curve.

Solution

Due to the horizontal loading, the horizontal deflection at B may not be equal to the horizontal deflection at D. A probable deflected shape is shown in Fig. 10.23b.

Let $BB' = \Delta_B$ and $DD' = \Delta_D$

The parallelograms BB'C"C and DD'C'"C are drawn so that B'C" and C'"D' are equal to BC and CD, respectively. Draw perpendiculars at C" and C'" to B'C" and C'"D' so that they intersect at C'. Thus,

length $B'C' = B'C'' = BC$

and $C'D' = C'''D' = CD$ (Deflections being small)

$$C'C'' = (\Delta_B - \Delta_D)\frac{\operatorname{cosec}\theta}{2} = (\Delta_B - \Delta_D)\frac{\sqrt{13}}{4} \quad \text{anti-clockwise sway}$$

$$C'C''' = (\Delta_B - \Delta_D)\frac{\sqrt{13}}{4} \quad \text{clockwise sway}$$

(a) *Fixed End Moments*

$$M_{FAB} = -\frac{wL^2}{12} = -10\times\frac{5^2}{12} = -20.83 \text{ kNm}, \quad M_{FBA} = 20.83 \text{ kNm}$$

$$M_{FBC} = -\frac{wL^2}{12} = -10\times\frac{4^2}{12} = -13.34 \text{ kNm}, \quad M_{FCB} = 13.34 \text{ kNm}$$

$$\theta_A = 0 = \theta_E \text{ being fixed ends.}$$

(b) *Slope-Deflection Equations*

$$M_{AB} = -20.83 + \frac{2EI}{5}\left(2\theta_A + \theta_B - \frac{3\Delta_B}{5}\right)$$

$$= -20.83 + 0.40EI\,\theta_B - 0.24\,EI\,\Delta_B$$

$$M_{BA} = 20.83 + \frac{2EI}{5}\left(\theta_A + 2\theta_B - \frac{3\Delta_B}{5}\right)$$

$$= 20.83 + 0.80EI\,\theta_B - 0.24\,EI\,\Delta_B$$

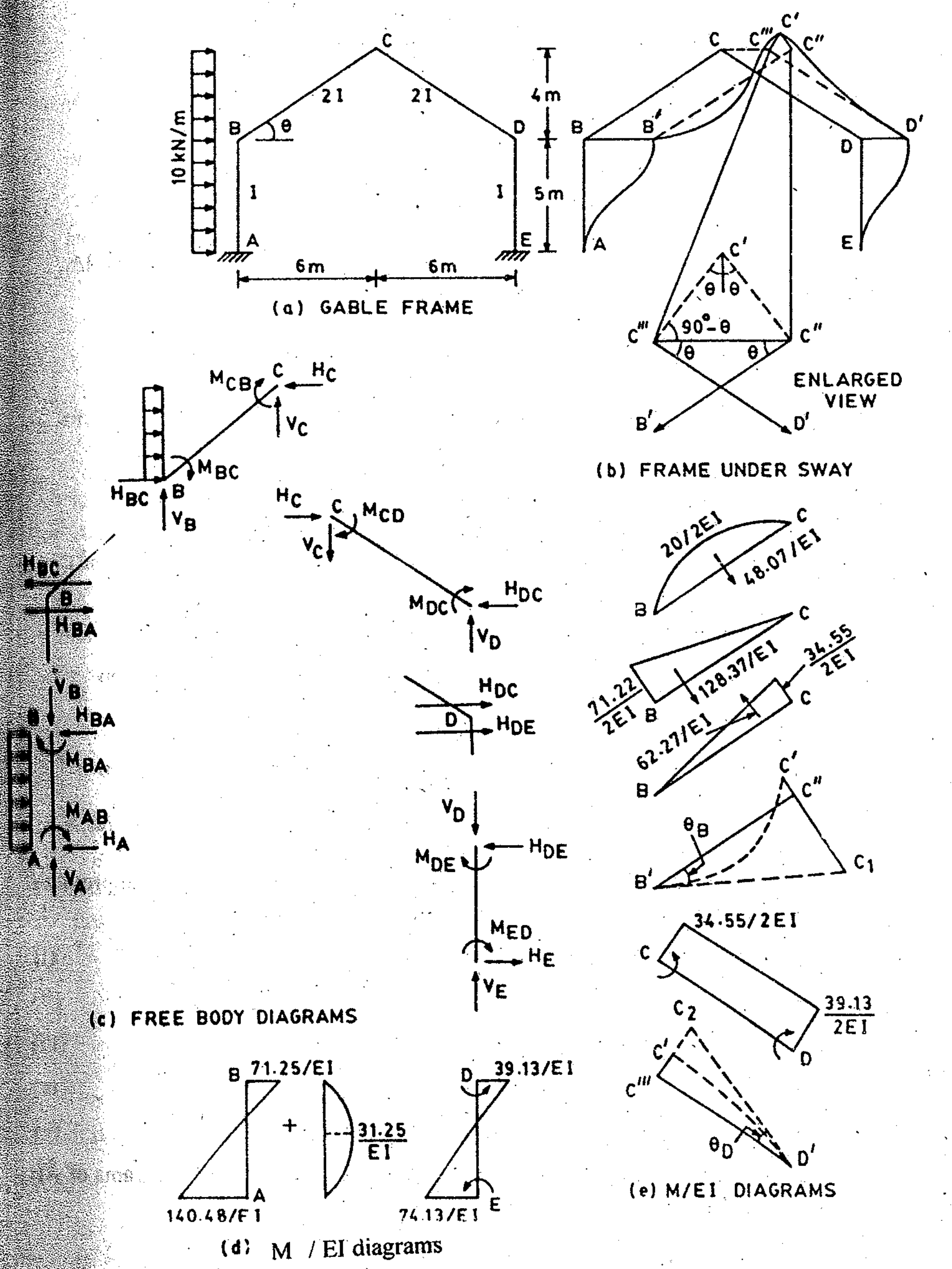


Fig. 10.23 Gable frame under wind loading

$$M_{BC} = -13.34 + \frac{2E(2I)}{2\sqrt{13}}\left[2\theta_B + \theta_C + \frac{3(\Delta_B - \Delta_D)}{2\sqrt{13}}\left(\frac{\sqrt{13}}{4}\right)\right]$$

or, $$= -13.34 + 1.11\,EI\,\theta_B + 0.554\,EI\,\theta_C + 0.208\,EI\,\Delta_B - 0.208\,EI\,\Delta_D$$

$$M_{CB} = 13.34 + \frac{2E(2I)}{2\sqrt{13}}\left[\theta_B + 2\theta_C + \frac{3(\Delta_B - \Delta_D)}{2\sqrt{13}}\left(\frac{\sqrt{13}}{4}\right)\right]$$

$$= 13.34 + 0.554\,EI\,\theta_B + 0.11\,EI\,\theta_C + 0.208\,EI\,\Delta_B - 0.208\,EI\Delta_D$$

$$M_{CD} = 0 + \frac{2E(2I)}{2\sqrt{13}}\left[2\theta_C + \theta_D - \frac{3(\Delta_B - \Delta_D)}{2\sqrt{13}}\left(\frac{\sqrt{13}}{4}\right)\right]$$

$$= 1.11\,EI\,\theta_C + 0.554\,EI\,\theta_D - 0.208\,EI\,\Delta_B + 0.208\,EI\,\Delta_D$$

$$M_{DC} = 0 + \frac{2E(2I)}{2\sqrt{13}}\left[\theta_C + 2\theta_D - \frac{3(\Delta_B - \Delta_D)}{2\sqrt{13}}\left(\frac{\sqrt{13}}{4}\right)\right]$$

or, $$= 0.554\,EI\,\theta_C + 1.11\,EI\,\theta_D - 0.208\,EI\,\Delta_B + 0.208\,EI\,\Delta_D$$

$$M_{DE} = 0 + \frac{2EI}{5}\left(2\theta_D + \theta_E - \frac{3\Delta_D}{5}\right) = 0.8\,EI\,\theta_D - 0.24\,EI\,\Delta_D$$

$$M_{ED} = 0 + \frac{2EI}{5}\left(\theta_D + 2\theta_E - \frac{3\Delta_D}{5}\right) = 0.4\,EI\,\theta_D - 0.24\,EI\,\Delta_D$$

(c) *Joint-Moment Equilibrium Equations*

There are 5 unknowns in θ_B, θ_C, θ_D, Δ_B and Δ_D. The moment equilibrium provides three equations and shear equations provide the other two equations :

Joint B,	$M_{BA} + M_{BC} = 0$	(i)
Joint C,	$M_{CB} + M_{CD} = 0$	(ii)
Joint D,	$M_{DC} + M_{DE} = 0$	(iii)
Joint B,	$H_{BC} - H_{BA} = 0$	(iv)
Joint D,	$H_{DC} + H_{DE} = 0$	(v)

The expressions for H_{BA} and H_{DE} can be obtained from the free body diagrams of AB and DE in Figs. 10.23c.

$\Sigma M_B = 0$ for the member BC gives,

$$M_{BC} + M_{CB} + 10 \times 4 \times 2 - V_C \times 6 - H_C \times 4 = 0, \qquad \text{(vi)}$$

$\Sigma M_D = 0$ for the member CD gives,

$$M_{DC} + M_{CD} + H_C \times 4 - V_C \times 6 = 0, \qquad \text{(vii)}$$

Adding these two equations,

or, $$V_C = \frac{M_{BC} + M_{CB} + M_{DC} + M_{CD} + 80}{12} \qquad \text{(viii)}$$

and $$H_C = \frac{M_{BC} + M_{CB} + 80 - M_{DC} - M_{CD}}{8} \qquad \text{(ix)}$$

$\Sigma H = 0$ for the member BC gives,

$$H_{BC} + 40 - H_C = 0 \text{ or, } H_{BC} = H_C - 40$$

or, $$H_{BC} = \frac{M_{BC} + M_{CB} - M_{DC} - M_{CD}}{8} - 30 \qquad \text{(x)}$$

$\Sigma H = 0$ for the member CD gives,

$$H_{DC} = H_C \qquad \text{(xi)}$$

The five simultaneous equations can be written as :

$$1.91\,EI\,\theta_B + 0.554\,EI\,\theta_C - 0.032\,EI\,\Delta_B - 0.208\,EI\,\Delta_D + 7.49 = 0 \qquad \text{(i)}$$

$$0.554\,EI\,\theta_B + 2.22\,EI\,\theta_C + 0.208\,EI\,\theta_D + 13.34 = 0 \qquad \text{(ii)}$$

$$0.554\,EI\,\theta_C + 1.91\,EI\,\theta_D - 0.208\,EI\,\Delta_B - 0.032\,EI\,\Delta_D = 0 \qquad \text{(iii)}$$

$$-0.032\,EI\,\theta_B - 0.208\,EI\,\theta_D + 0.20\,EI\,\Delta_B - 0.104\,EI\,\Delta_D - 55 = 0 \qquad \text{(iv)}$$

$$0.208\,EI\,\theta_B + 0.032\,EI\,\theta_D + 0.104\,EI\,\Delta_B - 0.20\,EI\,\Delta_D + 10 = 0 \qquad \text{(v)}$$

Rearranging in the matrix notation :

$$EI\begin{bmatrix} 1.91 & 0.554 & 0 & -0.032 & -0.208 \\ 0.554 & 2.22 & 0.554 & 0 & 0 \\ 0 & 0.554 & 1.91 & -0.208 & -0.032 \\ -0.032 & 0 & -0.208 & 0.20 & -0.104 \\ -0.208 & 0 & -0.032 & -0.104 & 0.20 \end{bmatrix}\begin{Bmatrix} \theta_B \\ \theta_C \\ \theta_D \\ \Delta_B \\ \Delta_D \end{Bmatrix} = \begin{Bmatrix} -7.49 \\ -13.34 \\ 0 \\ 55 \\ 10 \end{Bmatrix}$$

It can be seen that this is a symmetric square matrix. Its solution is :

$EI\,\theta_B = 68.938$, $EI\,\theta_C = -45.045$, $EI\,\theta_D = 87.49$

$EI\,\Delta_B = 613.46$, $EI\,\Delta_D = 454.693$

(d) *Back - Substitution*

$M_{AB} = -140.48$ kNm, $M_{BA} = -71.25$ kNm, $M_{BC} = 71.22$ kNm

$M_{CB} = 34.55$ kNm, $M_{CD} = -34.55$ kNm, $M_{DC} = 39.13$ kNm

$M_{DE} = -39.13$ kNm, $M_{ED} = -74.13$ kNm

(e) *Compatibility Check* (Figs. 10.23 d and e)

Member AB

Using the moment - area theorem,

$$\theta_B = \frac{1}{2} \times \frac{5}{EI} \times (140.48 - 71.25) - \frac{2}{3} \times \frac{5}{EI} \times \left(10 \times \frac{5^2}{8}\right)$$

$$\theta_B = \frac{68.91}{EI} \quad \text{(positive means clockwise)}$$

$$\Delta_B = \frac{351.2}{EI} \times \left(\frac{2}{3} \times 5\right) - \frac{178.12}{EI}\left(\frac{5}{3}\right) - \frac{104.17}{EI}(2.5)$$

$$= \frac{613.4}{EI} \quad \text{(positive means to right)}$$

Member BC

Moment - area theorem gives,

$$\theta_C = \theta_B - \frac{48.07}{EI} - \frac{128.37}{EI} + \frac{62.27}{EI} = -\frac{45.16}{EI} \text{(negative means anti-clockwise)}$$

$$C'C'' = C'C_1 - C''C_1$$

$$= \frac{48.07}{EI}(\sqrt{13}) + \frac{128.37}{EI}\left(\frac{2}{3} \times 2\sqrt{13}\right) - \frac{62.27}{EI}\left(\frac{1}{3} \times 2\sqrt{13}\right) - 2\sqrt{13} \times \frac{68.94}{EI}$$

$$= \frac{143.63}{EI}$$

$$\Delta_{CX} = CC'' - C'C'' \sin\theta = \frac{613.4}{EI} - \frac{143.63}{EI} \times \left(\frac{4}{2\sqrt{13}}\right) = \frac{534.10}{EI}$$

Member DE

$$\theta_D = \frac{1}{2} \times \frac{74.13}{EI} \times 5 - \frac{1}{2} \times \frac{39.13}{EI} \times 5 = \frac{87.5}{EI} \quad \text{(positive means clockwise)}$$

$$\Delta_D = \frac{1}{2} \times \frac{74.13 \times 5}{EI}\left(\frac{2}{3} \times 5\right) - \frac{1}{2} \times \frac{39.13 \times 5}{EI}\left(\frac{5}{3}\right) = \frac{454.70}{EI} \quad \text{(positive means to right)}$$

Member DC

$$\theta_C = \theta_D - \frac{1}{2} \times \frac{34.55}{2EI} \times 2\sqrt{13} - \frac{1}{2} \times \frac{39.13}{2EI} 2\sqrt{13}$$

$$= -\frac{45.2}{EI} \quad \text{(negative means anti-clockwise)}$$

$$C'C''' = C'''C_2 - C'C_2$$

$$= 2\sqrt{13} \times \frac{87.5}{EI} - \frac{1}{2} \times \frac{34.55}{2EI} \times 2\sqrt{13} \times \frac{2\sqrt{13}}{3} - \frac{1}{2} \times \frac{39.13}{2EI} \times 2\sqrt{13} \times \frac{2}{3} \times 2\sqrt{13}$$

$$C'C''' = \frac{143.45}{EI}$$

$$\Delta_{CX} = CC''' + C'''C' \sin\theta = \frac{454.70}{EI} + \frac{143.45}{EI} \times \frac{4}{2\sqrt{13}} = \frac{534.27}{EI}$$

Δ_{CX} as calculated using the left and right hand sides of the frame is the same. It proves that the calculations are correct.

10.8 FLEXIBILITY AND STIFFNESS MATRICES

Example 10.19

Develop the flexibility matrix and stiffness matrix for the beam elements with respect to the degrees of freedom shown in Figs. 10.24 a, b and c.

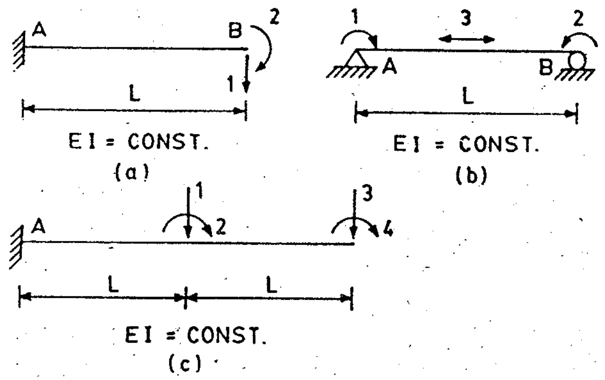


Fig. 10.24

Solution

The flexibility matrix can be developed by applying a unit force *successively* at degree of freedom (d.o.f.) each and *evaluating* the displacements at all the coordinates. Similarly, the stiffness matrix can be developed by giving a unit displacement successively at each d.o.f. while restraining all other degrees of freedom, and determining the forces required at all the d.o.f.

(a) Flexibility Matrix - Fig. 10.24a

Apply a unit force at d.o.f. 1, and the deflection and slope at B can be calculated with the help of standard results given in Appendix B.

$$\delta_{11} = \delta_B = \frac{PL^3}{3EI} = \frac{L^3}{3EI}$$

$$\delta_{21} = \theta_B = \frac{PL^2}{2EI} = \frac{L^2}{2EI}$$

Now apply a unit force at d.o.f. 2

$$\delta_{12} = \delta_B = \frac{ML^2}{2EI} = \frac{L^2}{2EI}$$

$$\delta_{22} = \theta_B = \frac{ML}{EI} = \frac{L}{EI}$$

The desired flexibility matrix can be written as:

$$[F]_{2\times 2} = \begin{bmatrix} \dfrac{L^3}{3EI} & \dfrac{L^2}{2EI} \\ \dfrac{L^2}{2EI} & \dfrac{L}{EI} \end{bmatrix} = \frac{1}{6EI}\begin{bmatrix} 2L^3 & 3L^2 \\ 3L^2 & 6L \end{bmatrix} \quad \text{F is a symmetric matrix.}$$

(a) Stiffness Matrix - Fig. 10.24a

Apply a unit deflection at d.o.f. 1, and the forces produced in the beam at end B can be determined with the help of standard results given in Table 10.1:

Table 10.1 Stiffness coefficients for a beam

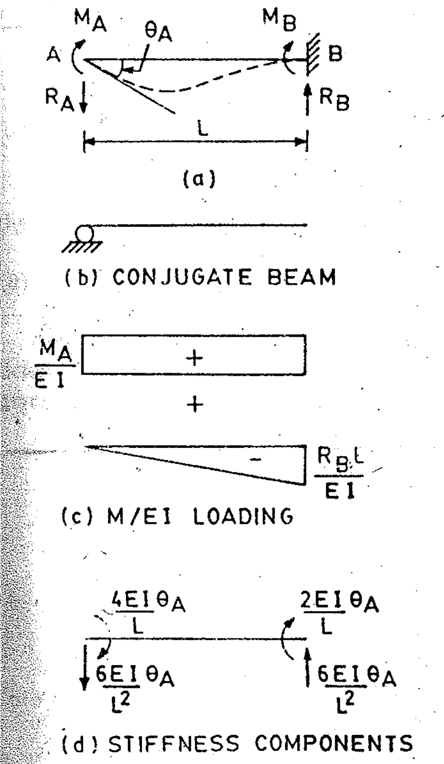
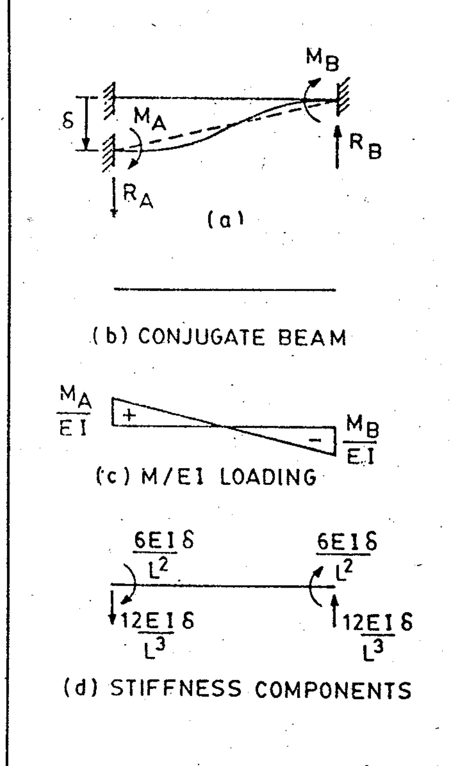
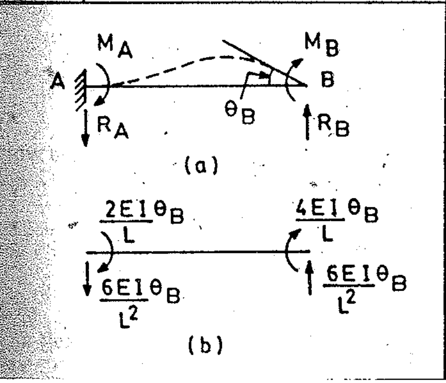
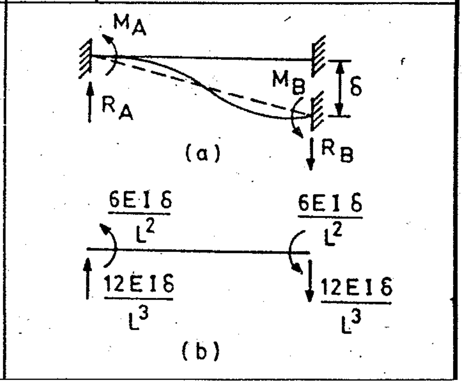
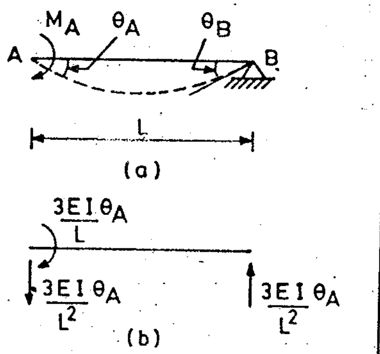
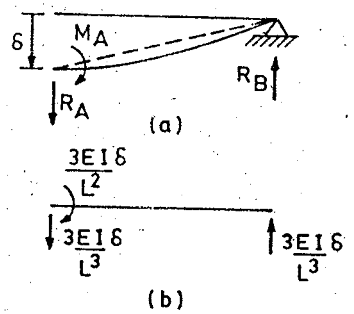
<table>
<tr><th>Load case</th><th>Stiffness coefficients</th><th>Load case</th><th>Stiffness coefficients</th></tr>
<tr><td>1</td><td>Rotation θ_A at A when for end is fixed</td><td>2</td><td>Settlement δ at A when for end is fixed</td></tr>
<tr><td colspan="2">
</td><td colspan="2">
</td></tr>
<tr><td>3</td><td>Rotation θ_B at B when for end is fixed</td><td>4</td><td>Settlement δ at B when for end is fixed</td></tr>
<tr><td colspan="2">
</td><td colspan="2">
</td></tr>
</table>

Table 10.1 Stiffness coefficients for a beam (contd.)

Load case	Stiffness coefficients	Load case	Stiffness coefficients
5	Rotation θ_A at A when for end is hinged	6	Settlement δ at A when for end is hinged
			

$$k_{11} = \frac{12EI}{L^3}\delta_B = \frac{12EI}{L^3}$$

$$k_{21} = \frac{6EI}{L^2}\delta_B = -\frac{6EI}{L^2}$$

Now apply a unit deformation at d.o.f. 2

$$k_{12} = \frac{6EI}{L^2}\theta_B = -\frac{6EI}{L^2}$$

$$k_{22} = \frac{4EI}{L}\theta_B = \frac{4EI}{L}$$

The desired stiffness matrix can be written as:

$$[K]_{2\times 2} = \begin{bmatrix} \dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} \\ -\dfrac{6EI}{L^2} & \dfrac{4EI}{L} \end{bmatrix} \quad \text{K is a symmetric matrix}$$

Multiplying the flexibility and stiffness matrices :

$$[F] \times [K] = \begin{bmatrix} \dfrac{L^3}{3EI} & \dfrac{L^2}{2EI} \\ \dfrac{L^2}{2EI} & \dfrac{L}{EI} \end{bmatrix} \times \begin{bmatrix} \dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} \\ -\dfrac{6EI}{L^2} & \dfrac{4EI}{L} \end{bmatrix}$$

$$FK = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I] \qquad \text{O. K.}$$

Since the product is an identity matrix, it can be concluded that the two matrices are the inverse of each other.

(b) Flexibility Matrix - Fig. 10.24b

Apply a unit force at d.o.f. 1, and the deflection and slope at all the d.o.f. can be determined using the moment-area theorem or unit load method or the standard results given in Appendix B.

$$\delta_{11} = \theta_A = \frac{ML}{3EI} = \frac{L}{3EI}$$

$$\delta_{21} = \theta_B = \frac{ML}{6EI} = \frac{L}{6EI}$$

$$\delta_{31} = 0 \quad \text{(axial deformation)}$$

Now apply a unit force at d.o.f. 2,

$$\delta_{12} = \theta_A = \frac{ML}{6EI} = \frac{L}{6EI}$$

$$\delta_{22} = \theta_B = \frac{ML}{3EI} = \frac{L}{3EI}$$

$$\delta_{32} = 0 \quad \text{(axial deformation)}$$

Now apply a unit force at d.o.f. 3,

$$\delta_{13} = 0 = \delta_{23}$$

$$\delta_{33} = \frac{PL}{AE} = \frac{L}{AE}$$

The flexibility matrix can be written as:

$$[F]_{3\times3} = \begin{bmatrix} \frac{L}{3EI} & \frac{L}{6EI} & 0 \\ \frac{L}{6EI} & \frac{L}{3EI} & 0 \\ 0 & 0 & \frac{L}{AE} \end{bmatrix} = \frac{L}{6EI}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 6r^2 \end{bmatrix} \quad \text{where } r^2 = I/A$$

(b) Stiffness Matrix -Fig. 10.24b

Apply a unit deformation at d.o.f. 1, while restraining all other d.o.f. The forces produced at all the d.o.f. can be determined using the conjugate beam method or the standard results given in Table 10.1.

$$k_{11} = M_A = \frac{4EI}{L}\theta_A = \frac{4EI}{L}$$

$$k_{21} = M_B = -\frac{2EI}{L}\theta_A = -\frac{2EI}{L}$$

$$k_{31} = 0 \quad \text{(axial force)}$$

Now apply a unit deformation at d.o.f. 2 and restrain all other d.o.f.

$$k_{12} = M_A = -\frac{2EI}{L}\theta_B = -\frac{2EI}{L}$$

$$k_{22} = M_B = \frac{4EI}{L}\theta_B = \frac{4EI}{L}$$

$$k_{32} = 0$$

Now apply a unit deformation at d.o.f. 3 and restrain all other d.o.f.,

$$k_{13} = 0 = k_{23} \text{ and } k_{33} = \frac{AE}{L} \text{ axial}$$

The stiffness matrix can be written as:

$$[K]_{3\times3} = \begin{bmatrix} \frac{4EI}{L} & -\frac{2EI}{L} & 0 \\ -\frac{2EI}{L} & \frac{4EI}{L} & 0 \\ 0 & 0 & \frac{4E}{L} \end{bmatrix}$$

Multiplying the flexibility and stiffness matrices :

$$FK = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \qquad \text{O. K.}$$

(c) Flexibility Matrix - Fig. 10.24c

Apply a unit force at d.o.f.1, and compute deflections at all the d.o.f. using the moment-area theorem- Fig.10.25 a.

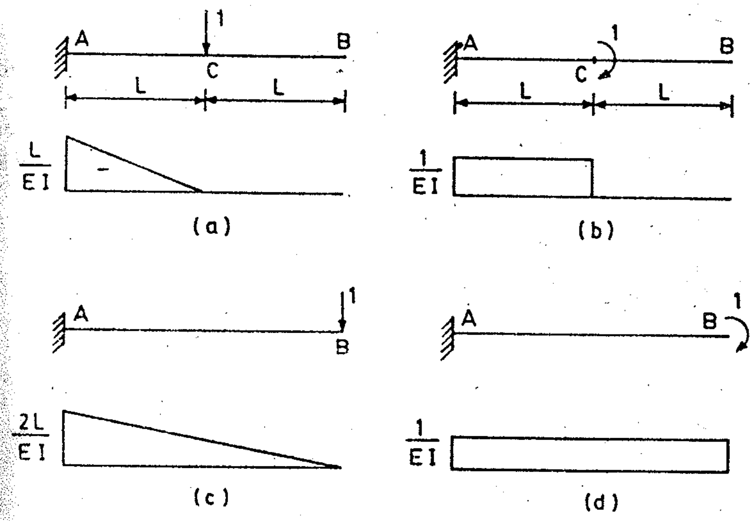


Fig. 10.25 Moment area diagram

$$\delta_{11} = \frac{1}{2}\frac{L^2}{EI} \times \frac{2}{3}L = \frac{L^3}{3EI}$$

$$\delta_{21} = \frac{L^2}{2EI} = \delta_{41}$$

$$\delta_{31} = \frac{1}{2}\frac{L^2}{EI} \times \frac{5}{3}L = \frac{5}{6}\frac{L^3}{EI}$$

Now apply a unit force at d.o.f. 2, Fig. 10.25b

$$\delta_{12} = \frac{L}{EI} \times \frac{L}{2} = \frac{L^2}{2EI}$$

$$\delta_{22} = \frac{L}{EI} = \delta_{42}$$

$$\delta_{32} = \frac{L}{EI} \times \frac{3L}{2} = \frac{3L^2}{2EI}$$

Now apply a unit force at d.o.f. 3, Fig. 10.25c

$$\delta_{13} = \frac{3L}{2EI} \times \left(\frac{L+4L}{L+2L}\right)\frac{L}{3} = \frac{5}{6}\frac{L^3}{EI}$$

$$\delta_{23} = \frac{3L^2}{2EI}$$

$$\delta_{33} = \frac{1}{2}2L\left(\frac{2L}{EI}\right) \times \frac{2}{3}(2L) = \frac{8}{3}\frac{L^3}{EI}$$

$$\delta_{43} = \frac{2L^2}{EI}$$

Now apply a unit force at d.o.f. 4, Fig. 10.25d

$$\delta_{14} = \frac{L^2}{2EI}, \quad \delta_{24} = \frac{L}{EI}$$

$$\delta_{34} = \frac{2L^2}{EI}, \quad \delta_{44} = \frac{2L}{EI}$$

The flexibility matrix can now be written as:

$$\mathbf{F} = \begin{bmatrix} \frac{L^3}{3EI} & & & \$ \\ \frac{L^2}{2EI} & \frac{L}{EI} & & \\ \frac{5}{6}\frac{L^3}{EI} & \frac{3L^2}{2EI} & \frac{8}{3}\frac{L^3}{EI} & \\ \frac{L^2}{2EI} & \frac{L}{EI} & \frac{2L^2}{EI} & \frac{2L}{EI} \end{bmatrix}$$

(c) Stiffness Matrix -Fig. 10.24c

Apply a unit displacement at d.o.f. 1 (Fig.10.26a), and compute the forces at all d.o.f. using the moment-area method or using Table 10.1.

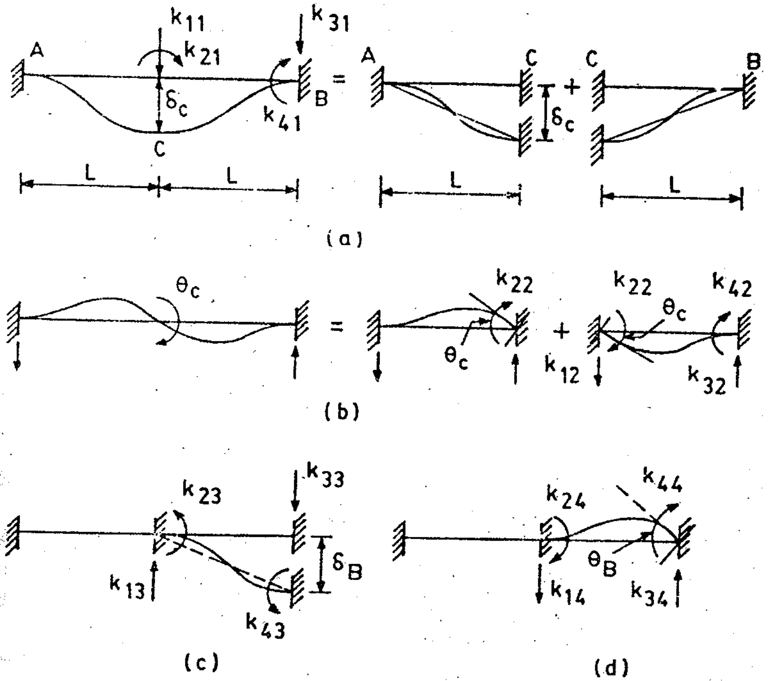


Fig. 10.26 Deflected shapes for stiffness coefficients

$$k_{11} = \left(\frac{12EI}{L^3} + \frac{12EI}{L^3}\right)\delta_C = \frac{24EI}{L^3}$$

$$k_{21} = -\frac{6EI}{L^2}\delta_C + \frac{6EI}{L^2}\delta_C = 0$$

$$k_{31} = -\frac{12EI}{L^3}$$

$$k_{41} = \frac{6EI}{L^2}\delta_C = \frac{6EI}{L^2}$$

Now apply a unit rotation at d.o.f. 2 (Fig. 10.26b)

$$k_{12} = -\frac{6EI}{L^2}\theta_C + \frac{6EI}{L^2}\theta_C = 0$$

$$k_{22} = \frac{4EI}{L}\theta_C + \frac{4EI}{L}\theta_C = \frac{8EI}{L} \qquad \because \theta_C = 1$$

$$k_{32} = -\frac{6EI}{L^2}\theta_C = -\frac{6EI}{L^2}$$

$$k_{42} = \frac{2EI}{L}\theta_C = \frac{2EI}{L}$$

Now apply a unit rotation at d.o.f. 3 (Fig. 10.26c)

$$k_{13} = -\frac{12EI}{L^3}\delta_B = -\frac{12EI}{L^3}$$

$$k_{23} = -\frac{6EI}{L^2}\delta_B = -\frac{6EI}{L^2} = k_{43}$$

$$k_{33} = \frac{12EI}{L^3}$$

Now apply a unit rotation at d.o.f. 4 (Fig. 10.26d)

$$k_{14} = \frac{6EI}{L^2}\theta_B = \frac{6EI}{L^2}$$

$$k_{24} = \frac{2EI}{L}\theta_B = \frac{2EI}{L}$$

$$k_{34} = -\frac{6EI}{L^2}$$

$$k_{44} = \frac{4EI}{L}\theta_B = \frac{4EI}{L}$$

The stiffness matrix can be written as:

$$\mathbf{K} = \begin{bmatrix} \frac{24EI}{L^3} & & & \text{\$} \\ 0 & \frac{8EI}{L} & & \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

It can be seen that the product $\mathbf{F} \times \mathbf{K}$ is an identity matrix.

PROBLEMS

10.1 Analyze the beams shown in Figs. P3.3 - P3.7 by the slope-deflection method. Draw shear force and bending moment diagrams and the deflected shapes.

10.2 Solve problems 3.3 and 3.4 by the slope-deflection method.

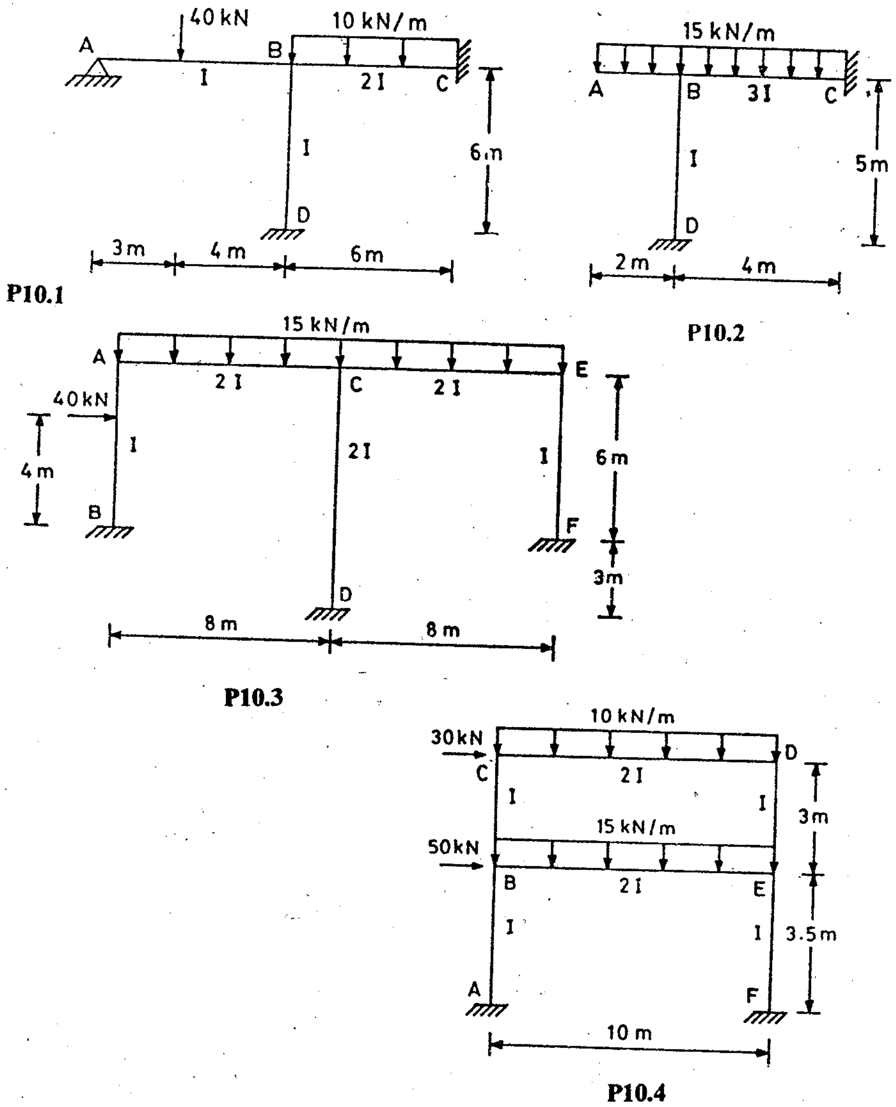
10.3 Analyze the beams shown in Figs. P4.2 and P4.3 by the slope-deflection method.

10.4 Solve problem 4.3 by the slope-deflection method.

10.5 Analyze the frames shown in Figs. P10.1 - P10.4 by the slope-deflection method. Draw shear force and bending moment diagrams and draw the deflected shapes.

10.6 Solve problems 3.6 and 3.7 by the slope-deflection method.

10.7 Analyze the frames shown in Figs. P5.2, P5.3 and P5.4 by the slope-deflection method.



Figs. P10.1 - P10.4

10.8 Analyze the gable frames shown in Figs. P6.3c and P6.3d by the slope-deflection method. Also draw the deflected shape

10.9 Solve problem 6.4 by the slope-deflection method.

10.10 Solve problem 6.5 by the slope-deflection method.

CHAPTER

eleven

MOMENT DISTRIBUTION METHOD

11.1 DEVELOPMENT OF THE METHOD

The moment distribution method was proposed by Professor Hardy Cross in 1932 for the analysis of statically indeterminate beams and rigid frames. It is an iterative method and is very useful when the degree of kinematic indeterminacy is very large. The slope-deflection method would require solution of a large number of simultaneous equations for such a problem. In the absence of a digital computer, it was very cumbersome to solve these equations. That is why, the moment distribution method attracted immediate attention.

It is necessary to measure the capacity of a member to rotate when a moment is applied at that end. Let us consider two cases :

(i) A beam fixed at one end,

and (ii) A beam hinged at one end.

Beam Fixed at One End

Let us consider a beam fixed at one end and simply supported at the other end as shown in Fig. 11.1a. If a clockwise moment M is applied at the end B, it is required to find the rotation at B and the moment produced at the fixed end A. This can be done by using the conjugate beam method. The moment produced at end A will be such that the rotation at A will be zero. It is, therefore, obvious that the moment at A should also be clockwise so that the slope at A is zero. The shear at A due to the M/EI loads is

$$V_A = \frac{1}{3} \times \frac{1}{2}\frac{ML}{EI} - \frac{2}{3}\frac{M_A L}{2EI} = 0$$

or,

$$M_A = \frac{M}{2} \tag{11.1}$$

The slope θ at B is equal to the shear at B due to the M/EI loads,

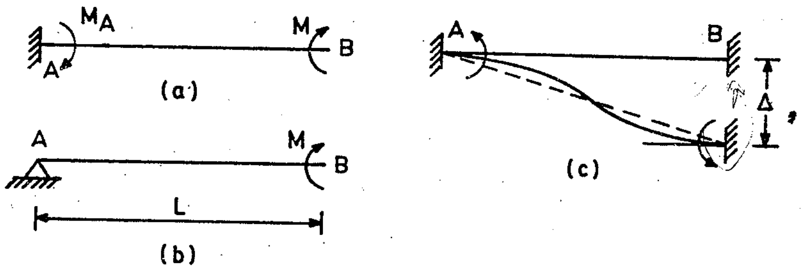


Fig. 11.1 Basic conditions

$$\theta = \frac{2}{3}\frac{ML}{2EI} - \frac{1}{3}\frac{M_A L}{2EI} = \frac{ML}{4EI} \tag{11.2}$$

or,

$$\frac{M}{\theta} = \frac{4EI}{L} \tag{11.3}$$

The moment M causes a slope ML/4EI at the same end of a beam when the far end is fixed. If the value of this rotation is unity, then the value of M required to produce this unit rotation is called the *stiffness* or *absolute stiffness*. The stiffness of a beam fixed at A and simply supported at B is equal to 4EI/L.

The moment created at the end A when a moment M_B is applied at end B is half in magnitude and has the same sign. The ratio M_A/M_B is equal to 1/2 and is called the *carry over factor*.

Beam Hinged at One End

Let us consider a beam AB hinged at one end and simply supported at the other end as shown in Fig. 11.1b. If a clockwise moment M is applied at the end B, it is required to find the rotation at B. The moment at A will be zero being a hinged end. The slope θ at B can be determined using conjugate beam method.

$$\theta = \frac{ML}{3EI} \tag{11.4}$$

or,

$$\frac{M}{\theta} = \frac{3EI}{L} \tag{11.5}$$

The moment M causes a slope equal to ML/3EI at the same end of a beam when the far end is hinged. The stiffness of a beam hinged at A and simply supported at B is equal to 3EI/L.

Beam with Relative End Translations

Let us consider a beam AB whose end B sinks by Δ as shown in Fig. 11.1c. It will create a moment at each end whose magnitude can be determined using the conjugate beam method. The moment at end B due to the M/EI loading will be equal to the deflection Δ as shown in the development of the slope- deflection method. Thus,

$$M = -\frac{6EI\Delta}{L^2} \tag{11.6a}$$

The negative sign indicates that if the beam rotates in the clockwise direction due to the relative end movements, the moment produced is in the anti-clockwise direction having equal magnitude at each end. These moments are treated as fixed end moments and can be easily taken into account in the analysis.

If one end of the beam is simply supported, the moment at the fixed end will be given by

$$M = -\frac{3EI\Delta}{L^2} \tag{11.6b}$$

11.2 DISTRIBUTION FACTORS

The concept of distribution factors is essential to understand the moment distribution method. Four members meet at joint O as shown in Fig. 11.2a. The joint O is rigid, that is, the angles between the members meeting at O remain unchanged. Let the members be fixed at A, C, D and the end B is hinged. The length and moment of inertia of the members are shown in the same figure. Under the application of a moment M at O, the frame will deflect as in Fig. 11.2a. It is desired to know what internal moments are developed in the ends of the various members at O. As the joint O is rigid, the angles

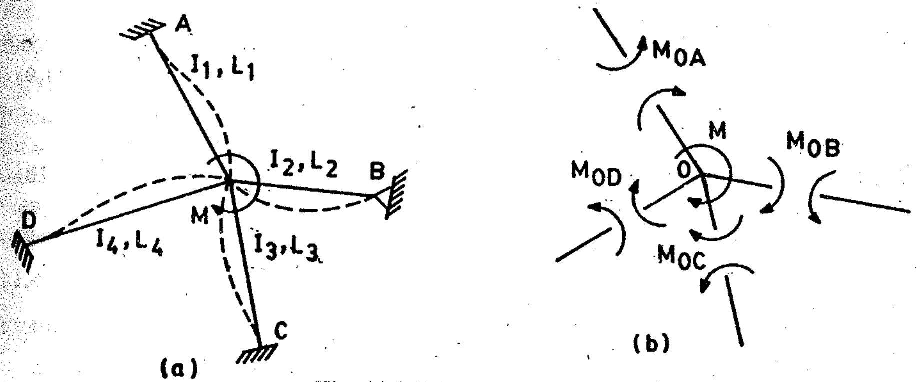


Fig. 11.2 Joint equilibrium

between the various members at O remain unchanged after the application of the moment M. The summation of the absolute stiffnesses of all the members meeting at O will be a measure of the resistance of joint O to rotation. It is obvious that each member will resist a proportion of the total applied moment such that $\Sigma M = 0$ at the joint. Let the moment shared by each member OA, OB, OC and OD be M_{OA}, M_{OB}, M_{OC} and M_{OD}, respectively as shown in Fig. 11.2b. Let the rotation created at O be θ. The moment required to produce a rotation θ in each of the four members is given by Eq.11.3 or 11.5, that is,

$$M_{OA} = \frac{4EI_1}{L_1}\theta$$

$$M_{OB} = \frac{3EI_2}{L_2}\theta$$

$$M_{OC} = \frac{4EI_3}{L_3}\theta$$

$$M_{OD} = \frac{4EI_4}{L_4}\theta \qquad (11.7)$$

Thus, $$M_{OA} : M_{OB} : M_{OC} : M_{OD} :: \frac{4EI_1}{L_1} : \frac{3EI_2}{L_2} : \frac{4EI_3}{L_3} : \frac{4EI_4}{L_4} \qquad (11.8)$$

In the above expression absolute values of the stiffness have been considered. It would be more convenient if they may be reduced to lowest terms by a common divisor. When such a division is made, the resulting numbers will be related in some constant ratio to the corresponding values of the absolute stiffness, and are called as *relative stiffnesses*. When a frame is composed of prismatic members it is easy to use values of I/L for the relative stiffness of each member. Thus, relative stiffness of a member whose far end is fixed is I/L, and that of a member whose far end is simply supported or hinged is 3I/4L or 0.75I/L.

∴ Eq. 11.8 can be rewritten as:

$$M_{OA} : M_{OB} : M_{OC} : M_{OD} :: \frac{I_1}{L_1} : \frac{0.75\,I_2}{L_2} : \frac{I_3}{L_3} : \frac{I_4}{L_4} \qquad (11.9)$$

Also, $$M_{OA} + M_{OB} + M_{OC} + M_{OD} = M \qquad (11.10)$$

or, $$M_{OA} = \frac{\frac{I_1}{L_1}}{\left[\frac{I_1}{L_1} + \frac{0.75\,I_2}{L_2} + \frac{I_3}{L_3} + \frac{I_4}{L_4}\right]} M = d_1 M$$

or, $$M_{OB} = \frac{\frac{0.75\,I_2}{L_2}}{\left[\frac{I_1}{L_1} + \frac{0.75\,I_2}{L_2} + \frac{I_3}{L_3} + \frac{I_4}{L_4}\right]} M = d_2 M$$

$$M_{OC} = \frac{\frac{I_3}{L_3}}{\left[\frac{I_1}{L_1} + \frac{0.75\,I_2}{L_2} + \frac{I_3}{L_3} + \frac{I_4}{L_4}\right]} M = d_3 M$$

and $$M_{OD} = \frac{\frac{I_4}{L_4}}{\left[\frac{I_1}{L_1} + \frac{0.75\,I_2}{L_2} + \frac{I_3}{L_3} + \frac{I_4}{L_4}\right]} M = d_4 M \qquad (11.11)$$

The quantities d_1, d_2, d_3, and d_4, etc. are called the distribution factors. The distribution factor for any member at a joint is equal to the relative stiffness of the member divided by the sum of the relative stiffnesses of all members meeting at the joint.

With the help of distribution factors, it is very convenient to calculate the bending moment created in various members meeting at a rigid joint under the action of an external moment at that joint.

11.3 SIGN CONVENTION

Any moment considered at the end of a member will always be the moment the joint or the support exerts on the member. If a support tends to rotate the member clockwise, the moment will be considered as positive, otherwise the moment will be considered as negative.

The following examples illustrate the moment distribution method on the lines discussed in section 1.4. It is an universal practice to show the iterative procedure directly on a sketch of the structure. This arrangement is quite convenient and helps in understanding the physical behaviour better. In case of a complicated frame, the computations are done in a tabular form which is quite different than the sketch of the structure.

11.4 BEAMS AND FRAMES WITH NO SIDE SWAY

Example 11.1

Analyze the propped cantilever beam shown in Fig. 11.3 by the moment distribution method.

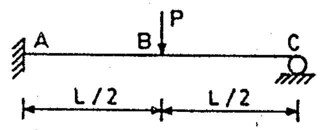


Fig. 11.3

Solution

(a) Fixed End Moments

$$M_{FAC} = -\frac{P(L/2)(L/2)^2}{L^2} = -\frac{PL}{8}$$

$$M_{FCA} = +\frac{P(L/2)^2(L/2)}{L^2} = +\frac{PL}{8}$$

(b) Distribution factors

Since only one member is meeting at joint C, the total unbalance moment will remain confined to the same joint. Hence, distribution factor is 1.00. Joint A is naturally fixed and takes any moment assigned to it. Hence distribution factor at a fixed joint is zero.

(c) Moment- Distribution (Table 11.1)

Table 11.1

Joint		A	C
Member		AC	CA
Distribution factor		0	1.0
Cycle 1	FEM	– PL/8	+ PL/8
	BAL	0	– PL/8
Cycle 2	CO	– PL/16	0
	BAL	0	
Total		– 3PL/16	0

End C is simply supported, hence the net bending moment at C will be zero. The initial locking moment at C is PL/8 (clockwise). Joint C can be released by applying an anti clockwise moment equal to – PL/8. Since support A is naturally fixed, it can readily carry the entire moment assigned or carry-over to it. This joint need not be released.

In the second cycle of moment distribution, the counter-clockwise moment at C induces a carry-over moment of 1/2 (– PL/8) = – PL/16 counter-clockwise at A. The

balancing moment at A and C are zero. The total balanced moments are obtained by adding all numbers in the corresponding joint columns.

The resulting moment diagram is the same as obtained by the consistent deformation method in Example 3.1.

Example 11.2 (f)

Analyze the two span continuous beam shown in Fig. 11.4a by the moment - distribution method.

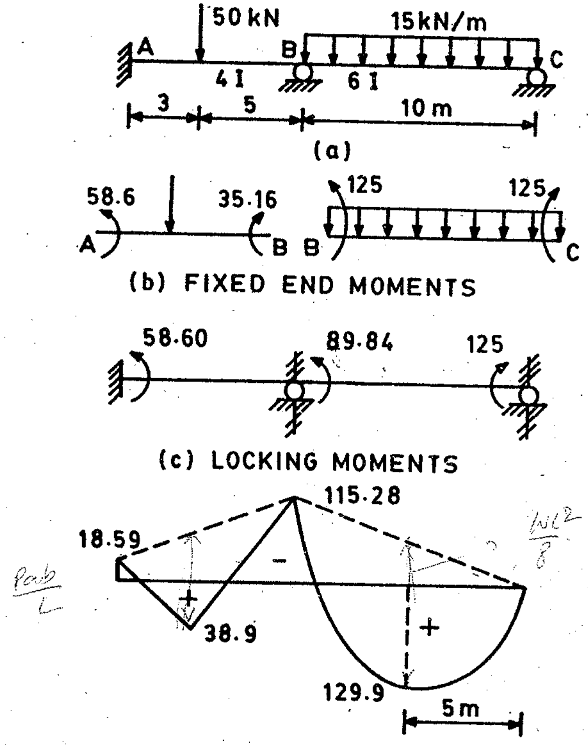


Fig. 11.4 Two span beam

Solution

(a) Fixed End Moments (Fig. 11.4b)

Span AB

$$M_{FAB} = -\frac{50(3)(5)^2}{8^2} = -58.60 \text{ kNm}$$

$$M_{FBA} = +\frac{50(3)^2(5)}{8^2} = +35.16 \text{ kNm}$$

Span BC

$$M_{FBC} = -15\times\frac{10^2}{12} = -125 \text{ kNm}$$

$$M_{FCB} = +15\times\frac{10^2}{12} = +125 \text{ kNm}$$

(b) Distribution Factors

Stiffness of member $AB = \frac{4EI}{L} = \frac{4E(4I)}{8} = 2EI$

$BC = \frac{4EI}{L} = \frac{4E(6I)}{10} = 2.4\,EI$

Stiffness just to the left of support $A = \infty$

DF at joint A in span $AB = \frac{2EI}{\infty + 2EI} = 0$

DF at joint B in span $BA = \frac{\text{Span stiffness}}{\Sigma \text{ stiffness of members meeting at joint B}}$

$$= \frac{2EI}{2EI + 2.4EI} = 0.4545$$

DF at joint B in span $BC = \frac{2.4EI}{2EI + 2.4EI} = 0.5455$

DF at joint C in span CB = 1.00, since there is only one member meeting at joint C.

(c) Moment Distribution

There is a counter-clockwise locking moment of 89.84 kNm at B and clockwise locking moment of 125 kNm at C as shown in Fig. 11.4c. In order to release the locking moment at B, a clockwise moment equal to 0.4545 × 89.84 = 40.83 kNm is applied at B in span BA, and a clockwise moment equal to 0.5455 × 89.84 = 49.01 kNm is applied at B in span BC. The locking moment at C is released by applying a counter-clockwise moment equal to 125 kNm at C itself since end C is simply supported. This completes the first cycle of moment distribution.

The clockwise moment 40.83 kNm applied at B in span BA induces a carry over moment equal to 1/2(40.83) at A in the same span. Similarly, the clockwise moment 49.01 kNm applied at B in span BC induces a carry over moment equal to 1/2 (49.01) = 24.50 kNm at C in the same span. The counter clockwise moment 125 kNm applied at C induces a counter-clockwise moment equal to 1/2(125) = 62.5 kNm at B in the same span BC. The carry over moment (CO) equal to + 20.41 – 62.5 and + 24.50 are the new set of fixed end moments required to lock all joints at the end of the first cycle.

FEM → Balance at the joint → multiply by ... with opposite sign

Table 11.2

Joint		A	B		C
Member		AB	BA	BC	CB
Stiffness		2EI	2EI	2.4EI	2.4EI
Distribution Factor		0	0.4545	0.5455	1.00
Cycle 1	FEM	– 58.60	+ 35.16	– 125	+ 125
	BAL	0	+ 40.83	+ 49.01	– 125
Cycle 2	CO	+ 20.41		– 62.50	+ 24.50
	BAL	0	+ 28.43	+ 34.10	– 24.50
Cycle 3	CO	+ 14.20		– 12.25	+ 17.05
	BAL	0	+ 5.56	+ 6.69	– 17.05
Cycle 4	CO	+ 2.78		– 8.52	+ 3.34
	BAL	0	+ 3.87	+ 4.65	– 3.34
Cycle 5	CO	+ 1.93		– 1.67	+ 2.32
	BAL	0	+ 0.76	+ 0.91	– 2.32
Cycle 6	CO	+ 0.38		– 1.16	+ 0.45
	BAL	0	+ 0.53	+ 0.63	– 0.45
Cycle 7	CO	+ 0.26		– 0.22	+ 0.31
	BAL	0	+ 0.10	+ 0.12	– 0.31
Cycle 8	CO	+ 0.05		– 0.15	+ 0.06
	BAL	0	+ 0.07	+ 0.08	– 0.06
Total		– 18.59	+ 115.28	– 115.28	0
Check	Change in moment	+ 40.01	+ 80.12	+ 9.72	– 125
	1/2(Change in moment)	40.06	+ 20.00	– 62.50	+ 4.86
	Difference	– 0.05	60.07	72.22	– 129.86
	E Iθ	– 0.03	40.08	40.12	– 72.14

In the second cycle, the joints B and C are released for the second time. The counter-clockwise moment of 62.5 kNm at B is released by applying a clockwise moments of 28.40 and 34.10 kNm at B in spans BA and BC. Clockwise moment of 24.50 kNm at C is released by applying a counter-clockwise moment of 24.50 kNm at C. Joint A is naturally fixed, hence, need not be released. This completes the second cycle of moment distribution.

The carry over moment of + 14.20, – 12.25 and + 17.05 kNm are the new set of fixed end moments required to lock all joints at the beginning of the third cycle. The above process can be repeated for as many cycles to achieve the desired accuracy.

The total moments are obtained by adding all numbers in the corresponding joint columns. The free span moments are superimposed on the support moments. The net bending moment diagram is shown in Fig. 11.4d.

(d) Check

The moment distribution method is an iterative method. It is essential to check the accuracy of the final result. The first check is to see if the algebraic sum of all the end moments meeting at a joint is zero or not, except at a fixed support. This check is on the joint moment equilibrium.

There can be another independent check on the calculations. This makes use of the conditions of continuity of slopes at each joint. The slope-deflection equations for a span AB may be written as :

$$M_A = M_{FAB} + \frac{2EI}{L}(2\theta_A + \theta_B)$$

$$M_B = M_{FBA} + \frac{2EI}{L}(\theta_A + 2\theta_B)$$

Solution of these two equations give

$$\theta_A = \frac{(M_A - M_{FAB}) - \frac{1}{2}(M_B - M_{FBA})}{3EI/L}$$

$$\theta_B = \frac{(M_B - M_{FBA}) - \frac{1}{2}(M_A - M_{FAB})}{3EI/L}$$

These equations may be rewritten as:

$$\text{Slope at near end} = \frac{\text{Change at near end} - \frac{1}{2}(\text{Change at far end})}{3EI/L} \qquad (11.12)$$

Using the above equation, the compatibility condition can be satisfied. The slopes at the end of each member meeting at a joint must be equal to each other. The slope at a fixed end should be equal to zero.

It can be seen in Table 11.2 that the slope at the fixed end A is nearly zero. The slopes at joint B are nearly equal. For a better accuracy, the number of cycles should be increased.

(e) Final Moments

The total moments obtained at the end of cycle 8 are the end moments or support moments. These moments and the free span moments are superimposed algebraically as shown in Fig. 11.4d.

Example 11.3

Analyze the continuous beam of Fig. 11.5a by the method of moment distribution. Draw the shear force and bending moment diagrams and sketch the deflected structure.

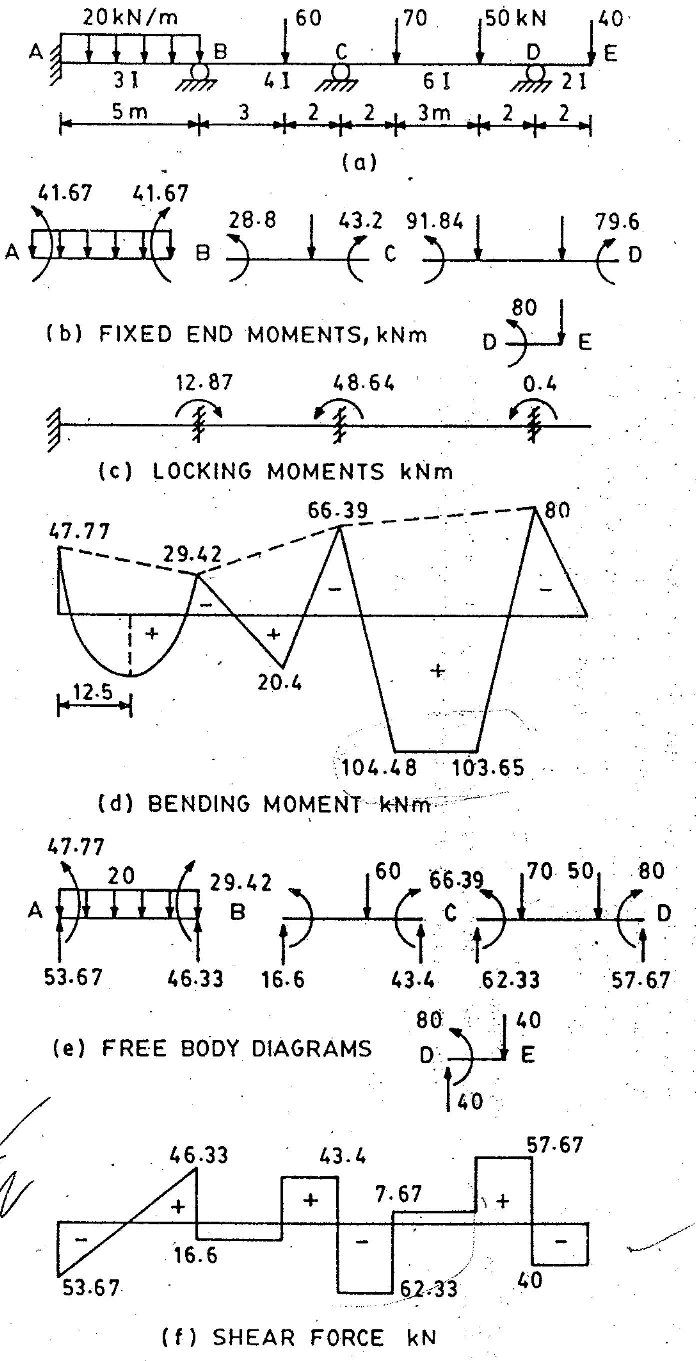


Fig. 11.5 Three span beam with over hang

Solution

(a) Fixed End Moment (Fig. 11.5b)

Span AB $\quad M_{FAB} = -\dfrac{20\times 5^2}{12} = -\ 41.67\ \text{kNm}, \quad M_{FBA} = +\ 41.67\ \text{kNm}$

Span BC $\quad M_{FBC} = -\ 60\times\dfrac{3\times 2^2}{5^2} = -\ 28.80\ \text{kNm}$

$$M_{FCB} = +\ 60\times\frac{3^2\times 2}{5^2} = +\ 43.20\ \text{kNm}$$

Span CD $\quad M_{FCD} = -\ 70\times\dfrac{2\times 5^2}{7^2} - 50\times\dfrac{5\times 2^2}{7^2} = -\ 91.84\ \text{kNm}$

$$M_{FDC} = +\ 70\times\frac{2^2\times 5}{7^2} + 50\times\frac{5^2\times 2}{7^2} = +\ 79.60\ \text{kNm}$$

Span DE $\quad M_{DE} = -\ 80\ \text{kNm}$

(b) Distribution Factors

Stiffness $\quad k_{AB} = \dfrac{4E(3I)}{5} = 2.40\ EI, \quad k_{BC} = \dfrac{4E(4I)}{5} = 3.20\ EI$

$$k_{CD} = \frac{4E(6I)}{7} = 3.43\ EI$$

DF at joint A = 0, since joint A is a fixed end.

DF at joint B in span $\quad BA = \dfrac{2.40EI}{2.40EI + 3.20EI} = 0.4286$

$$BC = \frac{3.20EI}{5.65EI} = 0.5714$$

DF at joint C in span $\quad CB = \dfrac{3.20EI}{3.20EI + 3.43EI} = 0.4826$

$$CD = \frac{3.43EI}{6.63EI} = 0.5174$$

The cantilever span DE is not a *usual* member. It can be removed by imposing a fixed end moment at D in the span DE equal to – 80 kNm. Its nature is negative since it acts in anti-clockwise direction. Now there remains only one member at joint D. Hence distribution factor at D in the span DC is 1.0000.

(c) Moment Distribution

Table 11.3

Joint		A	B		C		D	
Member		AB	BA	BC	CB	CD	DC	DE over hang
DF		0	0.4286	0.5714	0.4826	0.5714	1.0000	-
Cycle 1	FEM	–41.67	41.67	–28.80	+43.20	–91.84	+79.60	–80.00
	BAL	0	– 5.52	– 7.35	+23.47	+25.17	+ 0.40	
Cycle 2	CO	– 2.76		+11.74	– 3.68	+ 0.20	+12.58	
	BAL	0	– 5.03	– 6.71	+ 1.68	+ 1.80	–12.58	
Cycle 3	CO	– 2.51		+ 0.84	– 3.36	– 6.29	+ 0.90	
	BAL	0	– 0.36	– 0.48	+ 4.66	+ 4.99	– 0.90	
Cycle 4	CO	– 0.18		+ 2.33	– 0.24	– 0.45	+ 2.50	
	BAL	0	– 1.00	– 1.33	+ 0.33	+ 0.36	– 2.50	
Cycle 5	CO	– 0.50		+ 0.17	– 0.67	– 1.25	+ 0.18	
	BAL	0	– 0.07	– 0.10	+ 0.93	+ 0.99	– 0.18	
Cycle 6	CO	– 0.04		+ 0.47	– 0.05	– 0.09	+ 0.50	
	BAL	0	– 0.20	– 0.27	+ 0.07	+ 0.07	– 0.50	
Cycle 7	CO	– 0.10		+ 0.04	– 0.14	– 0.25	+ 0.04	
	BAL	0	– 0.02	– 0.02	+ 0.19	+ 0.20	– 0.04	
Cycle 8	CO	– 0.01		+ 0.10	– 0.01	– 0.02	+ 0.10	
	BAL	0	– 0.05	– 0.05	+0.015	+0.015	– 0.10	
Total moment		–47.77	+29.42	–29.42	+66.39	–66.39	+80.00	–80.00
Check	change in moment	– 6.1	–12.25	– 0.62	+23.19	+25.45	+ 0.40	
	1/2 (change in moment	– 6.1	– 3.05	11.59	– 0.31	+ 0.20	+12.73	
	Diff.	0	– 9.20	–12.21	+23.50	+25.25	–12.69	
	EIθ	0	– 5.10	– 5.09	+ 9.79	+ 9.79	– 4.93	

The fixed end moments are written in the first line of cycle 1. The locking moments applied at each support are shown in Fig. 11.5c. The joint B is released by applying counter clockwise moments equal to 5.52 and 7.35 kNm at B in the spans BA and BC, respectively. Similarly, joint C is released. Joint D has an unbalanced moment of – 0.40 kNm. A clockwise moment equal to 0.40 kNm is applied at D in the span DC. Span DE is an over hang whose moment equilibrium depends on the loading in the same span DE only. If there is no load in the span DE, the moment at joint D will be zero.

In the second cycle of moment distribution, the carry over moments are applied at joints A, B, C and D as shown in the table. They are the new locking moments. These joint are again released in proportion to the distribution factors at the respective joints.

The iteration process is continued till the locking moments become sufficiently small. At the end of cycle 8, the total moments are obtained by adding respective columns. For equilibrium, the net moment at each joint must be zero.

(d) Check

The moment equilibrium at each joint is satisfied. Now let us apply the check on the continuity condition at each joint as discussed in Example 11.2. The change in moments at the near end is shown in the first line, the change in moments at the far end is shown in the second line. Their difference is shown in the third line. These values are divided by the corresponding value of 3EI/L. It can be seen that E I θ value at the end of each member meeting at a joint is the same. This shows that the moment distribution method has converged.

(e) Bending Moment and Shear Force Diagrams

The free span moments are as follows:

$$M_{0AB} = 20\times\frac{5^2}{8} = 62.5 \text{ kNm}$$

$$M_{0BC} = 60\times\frac{3\times2}{5} = 72 \text{ kNm}$$

$$M^1_{0CD} = 70\times\frac{2\times5}{7} + 50\times\frac{5\times2}{7} = 171.43 \text{ kNm under 70 kN load}$$

$$M^2_{0CD} = 70\times\frac{5\times2}{7} + 50\times\frac{2\times5}{7} = 171.43 \text{ kNm under 70 kN load}$$

The total moments obtained at the end of cycle 8 are the end moments. These moments and the free span moments are superimposed algebraically as shown in Fig. 11.5d.

The free body diagram of each span is shown in Fig. 11.5e. The resulting shear force diagram is shown in Fig. 11.5f.

Example 11.4

Analyze the frame shown in Fig. 11.6a by the moment distribution method and draw moment diagrams.

Solution

(a) Fixed End Moments

Span AB $\quad M_{FAB} = -\dfrac{50\times5^2}{12^2} = -\ 104.17 \text{ kNm}, \quad M_{FBA} = +\ 104.17 \text{ kNm}$

Span BC $\quad M_{FBC} = -\ 15\times1 = -\ 15 \text{ kNm}$

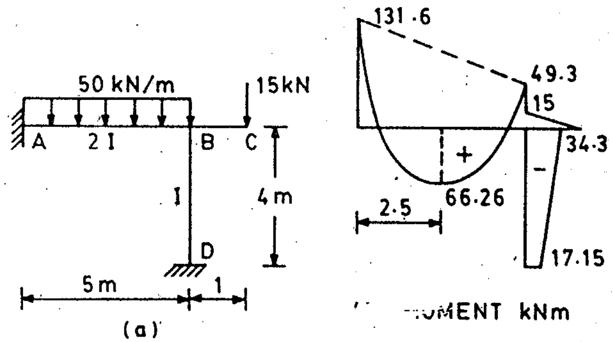


Fig. 11.6

Span BD $\quad M_{FBD} = 0 = M_{FDB}$

(b) Distribution Factors

DF at joint A in span AB = 0, A being a fixed end

Relative stiffness of member AB = 2 I/5 = 0.4000 I

Relative stiffness of member BD = I/4 = 0.2500 I

Member BC is over hang member and it can be replaced by imposing a counter clockwise moment at B equal to 15 × 1 = 15 kNm.

DF at joint B in the span BA $= \dfrac{0.4000\text{I}}{0.4000\text{I} + 0.2500\text{I}} = 0.6154$

DF at joint B in the span BD $= 1 - 0.6154 = 0.3845$

DF at joint D in the span DB = 0, D being a fixed end.

(c) Moment Distribution

In the case of frames, it is usual to arrange all the joints in alphabetic order. The members meeting at each joint are written under the joint as shown in Table 11.5. Since there may be more than two members meeting at a joint, the carry overs are not necessarily between adjacent columns. The carry overs are always from near end to far end of the same member.

The iterations stop at the end of cycle 2 since there is only one unknown joint rotation in the given rigid frame. A linear equation of one unknown can be solved directly without any iterations.

(d) Check

The joint moment equilibrium is satisfied at joint B. The compatibility check is also shown. The slopes at fixed ends A and D are zero. The slope at joint B in the spans BA and BD is equal.

(e) The bending moment diagram is shown in Fig. 11.6b.

Table 11.5

Joint		A	B			D
Member		AB	BA	BD	BC	DB
Relative Stiffness I/L		0.4000	0.4000	0.2500	-	0.2500
DF		0	0.6154	0.3845	-	0
Cycle 1	FEM	–104.17	104.17	0	– 15	0
	BAL	0	– 54.87	– 34.30	0	0
Cycle 2	CO	–27.43				– 17.15
	BAL	0				0
Total Moments		–131.60	49.30	– 34.30	– 15	– 17.15
Check	Change in moment	–27.43	– 54.87	– 34.30		– 17.15
	1/2 (change in moment	–27.43	– 13.71	– 8.57		– 17.15
	Difference	0	– 41.16	– 25.73		0
	EIθ	0	– 34.30	– 34.30		0

Example 11.5

Analyze the frame shown in Fig. 11.7a by the moment distribution method.

Solution

(a) Fixed End Moment

Span BC $\quad M_{FBC} = -\ 50 \times \dfrac{2 \times 2^2}{4^2} = -\ 25 \text{ kNm}, \quad M_{FCB} = +\ 25 \text{ kNm}$

Span EC $\quad M_{FEC} = -\ 40 \times \dfrac{1.5 \times 2.5^2}{4^2} = -\ 23.44 \text{ kNm}$

$$M_{FCE} = +\ 40 \times \frac{1.5^2 \times 2.5}{4^2} = +\ 14.10 \text{ kNm}$$

There is no external load on spans AB and CD, hence fixed end moment are zero on these spans.

(b) Distribution Factors

Relative stiffness of members

$$k_{AB} = 4E \times \frac{I}{3} = 1.3334 \text{ E I}, \quad k_{BC} = 4E \times \frac{2I}{4} = 2.0000 \text{ E I}$$

$$k_{CD} = 3E \times \frac{2I}{5} = 1.2000 \text{ E I}, \quad k_{CE} = 4E \times \frac{1.5I}{4} = 1.5000 \text{ E I}$$

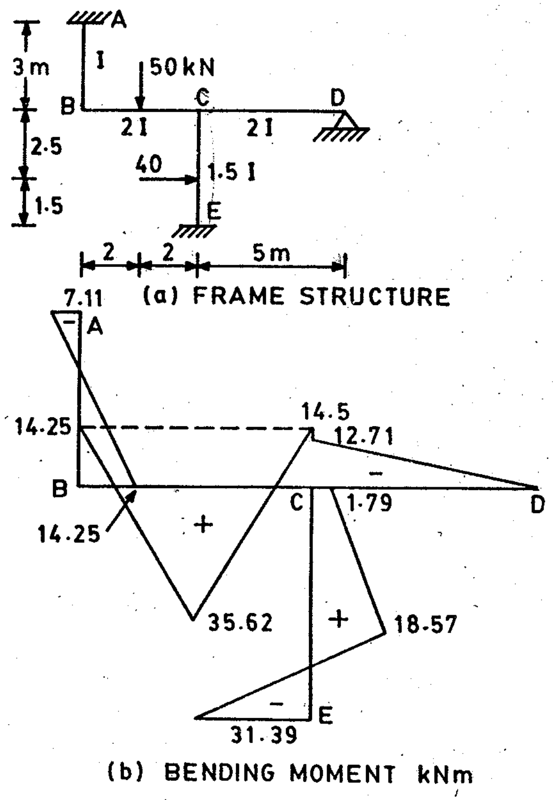


Fig. 11.7

Since joint D is hinged, either the regular stiffness factor (= 4EI/L) or the modified stiffness factor (= 3/4 × 4EI/L = 3EI/L) can be used for end D of member CD. They will lead to identical results with a slight change in the moment distribution. In the present case, the modified stiffness of member CD will be used and the moment distribution process is shown in Table 11.5.

DF at joint B in span $\quad BA = \dfrac{1.3334EI}{1.3334EI + 2.0000EI} = 0.4000$

span $\quad BC = \dfrac{2.0000EI}{3.3334EI} = 0.6000$

DF at joint C in span $\quad CB = \dfrac{2.0000EI}{(2.0000 + 1.2000 + 1.5000)EI} = 0.4255$

$$CD = \frac{1.2000EI}{4.7000EI} = 0.2553$$

$$CE = \frac{1.5000EI}{4.7000EI} = 0.3192$$

Check $\sum DF$ at joint C = 1 O.K.

DF at joint D is not applicable since modified stiffness of member CD is used.

DF at joints A and E are zero, since they are fixed ends.

(c) Moment Distribution

The moment distribution process is shown in Table 11.5. It can be seen that since modified stiffness is used for member CD whose far end D is hinged, no carry over moment is transferred to end D. In this case 6 cycles are required to achieve a reasonable accuracy. Both the joint moment equilibrium and continuity of slopes conditions at each joint are satisfied.

The bending moment diagram is shown in Fig. 11.7b.

Table 11.5

Joint		A	B		C			D.	E
Member		AB	BA	BC	CB	CD	CE	DC	EC
DF		0	0.4000	0.6000	0.4255	0.2553	0.3192	–	0
Cycle 1	FEM	0	0	– 25	+ 25	0	+ 14.10	0	– 23.44
	BAL		+ 10.00	+ 15.00	– 16.64	– 9.98	– 12.48	0	0
Cycle 2	CO	+ 5.00		– 8.32	+ 7.50				– 6.24
	BAL	0	+ 3.33	+ 4.99	– 3.19	– 1.91	– 2.40		0
Cycle 3	CO	+ 1.67		– 1.60	+ 2.50				– 1.20
	BAL	0	+ 0.64	+ 0.96	– 1.06	– 0.64	– 0.80		0
Cycle 4	CO	+ 0.32		– 0.53	+ 0.48				– 0.40
	BAL	0	+ 0.21	+ 0.31	– 0.20	– 0.13	– 0.15		0
Cycle 5	CO	+ 0.10		– 0.10	+ 0.16				– 0.08
	BAL	0	+ 0.04	+ 0.06	– 0.07	– 0.04	– 0.05		0
Cycle 6	CO	+ 0.02		– 0.04	+ 0.03				– 0.03
	BAL	0	+ 0.02	+ 0.02	– 0.01	– 0.01	– 0.01		0
Total		+ 7.11	+ 14.24	– 14.25	+ 14.50	– 12.71	– 1.79	0	– 31.39
Change in moment		+ 7.11	+ 14.24	+ 10.75	– 10.5	– 12.71	– 15.89	0	– 7.95
1/2 (Change in moment)		+ 7.12	+ 3.56	– 5.25	+ 5.38		– 3.98	– 6.35	– 7.95
Difference		0	+ 10.68	+ 16.00	– 15.88	– 12.71	– 11.91	+ 6.35	0
EIθ		0	+ 10.68	+ 10.67	– 10.58	– 10.59	– 10.59	+ 5.29	0

11.5 BEAMS WITH UNEVEN SUPPORT SETTLEMENT

Example 11.6

Analyze the beam shown in Fig. 11.8a due to sinking of support B by 10 mm by the moment distribution method. Take E= 200 GPa and I = 200×10^{-6} m^4.

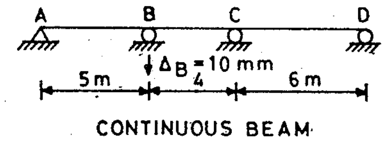


Fig. 11.8

Solution

(a) Fixed End Moments

$$M_{FAB} = M_{FBA} = -\frac{6EI\Delta}{L^2} = -\frac{6\left(200\times10^{+6}\times200\times10^{-6}\right)0.010}{5^2} = -96 \text{ kNm}$$

$$M_{FBC} = M_{FCB} = -\frac{6EI\Delta}{L^2} = -\frac{6\times\left(200\times10^{+6}\times200\times10^{-6}\right)\times(-0.010)}{4^2}$$

$$= 150 \text{ kNm}$$

$$M_{FCD} = M_{FDC} = 0$$

(b) Distribution Factors

Ends A and D are simply supported. The distribution factors may be computed using either regular stiffnesses of members AB and CD, or their modified stiffnesses.

(i) Let us first compute the distribution factors using the regular stiffnesses of members AB and CD.

$$k_{AB} = \frac{I}{5} = 0.2\text{ I},\quad k_{BC} = \frac{I}{4} = 0.25\text{ I},\quad k_{CD} = \frac{I}{6} = 0.1667\text{ I}$$

DF at joint A in the span AB = 1.0

DF at joint B in the span BA $= \dfrac{0.2I}{0.2I + 0.25I} = 0.4445$

BC $= \dfrac{0.25I}{0.45I} = 0.5555$

DF at joint C in the span CB $= \dfrac{0.25I}{0.25I + 0.1667I} = 0.6000$

$$CD = 0.4000$$

DF at joint D in the span $\quad DC = 1.0$

Table 11.6a Moment distribution using regular stiffnesses

Joint		A	B		C		D
Member		AB	BA	BC	CB	CD	DC
DF		1.0000	0.4445	0.5555	0.6000	0.4000	1.000
Cycle 1	FEM	− 96	− 96	+ 150	+150	0	0
	BAL	+ 96	− 24.00	− 30.00	− 90.00	− 60.00	0
Cycle 2	CO	− 12.00	+ 48.00	− 45.00	− 15.00	0	− 30.00
	BAL	+ 12.00	− 1.34	− 1.66	+ 9.00	+ 6.00	+ 30.00
Cycle 3	CO	− 0.67	+ 6.00	+ 4.50	− 0.83	+ 15.00	+ 3.00
	BAL	+ 0.67	− 4.67	− 5.83	− 8.50	− 5.67	− 3.00
Cycle 4	CO	− 2.34	+ 0.34	− 4.25	− 2.92	− 1.50	− 2.84
	BAL	+ 2.34	+ 1.74	+ 2.17	+ 2.65	+ 1.77	+ 2.84
Cycle 5	CO	+ 0.87	+ 1.17	+ 1.32	+ 1.08	+ 1.42	+ 0.88
	BAL	− 0.87	− 1.10	− 1.39	− 1.50	− 1.00	− 0.88
Cycle 6	CO	− 0.55	− 0.43	− 0.75	− 0.70	− 0.44	− 0.50
	BAL	+ 0.55	+ 0.52	+ 0.66	+ 0.68	+ 0.46	+ 0.50
Cycle 7	CO	+ 0.26	+ 0.20	+ 0.34	+ 0.33	+ 0.25	+ 0.23
	BAL	− 0.26	− 0.24	− 0.30	− 0.35	− 0.23	− 0.23
Cycle 8	CO	− 0.12	− 0.13	− 0.18	− 0.15	− 0.12	− 0.12
	BAL	+ 0.12	+ 0.14	+ 0.17	+ 0.16	+ 0.11	+ 0.12
Total moment		0	− 69.80	+ 69.80	+ 43.95	− 43.95	0
Change in moment		+ 96	+ 26.20	− 80.20	−106.05	− 43.95	0
1/2 (Change in moment)		+ 13.10	+ 48	− 53.02	− 40.10	0	
Difference		+ 82.90	− 21.80	− 27.18	− 65.95	− 43.95	
E I θ		+138.17	− 36.33	− 36.24	− 87.93	− 87.90	

(ii) Let us now compute the distribution factors using the modified stiffnesses of members AB and CD.

$$k_{AB} = \frac{3EI}{L} = \frac{3EI}{5} = 0.6\,EI \qquad k_{BC} = \frac{4EI}{L} = \frac{4EI}{4} = 1.0\,EI$$

$$k_{CD} = \frac{3EI}{L} = \frac{3EI}{6} = 0.5\,EI$$

DF at joint B in the span $\quad BA = \dfrac{0.6EI}{0.6EI + 1.0EI} = 0.3750$

$$BC = 0.6250$$

DF at joint C in the span $\quad CB = \dfrac{1.0EI}{1.0EI + 0.5EI} = 0.6667$

$$CD = 0.3333$$

(c) Moment Distribution

The moment distribution process using regular and modified stiffnesses is shown in Table 11.6a and b.

Modified Moment Distribution

The distribution factors are computed using the modified stiffnesses of spans AB and CD. The modified stiffness of a span signify that its far end is hinged or simply supported. Hence bending moment is zero at the far end. Initially joint A is balanced by applying +96 kNm at A. One-half of this moment is carried over to support B in the span BA. The net moments are written by adding the respective columns. It can be seen that the moments at supports A and D are zero. These are referred to as the modified fixed end moments. The moment distribution cycles are assumed to begin from this point in the usual manner. The bending moment diagram was shown in Fig. 10.5b.

Table 11.6b Moment distribution using modified stiffnesses

Joint		A	B		C		D
Member		AB	BA	BC	CB	CD	DC
DF			0.3750	0.6250	0.6667	0.3333	
	FEM	−96	− 96	+150	+150	0	0
	BAL	+96	0	0	0		0
	CO	0	+ 48	0	0		
	BAL	0	0	0	0	0	0
Cycle 1	MFEM		− 48	+150	+150	0	0
	BAL		− 38.25	− 63.75	−100.00	− 50.00	
Cycle 2	CO			− 50.00	− 31.88		
	BAL		+ 18.75	+ 31.25	+ 21.25	+ 10.63	
Cycle 3	CO			+ 10.62	+ 15.62		
	BAL		+ 3.98	− 6.64	− 10.41	− 5.21	
Cycle 4	CO			− 5.20	− 3.32		
	BAL		+ 1.95	+ 3.25	+ 2.21	+ 1.11	
Cycle 5	CO			+ 1.11	+ 1.62		
	BAL		− 0.42	− 0.69	− 1.08	− 0.54	
Cycle 6	CO			− 0.54	− 0.34		
	BAL		+ 0.20	+ 0.34	+ 0.23	+ 0.11	
Cycle 7	CO			+ 0.11	+ 0.17		
	BAL		− 0.04	− 0.07	− 0.11	− 0.06	
Cycle 8	CO			− 0.06	− 0.04		
	BAL		+ 0.02	+ 0.04	+ 0.03	+ 0.01	
Total moment		0	− 69.77	+ 69.77	+ 43.95	− 43.95	0

11.6 FRAMES WITH SIDE SWAY

The frames with side sway were analyzed using the slope-deflection method. The fixed end moments were computed due to an arbitrary side sway Δ and using additional shear equilibrium equations, the magnitude of side sway was determined. It is not possible to repeat this procedure in the moment distribution method which is iterative in nature. The moment distribution method is carried out in two stages:

Consider the frame shown in Fig. 11.9a. Its true deflected shape is shown in Fig. 11.9b. In stage 1, an imaginary support is provided at the level of the beam (s) to prevent the side sway. This problem can be solved using the moment distribution procedure as discussed earlier. In stage 2, an arbitrary lateral displacement Δ is imposed at the level of beam (s), and the fixed end moments are computed as before. These moments are written in terms of the arbitrary displacement Δ. The external loads are withdrawn from the frame for stage 2 analysis. The moment distribution procedure is carried out for the arbitrary values of the sway.

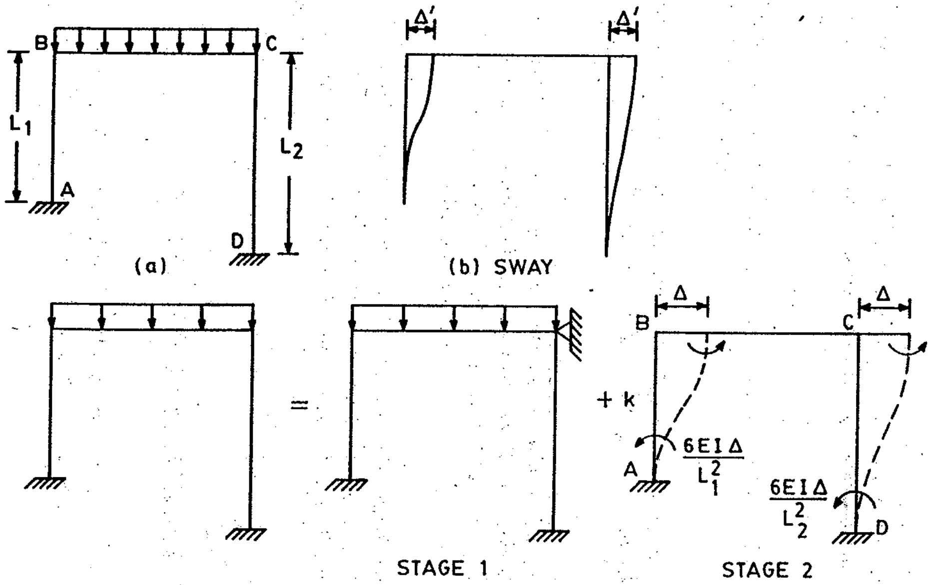


Fig. 11.9 Single storey frame with side sway

The true situation is obtained by superposition of the results of stage 1 and 2. The equation of shear equilibrium is written assuming that the correct moments in stage 2 are k times those obtained due to the arbitrary sway Δ. The factor k tells us how much to scale the moments of stage 2.

$$\frac{M^1_{AB}+M^1_{BA}}{L_1}+\frac{M^1_{DC}+M^1_{CD}}{L_2}+k\left[\frac{M^2_{AB}+M^2_{BA}}{L_1}+\frac{M^2_{DC}+M^2_{CD}}{L_2}\right]=0$$

where, the superscript 1 refers to the first stage and 2 refers to the second stage.

Thus, the shear equation provides the value of the factor k. The scaled moments of stage 2 are then added to the moments of the stage 1 to obtain the actual moments in the structure. The true sway Δ' is equal to k times the arbitrary sway Δ.

Consider a two storey portal frame shown in Fig. 11.10a. It is a typical situation in which there is more than one degree of freedom in side sway. The key to the solution is again the principle of superposition. The actual equilibrium and kinematic state of the structure is considered to be the superposition of the three stages shown in Figs. 11.10b. In the first stage, all lateral motions are restrained, one at each floor. The moment distribution is performed for this case. In stage 2a, a lateral sway of displacement Δ_1 is imposed on the top floor while the other floor is prevented from any lateral motion. Fixed end moments are computed in terms of Δ_1 and a moment distribution is performed. In stage 2b, a lateral sway of Δ_2 is imposed on the first floor while the top floor is prevented from any lateral motion. Fixed end moments are computed in terms of Δ_2 and a moment distribution is performed. The principle of superposition requires that

$$Q^1_1+k_1Q^{2a}_1+k_2Q^{2b}_1+P_1=0$$

and $$Q^1_2+k_1Q^{2a}_2+k_2Q^{2b}_2+P_1+P_2=0$$

where Q_1 and Q_2 are the storey shears in the top storey and first storey, respectively, and computed as:

$$Q^i_1 = \left[\frac{M^i_{BC}+M^i_{CB}}{L_1}+\frac{M^i_{ED}+M^i_{DE}}{L_1}\right]$$

$$Q^i_2 = \left[\frac{M^i_{AB}+M^i_{BA}}{L_2}+\frac{M^i_{FE}+M^i_{EF}}{L_2}\right]$$

L_1 = length BC

L_2 = length AB

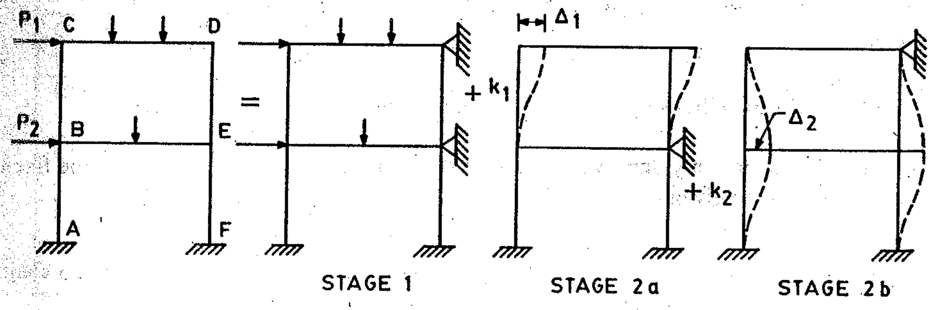


Fig. 11.10 Two storey frame with side sway

These equations are now solved for k_1 and k_2. These are the correction or scale factors to be applied to the moments of stages 2a and 2b to obtain the correct moments. The sum of all three cases give the correct moments in the actual structure.

The following examples illustrate the application of the moment distribution procedure to the sway problems.

Example 11.7

Analyze the portal frame shown in Fig. 11.11a by the moment distribution method and sketch the deflected shape.

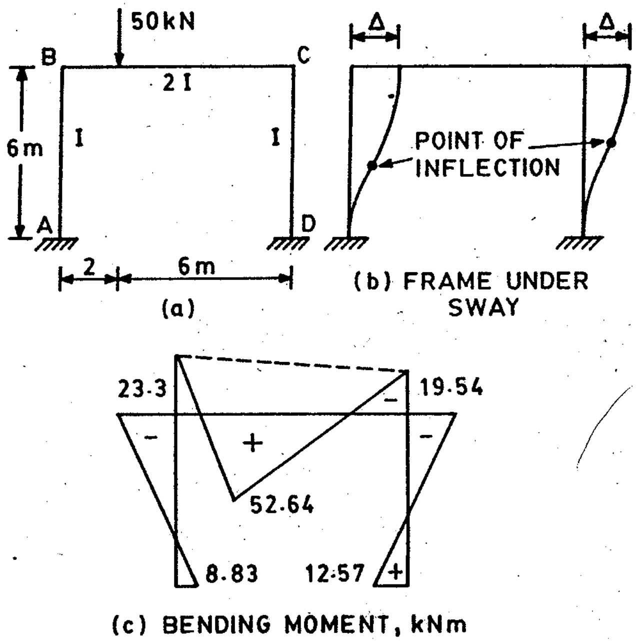


Fig. 11.11

Solution

I. Moment distribution for the applied loads with side sway prevented

This frame has only one sway, that is, to the right. If this sway is restrained, the fixed end moments are given by

$$M_{FBC} = -\frac{50 \times 2 \times 6^2}{8^2} = -56.25 \text{ kNm}, \quad M_{FCB} = \frac{50 \times 2^2 \times 6}{8^2} = +18.75 \text{ kNm}$$

$$M_{FAB} = M_{FBA} = 0, \qquad M_{FDC} = M_{FCD} = 0$$

Distribution Factors

Stiffnesses of members $AB = \frac{EI}{6}, \quad BC = \frac{E(2I)}{8}$

DF at joint B in the span $BA = \frac{EI/6}{EI/6 + EI/4} = 0.4$

DF at joint B in the span $BC = \frac{EI/4}{EI/6 + EI/4} = 0.6$

Same will be the distribution factors at joint C. The moment distribution for the above fixed end moments is shown in Table 11.7a. The values of H_A and H_D are computed by considering the free body diagrams of the columns.

$$H_A = \frac{M_{AB} + M_{BA}}{L_{AB}} = \frac{13.57 + 27.18}{6} = 6.79 \text{ kN}\rightarrow$$

$$H_D = \frac{M_{DC} + M_{CD}}{L_{CD}} = \frac{-7.83 - 15.66}{6} = -3.92 \text{ kN}\leftarrow$$

Table 11.7a

Joint		A	B		C		D
Member		AB	BA	BC	CB	CD	DC
DF		–	0.4	0.6	0.6	0.4	–
Cycle 1	FEM	0	0	– 56.25	+ 18.75	0	0
	BAL	0	+ 22.50	+ 33.75	– 11.25	– 7.50	0
Cycle 2	CO	+ 11.25		– 5.62	+ 16.87		– 3.75
	BAL	0	+ 2.25	+ 3.37	– 10.12	– 6.75	0
Cycle 3	CO	+ 1.12		– 5.02	+ 1.68		– 3.38
	BAL	0	+ 2.01	+ 3.01	– 1.00	– 0.68	0
Cycle 4	CO	+ 1.00		– 0.50	+ 1.50		– 0.34
	BAL	0	+ 0.20	+ 0.30	– 0.90	– 0.60	0
Cycle 5	CO	+ 0.10		– 0.45	+ 0.15		– 0.30
	BAL	0	+ 0.18	+ 0.27	– 0.09	– 0.06	0
Cycle 6	CO	+ 0.09		– 0.05	+ 0.14		– 0.03
	BAL	0	+ 0.02	+ 0.03	– 0.08	– 0.06	0
Cycle 7	CO	+ 0.01		– 0.04	+ 0.02		– 0.03
	BAL	0	+ 0.02	+ 0.02	– 0.01	– 0.01	0
Total moment		+ 13.57	+ 27.18	– 27.18	+ 15.66	– 15.66	– 7.83

II. Moment distribution for arbitrary sway Δ'

Fixed end moment $M_{FAB} = -\dfrac{6EI\Delta'}{L^2} = M_{FBA}$

$$= -\frac{6EI\Delta'}{L^2} = -\frac{6\times 100}{6^2} = -16.67\text{ kNm}$$

Assuming $EI\,\Delta' = 100\text{ kNm}^3$

Similarly, $M_{FCD} = -\dfrac{6EI\Delta'}{L^2} = M_{FDC} = -16.67\text{ kNm}$

There is no external load on the frame in this case. The moment distribution is shown in Table 11.7b.

Table 11.7b

Joint		A	B		C		D
Member		AB	BA	BC	CB	CD	DC
DF		–	0.4	0.6	0.6	0.4	–
Cycle 1	FEM	–16.67	– 16.67	0	0	– 16.67	–16.67
	BAL	0	+ 6.67	+ 10.0	+ 10.0	+ 6.67	0
Cycle 2	CO	+ 3.33		+ 5.0	+ 5.0		+ 3.33
	BAL	0	– 2.0	– 3.0	– 3.0	– 2.0	0
Cycle 3	CO	– 1.00		– 1.5	– 1.5		– 1.00
	BAL	0	+ 0.6	+ 0.9	+ 0.9	+ 0.6	0
Cycle 4	CO	+ 0.30		+ 0.45	+ 0.45		+ 0.30
	BAL	0	– 0.18	– 0.27	– 0.27	– 0.18	0
Cycle 5	CO	– 0.09		– 0.14	– 0.14		– 0.09
	BAL	0	+ 0.06	+ 0.08	+ 0.08	+ 0.06	0
Cycle 6	CO	+ 0.03		+ 0.04	+ 0.04		+ 0.03
	BAL		– 0.02	– 0.02	– 0.02	– 0.02	
Total moment		–14.10	– 11.54	+ 11.54	+ 11.54	– 11.54	–14.10
Corrected Moments		– 4.74	– 3.88	+ 3.88	+ 3.88	– 3.88	– 4.74
Moment (1)		+13.57	+ 27.18	– 27.18	+ 15.66	– 15.66	– 7.83
Net Moment		+ 8.83	+ 23.30	+ 23.30	+ 19.54	– 19.54	–12.57

The values of H_A' and H_D' are computed by considering the free body diagrams of the columns.

$$H_A' = -\left(\frac{14.10 + 11.54}{6}\right) = -4.27\text{ kN}\leftarrow = H_D'$$

III. Correction in sway

The shear condition gives,

$$(H_A + H_D) + k(H_A' + H_D') = 0$$

or, $(6.79 - 3.92) + k(-4.27 - 4.27) = 0$, or, $k = 0.336$

Correct sway $\Delta = k\Delta' = 0.336 \times \dfrac{100}{EI} = \dfrac{33.6}{EI}$ m

The moments obtained in Table 11.7b can be corrected by multiplying with factor k as shown in the same table. The total moments obtained in Table 11.7a can be added algebraically to the corrected moments of Table 11.7b and the net moments are obtained. The bending moment diagram is shown in Fig. 11.11c. The elastic curve is shown in Fig. 11.11b.

Example 11.8

Analyze the portal frame shown in Fig. 11.12 by the moment distribution method.

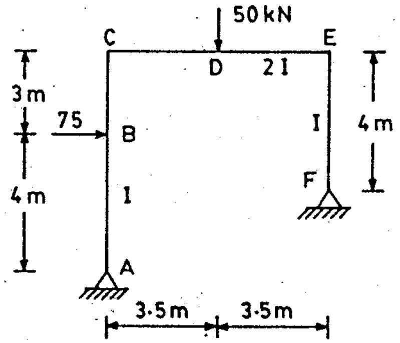


Fig. 11.12

Solution

I. Moment distribution for the applied loads with side sway prevented.

This frame has only one sway, that is, to the right. If this sway is restrained, the fixed end moments are given by :

$$M_{FAC} = -\frac{75\times 4\times 3^2}{7^2} = -55.1\text{ kNm},\quad M_{FCA} = \frac{75\times 4^2\times 3}{7^2} = 73.47\text{ kNm}$$

$$M_{FCE} = -50\times\frac{3.5^3}{7^2} = -43.75\text{ kNm} = -M_{FEC}$$

Member stiffnesses

$$k_{AC} = \frac{4EI}{7}$$

A being a hinged end, modified stiffness $= \dfrac{3}{4}\times\dfrac{4EI}{7} = 0.428\,EI$

$$k_{CE} = \frac{4E(2I)}{7} = 1.143\,EI, \quad k_{EF} = \frac{4EI}{4} = EI$$

F being a hinged end, modified stiffness $= \frac{3}{4} EI = 0.75\,EI$

DF at joint C in span $CA = \frac{0.428EI}{0.428EI + 1.143EI} = 0.27$

$CE = 0.73$

DF at joint E in span $EC = \frac{1.143EI}{(1.143+0.75)EI} = 0.60$

$EF = 0.40$

Table 11.8a

Joint		A	C		E		F
Member		AC	CA	CE	EC	EF	FE
DF		–	0.27	0.73	0.60	0.40	–
FEM		–55.1	+ 73.47	– 43.75	+ 43.75	0	0
BAL		+55.1					
CO			+ 27.55				
BAL		0	0	0			
Cycle 1	MdFE	0	+101.02	– 43.75	+ 43.75	0	0
	M		– 15.46	– 41.81	– 26.25	– 17.50	
	BAL						
Cycle 2	CO			– 13.13	– 20.90		
	BAL		+ 3.54	+ 9.59	+ 12.60	+ 8.30	
Cycle 3	CO			+ 6.30	+ 4.80		
	BAL		– 1.70	– 4.60	– 2.88	– 1.92	
Cycle 4	CO			– 1.44	– 2.30		
	BAL		+ 0.38	+ 1.06	+ 1.38	+ 0.92	
Cycle 5	CO			+ 0.69	+ 0.53		
	BAL		– 0.19	– 0.50	– 0.32	– 0.21	
Cycle 6	CO			– 0.16	– 0.25		
	BAL		+ 0.04	+ 0.12	+ 0.15	+ 0.10	
Total moment		0	+ 87.63	– 87.63	+ 10.31	– 10.31	0

The moment distribution for these fixed end moments is shown in Table 11.8a. The values of H_A and H_F are computed by considering the free body diagrams of the columns.

$$H_A = \frac{M_{AC} + M_{CA}}{L_{AC}} - 75 \times \frac{3}{7} = \frac{87.63}{7} - 75 \times \frac{3}{7} = -19.62 \text{ kN} \leftarrow$$

$$H_F = -\frac{10.31}{4} = -2.58 \text{ kN} \leftarrow$$

II. Moment distribution for arbitrary sway Δ'

Fixed end moment $M_{FAC} = -\frac{6EI\Delta'}{L^2}$

$$= -\frac{6EI\Delta'}{7^2} = -\frac{6 \times 100}{7^2} = -12.25 \text{ kNm} = M_{FCA}$$

Assuming $EI\Delta' = 100 \text{ kNm}^3$

Similarly, $M_{FFE} = M_{FEF} = -\frac{6EI\Delta'}{4^2} = -\frac{6 \times 100}{4^2} = -37.5 \text{ kNm}$

The moment distribution is shown in Table 11.8b.

Table 11.8b

Joint		A	C		E		F
Member		AC	CA	CE	EC	EF	FE
DF		–	0.27	0.73	0.60	0.40	–
FEM		–12.25	–12.25	0	0	– 37.50	–37.50
BAL		+12.25					+37.50
CO			+ 6.13			+ 18.75	
BAL			0	0	0	0	
Cycle 1	FEM	0	– 6.12	0	0	– 18.75	0
	BAL		+ 1.65	+ 4.47	+ 11.25	+ 7.50	
Cycle 2	CO			+ 5.63	+ 2.24		
	BAL		– 1.52	– 4.11	– 1.34	– 0.90	
Cycle 3	CO			– 0.67	– 2.06		
	BAL		+ 0.18	+ 0.49	+ 1.23	+ 0.83	
Cycle 4	CO			+ 0.61	+ 0.25		
	BAL		– 0.16	– 0.45	– 0.15	– 0.10	
Cycle 5	CO			– 0.08	– 0.23		
	BAL		+ 0.02	+ 0.06	+ 0.14	+ 0.09	
Cycle 6	CO			+ 0.07	+ 0.03		
	BAL		– 0.02	– 0.05	– 0.02	– 0.01	
Total moment		0	– 5.97	+ 5.97	+ 11.34	– 11.34	0
Corrected Moment		0	–85.37	+85.37	+162.16	–162.16	0
Moment (1)		0	+87.63	–87.63	+ 10.31	– 10.31	0
Net Moment		0	+ 2.26	– 2.26	+172.47	–172.47	0

The values of H_A' and H_D' are given by

$$H_A' = -\frac{5.97}{7} = -0.85\ \text{kN}\leftarrow, \quad H_F' = -\frac{11.34}{4} = -2.84\ \text{kN}\leftarrow$$

III. Correction in sway

The shear equation gives,

$$\boxed{-(19.62 + 2.58) - k(0.85 + 2.84) + 75 = 0 \text{ or,}} \quad k = 14.30$$

The corrected sway moments and final moments are shown in Table 11.8b.

$$\text{Net } H_A = -19.62 - 14.30 \times 0.85 = -31.78\ \text{kN}\leftarrow$$

$$\text{and } H_F = -2.58 - 14.30 \times 2.84 = -43.22\ \text{kN}\leftarrow$$

These are nearly the same values as obtained by the slope - deflection method, Example 10.9.

11.7 FRAMES WITH UNEVEN SUPPORT SETTLEMENTS

Imp

Example 11.9

Analyze the frame shown in Fig. 11.13 due to sinking of support D by 10 mm. The support D also rotates clockwise by 0.004 radians. Take E = 200 GPa and I = 300 × 10^{-6} m^4. Take E I = constant.

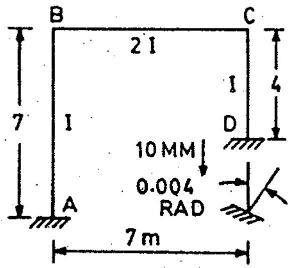


Fig. 11.13

Solution

I. Moment distribution due to yielding of supports when side sway is prevented

(a) Fixed End Moments

$$M_{FAB} = 0 = M_{FBA}$$

$$M_{FBC} = -\frac{6EI\Delta}{L^2} = -\frac{6 \times 200 \times 10^6 \times 2 \times 300 \times 10^{-6} \times (0.01)}{7^2}$$

$$= -146.94\ \text{kNm} = M_{FCB}$$

clockwise rotation $+\frac{4EI}{L}\theta$, $+\frac{2EI}{L}\theta$
anti clockwise " $+$, $-$

where Δ = 0.01 m which tries to rotate the beam BC in clockwise direction.

$$M_{FDC} = +\frac{4EI}{L} \times \theta = \frac{4 \times 200 \times 10^6 \times 300 \times 10^{-6} \times 0.004}{4} = +240\ \text{kNm}$$

$$M_{FCD} = +\frac{2EI}{L} \times \theta = +120\ \text{kNm}$$

(b) Distribution Factors

$$k_{AB} = \frac{4EI}{L} = \frac{4 \times EI}{7} = 0.57\,EI, \quad k_{BC} = \frac{4EI}{L} = \frac{4E \times 2I}{7} = 1.14\,EI$$

$$k_{CD} = \frac{4EI}{L} = \frac{4E \times I}{4} = EI$$

DF at joint B in the span BA $= \frac{0.57}{0.57 + 1.14} = 0.33$

BC = 0.67

DF at joint C in the span CB $= \frac{1.14}{1.14 + 1} = 0.53$

CD = 0.47

Table 11.9a

Joint		A	B		C		D
Member		AB	BA	BC	CB	CD	DC
DF		−	0.33	0.67	0.53	0.47	−
Cycle 1	FEM	0	0	−146.94	− 146.94	+ 120.0	+ 240.0
	BAL	0	+ 48.49	+ 98.45	+ 14.28	+ 12.66	
Cycle 2	CO	+24.25		+ 7.14	+ 49.22		+ 6.33
	BAL		− 2.36	− 4.78	− 26.08	− 23.14	
Cycle 3	CO	− 1.18		− 13.04	− 2.39		− 11.57
	BAL		+ 4.31	+ 8.73	+ 1.27	+ 1.12	
Cycle 4	CO	+ 2.16		+ 0.64	+ 4.37		+ 0.56
	BAL		− 0.21	− 0.43	− 2.32	− 2.05	
Cycle 5	CO	− 0.10		− 1.16	− 0.21		− 1.02
	BAL		+ 0.39	+ 0.77	+ 0.11	+ 0.10	
Cycle 6	CO	+ 0.20		+ 0.06	+ 0.38		+ 0.05
	BAL		− 0.02	− 0.04	− 0.20	− 0.18	
Cycle 7	CO	− 0.01		− 0.10	− 0.02		− 0.09
	BAL		+ 0.03	+ 0.07	+ 0.01	+ 0.01	
Total moment (1)		+25.34	+ 50.63	− 50.63	− 108.52	+ 108.52	+ 234.26

The moment distribution for these fixed end moments is shown in Table 11.9a. The reactions H_A and H_D can be determined from the free body diagrams of the columns.

$$H_A = \frac{M_{AB} + M_{BA}}{L_{AB}} = \frac{25.34 + 50.63}{7} = 10.85 \text{ kN} \rightarrow$$

$$H_D = \frac{108.52 + 234.26}{4} = 85.70 \text{ kN} \rightarrow$$

II. Moment distribution for arbitrary sway

(a) Fixed End Moments

$$M_{FAB} = M_{FBA} = -\frac{6EI\Delta'}{L^2} = -\frac{6 \times 1000}{7^2} = -122.50 \text{ kNm}$$

$$M_{FDC} = M_{FCD} = -\frac{6EI\Delta'}{L^2} = -\frac{6 \times 1000}{4^2} = -375 \text{ kNm}$$

where, it is assumed that $EI\Delta' = 1000 \text{ kNm}^3$

The fixed end moments so obtained are of the same order as obtained in case I. The moment distribution is shown in Table 11.9b

Table 11.9b

Joint		A	B		C		D
Member		AB	BA	BC	CB	CD	DC
DF			0.33	0.67	0.53	0.47	–
Cycle 1	FEM	– 122.50	– 122.50			– 375.00	– 375.0
	BAL		+ 40.40	+ 82.10	+ 198.75	+ 176.25	
Cycle 2	CO	+ 20.20		+ 99.38	+ 41.05		+ 8.13
	BAL		– 32.80	– 66.58	– 21.75	– 19.30	
Cycle 3	CO	– 16.40		– 10.87	– 33.29		– 9.65
	BAL		+ 3.59	+ 7.28	+ 17.64	+ 15.65	
Cycle 4	CO	+ 1.79		+ 8.82	+ 3.64		+ 7.83
	BAL		– 2.92	– 5.90	– 1.93	– 1.71	
Cycle 5	CO	– 1.46		– 0.96	– 2.95		– 0.86
	BAL		+ 0.32	+ 0.64	+ 1.56	+ 1.39	
Cycle 6	CO	+ 0.16		+ 0.78	+ 0.32		+ 0.70
	BAL		– 0.26	– 0.52	– 0.17	– 0.15	
Cycle 7	CO	– 0.13		– 0.08	– 0.26		– 0.08
	BAL		+ 0.03	+ 0.05	+ 0.14	+ 0.12	
Total		– 118.34	– 114.14	+ 114.14	+ 202.75	– 202.75	– 288.93
Corrected moment (2)		– 73.13	– 73.13	+ 70.54	+ 125.30	+ 125.30	– 178.56
Moment (1)		+ 25.34	+ 50.63	– 50.63	– 108.52	+ 108.52	+ 234.26
Net moment		– 47.79	– 19.91	+ 19.91	+ 16.78	– 16.78	+ 55.70

The values of H_A' and H_D' are given by

$$H_A' = \frac{-118.34 - 114.14}{7} = -33.21 \text{ kN} \leftarrow$$

$$H_D' = \frac{-202.75 - 288.93}{4} = -122.92 \text{ kN} \leftarrow$$

The shear condition is that the sum of the horizontal reactions at A and D must be equal to zero,

$$(H_A + H_D) + k(H_A' + H_D') = 0$$

or, $\quad (10.85 + 85.70) + k(-33.21 - 122.92) = 0$ or, $k = 0.618$

$$\text{Correct sway } \Delta = k\Delta' = 0.618 \times \frac{1000}{EI} = 0.618 \times \frac{1000}{200 \times 300} = 0.0103 \text{ m}$$

The corrected sway moments and final moments are shown in Table 11.9b.

11.8 SYMMETRY AND ANTI-SYMMETRY

Many different types of modifications have been suggested by various researchers in the moment distribution method for faster convergence. Use of symmetry or anti-symmetry is one such modification. When a structure is perfectly symmetrical with respect to geometry and boundary conditions about its centre line, the applied loads may cause a condition of symmetry or anti-symmetry in the bending moment diagram in the central span. If either of these conditions exist, it is possible to modify the stiffness of the central span that will permit the moment distribution to be performed for only one-half of the structure. The convergence becomes faster and the efforts are reduced considerably.

Consider a five span beam shown in Fig.11.14a. It carries symmetrical loading and the deflected shape is also shown in the same figure. In the central span CD, the moment diagram will be symmetrical. The conjugate beam for span CD is shown in Fig. 11.14c. The slopes at the ends C and D are equal to

$$\theta = \pm \frac{ML}{2EI} \quad \text{or,} \quad M = \frac{2EI\theta}{L}$$

Therefore, the moment per unit rotation at either end, that is, the stiffness of the beam is equal to 2EI/L. The usual stiffness of the beam is 4EI/L. Thus, relative stiffness of a symmetrical beam is equal to one-half of that of the usual stiffness. Since both ends rotate simultaneously, the net moment at each end is 2EI/L and the carry over moment across the beam is zero. Therefore, only one-half the structure on either side of the beam need be considered for analysis.

Consider a five span beam shown in Fig. 11.14b. It carries an anti-symmetrical loading and the resulting deflected shape is also shown in the same figure. In the central

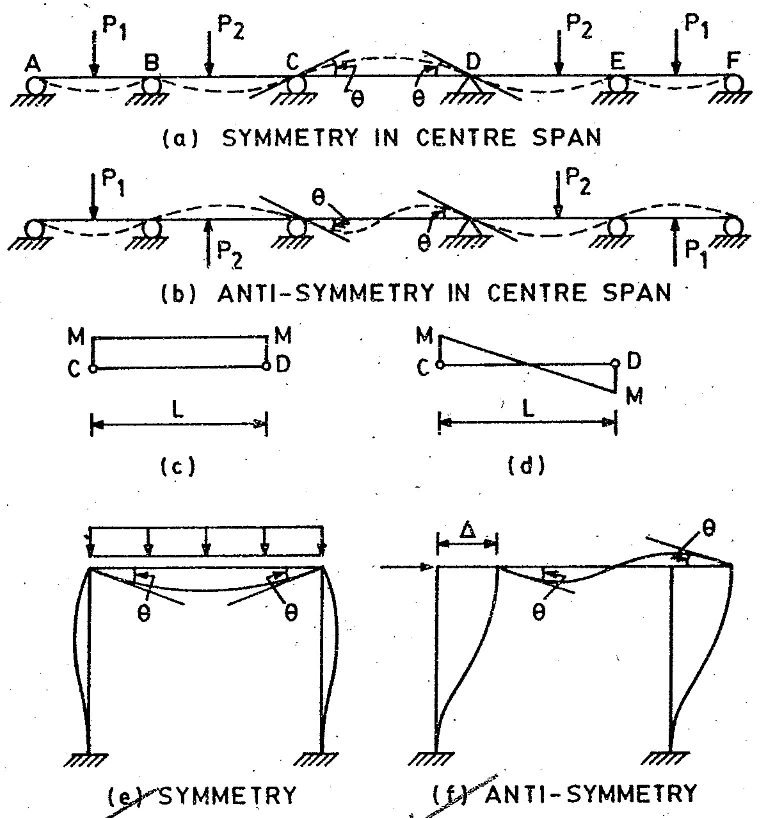


Fig. 11.14

span CD, the moment diagram will be anti-symmetrical. The conjugate beam for span CD is shown in Fig. 11.14d. It can easily be shown that

$$\theta = \pm \frac{ML}{6EI} \quad \text{or,} \quad M = \frac{6EI\theta}{L}$$

Thus, anti-symmetry in the bending moment diagram in the central span increases the effective stiffness of that span by 50%. The relative stiffness of the central span is increased to 1.5 times the usual stiffness. The moments on the two sides of the beam will be of the same magnitude and sign. Hence the carry over moment across the beam is zero. Again, only one-half the structure on either side of the beam need be considered for analysis.

The frame shown in Fig. 11.14e represents a symmetrical case, while the frame shown in Fig. 11.14f represents an anti-symmetrical case. In both these cases only one-half frame need be considered. The following examples illustrate the applications of symmetry or anti-symmetry in the moment distribution method. The same concept can also be conveniently used in other methods of structural analysis.

Example 11.10

Analyze the portal frame shown in Fig. 11.15a by the moment distribution method.

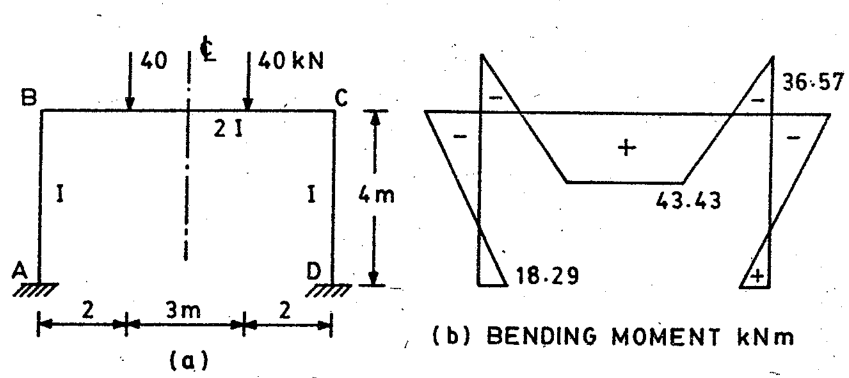


Fig. 11.15

Solution

The frame is symmetrical about its central line with respect to geometry and loading. There will be no side sway. Only half of the frame need be considered for moment distribution

(a) Fixed End Moments

$$M_{FBC} = -\frac{40 \times 2 \times 5^2}{7^2} - \frac{40 \times 5 \times 2^2}{7^2} = -\ 57.14 \text{ kNm}$$

$$= -\ M_{FCB}$$

(b) Distribution Factors

Stiffness of column AB $= \dfrac{4EI}{L} = EI$

beam BC $= \dfrac{1}{2} \times \dfrac{4EI}{L} = \dfrac{1}{2} \times \dfrac{4E(2I)}{7} = 0.57\,EI$

DF at joint B in span BA $= \dfrac{1}{1+0.57} = 0.64$

span BC $= 0.36$

The moment distribution is shown in Table 11.10.

Table 11.10

Joint	A	B	
Member	AB	BA	BC
DF	–	0.64	0.36
FEM BAL		+36.57	–57.14 +20.57
CO BAL	+18.29 0	–	–
Total moment	+18.29	+36.57	–36.57

The moment in the other half frame can be written by symmetry. The resulting moment diagram is shown in Fig.11.15b.

Example 11.11

Analyze the two bay frame shown in Fig. 10.12a by the moment distribution method and draw bending moment diagram.

Solution

The frame is symmetrical about its centre line passing through the central column with respect to geometry as well as loading. Hence, there will be no side sway. Moreover, there will be no moment in the central column due to symmetry. Hence, consider the frame shown in Fig.11.16.

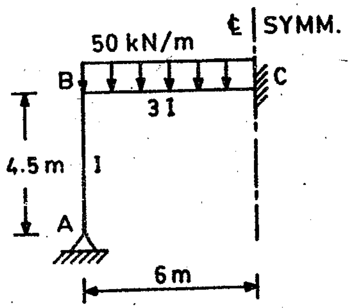


Fig. 11.16 One-half frame due to symmetry

(a) Fixed End Moments

$$M_{FBC} = -\frac{wL^2}{12} = -\frac{50 \times 6^2}{12} = -150 \text{ kNm}$$

$$M_{FCB} = +150 \text{ kNm}$$

(b) Distribution Factors

Stiffness of column $\quad AB = \dfrac{4EI}{L}$

Modified stiffness due to end A being hinged $= \dfrac{3}{4} \times \dfrac{4EI}{L} = \dfrac{3EI}{4.5} = 0.67EI$

Stiffness of beam $\quad \cancel{AB}$ BC $= \dfrac{4EI}{L} = \dfrac{4E(3I)}{6} = 2EI$

DF at joint B in span $\quad BA = \dfrac{0.67}{0.67+2} = 0.25$

span $\quad BC = 0.75$

The moment distribution for these fixed end moments is shown in Table 11.11.

Table 11.11

Joint		A	B		C
Member		AB	BA	BC	CB
DF		–	0.25	0.75	–
Cycle 1	FEM BAL		+ 37.50	– 150.00 + 112.50	+ 150.00
Cycle 2	CO BAL	0 0	–	–	+ 56.25 0
Final moment		0	+ 37.50	– 37.50	+ 206.25

The moment in the other half can be found by symmetry, that is,

$$M_{CE} = -M_{CB} = -206.25 \text{ kNm}$$
$$M_{EC} = -M_{BC} = +37.50 \text{ kNm}$$
$$M_{EF} = -M_{BA} = -37.50 \text{ kNm}$$

These are the same values as obtained in Ex.10.7, by the slope-deflection method.

Example 11.12

Analyze the two storey-single bay frame shown in Fig. 10.17a by the moment distribution method and draw the elastic curve.

Solution

The frame is symmetrical about its central line both with respect to geometry as well as loading. There will be no side sway in the frame. Hence, consider one -half the frame shown in Fig. 11.17.

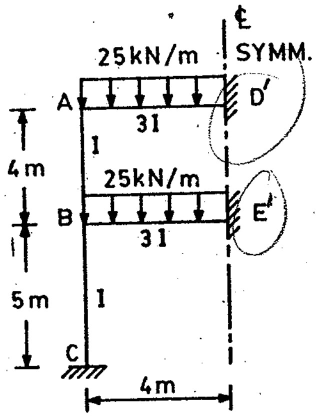


Fig. 11.17 One-half frame due to symmetry

(a) Fixed End Moments

$$M_{FAD} = -\frac{wL^2}{12} = -\frac{25\times 8^2}{12} = -133.4 \text{ kNm} = M_{FBE}$$

$$M_{FDA} = +133.4 \text{ kNm} = M_{FEB}$$

(b) Distribution Factors

Stiffness of column $\quad BC = \frac{4EI}{L} = \frac{4EI}{5} = 0.8\,EI$

$$AB = \frac{4EI}{4} = EI$$

Stiffness of beam $\quad AD = \frac{1}{2}\times\frac{4EI}{L} = \frac{1}{2}\times\frac{4E(3I)}{8} = 0.75EI$

$= $ stiffness of beam BE

Distribution factor at joint B in span $\quad BC = \frac{0.8}{0.8+1.0+0.75} = 0.31$

in span $\quad BA = \frac{1.0}{0.8+1.0+0.75} = 0.39$

in span $\quad BE = \frac{0.75}{0.8+1.0+0.75} = 0.30$

Distribution factor at A in span $\quad AD = \frac{0.75}{0.75+1.0} = 0.43$

in span $\quad AB = \frac{1.0}{0.75+1.0} = 0.57$

To carryout the moment distribution process only half the frame needs to be considered as shown in Table 11.12.

Table 11.12

Joint		C	B			A	
Member		CB	BC	BE	BA	AB	AD
DF		–	0.31	0.30	0.39	0.57	0.43
Cycle 1	FEM			–133.40			–133.40
	BAL		+ 41.35	+ 40.00	+52.05	+ 76.04	+ 57.36
Cycle 2	CO	+ 20.70			+38.02	+ 26.02	
	BAL		– 11.79	– 11.40	–14.83	– 14.83	– 11.19
Cycle 3	CO	– 5.89			–7.41	– 7.41	
	BAL		+ 2.30	+ 2.22	+2.89	+ 4.22	+ 3.19
Cycle 4	CO	+ 1.15			+2.11	+ 1.45	
	BAL		– 0.65	– 0.63	– 0.83	– 0.83	– 0.62
Cycle 5	CO	– 0.32			– 0.41	– 0.41	
	BAL		+ 0.13	+ 0.12	+0.16	+ 0.23	+ 0.18
Cycle 6	CO	+ 0.07			+0.11	+ 0.08	
	BAL		– 0.03	– 0.03	– 0.05	– 0.05	– 0.03
Total moments		+ 15.71	+ 31.31	–103.12	+71.81	+ 84.51	– 84.51

The final moments are nearly the same as obtained in Ex. 10.12 by the slope-deflection method.

Example 11.13

Analyze the box frame shown in Fig.10.18a by the moment-distribution method.

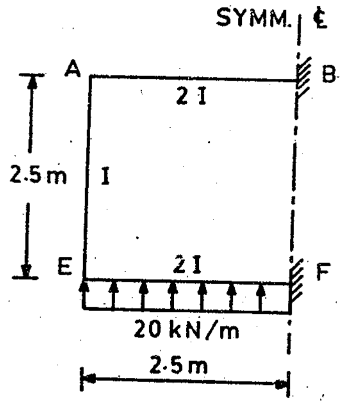


Fig. 11.18 One-half frame due to symmetry

Solution

(a) Fixed End Moments

Since the frame is symmetrical about its centre line consider one-half of the frame as shown in Fig. 11.18

$$M_{FAC} = -\frac{Pab^2}{L^2} = -\frac{PL}{8} = -\frac{100\times 5}{8} = -62.5 \text{ kNm}$$

$$M_{FED} = +\frac{wL^2}{12} = \frac{20\times 5^2}{12} = 41.67 \text{ kNm}$$

(b) ***Distribution Factors***

Joint A

Modified stiffness of member AC $= \frac{1}{2}\times\frac{2I}{5}$

Stiffness of member AE $= \frac{I}{2.5}$

DF at A in the span AE and AC $= \frac{I}{2.5} : \frac{1}{2}\times\frac{2I}{5} = \frac{0.4}{0.4+0.2} : \frac{0.2}{0.4+0.2}$

$= 0.67 : 0.33$

Joint B

Stiffness of member AE $= \frac{I}{2.5}$

Modified stiffness of member ED $= \frac{1}{2}\times\frac{2I}{5} = \frac{I}{5}$

DF at E $= \frac{I}{2.5} : \frac{I}{5} = 0.67 : 0.33$

Carry over factors

between A and E = 0.5, between A and B = 0, between E and F = 0

The moment distribution for these fixed end moments is shown in Table 11.13. The moments in the remaining one-half frame can be written directly by symmetry. The final end moments are nearly the same as obtained from slope-deflection method in Ex.10.13.

Table 11.13

Joint		A		E	
Side		CA	AE	EA	ED
DF		0.33	0.67	0.67	0.33
Cycle 1	FEM	– 62.50	0	0	41.67
	BAL	20.63	41.87	– 27.92	– 13.75
Cycle 2	CO		– 13.96	20.93	
	BAL	4.60	9.36	– 14.03	– 6.90
Cycle 3	CO		– 7.01	4.68	
	BAL	2.32	4.69	– 3.13	– 1.55
Cycle 4	CO		– 1.57	2.35	
	BAL	0.52	1.05	– 1.57	– 0.78
Cycle 5	CO		– 0.79	0.52	
	BAL	0.26	0.53	– 0.35	– 0.17
Total moment		– 34.17	34.17	– 18.52	+ 18.52

Example 11.14

Analyze the portal frame shown in Fig. 11.19a. Draw the bending moment diagram.

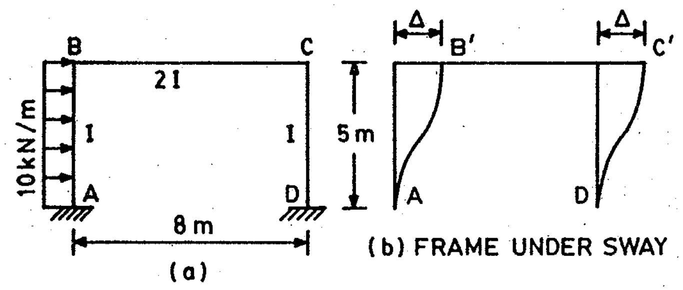


Fig. 11.19

Solution

The frame is symmetrical about its centre line with respect to geometry. However, it is not a case of anti-symmetrical loading with respect to the beam BC. Hence, this frame will be analyzed by the usual moment-distribution method.

I. Moment distribution when side sway is prevented

(a) Fixed End Moments

$$M_{FAB} = -\frac{wL^2}{12} = -\frac{10\times 5^2}{12} = -20.84 \text{ kNm}, \quad M_{FBA} = +20.84 \text{ kNm}$$

(b) Distribution Factors

$$\text{Stiffness of column} \quad AB = \frac{4EI}{L} = \frac{4EI}{5} = 0.8\,EI$$

$$\text{beam} \quad BC = \frac{4EI}{L} = \frac{4E \times 2I}{8} = EI$$

$$\text{DF at B in span} \quad BA = \frac{0.8}{0.8+1.0} = 0.44$$

$$\text{span} \quad BC = \frac{1.0}{0.8+1.0} = 0.56$$

The moment distribution for these fixed end moments is shown in Table 11.14a.

Table 11.14a

Joint		A	B		C		D
Member		AB	BA	BC	CB	CD	DC
DF		–	0.44	0.56	0.56	0.44	–
Cycle 1	FEM	–20.84	+20.84				
	BAL		– 9.17	–11.67			
Cycle 2	CO	– 4.58			– 5.83		
	BAL				+ 3.26	+2.57	
Cycle 3	CO			+1.63			+ 1.28
	BAL		– 0.72	– 0.91			
Cycle 4	CO	– 0.36			– 0.45		
	BAL				+ 0.25	+0.20	
Cycle 5	CO			+0.12			+ 0.10
	BAL		– 0.05	– 0.07			
Total moment, M (1)		–25.78	+10.90	–10.90	– 2.77	+2.77	+ 1.38

The horizontal reactions H_A and H_D are given by the free body diagrams of the columns AB and CD,

$$H_A = \frac{25.78+10.90}{5} - 25 = -27.98\text{ kN}\leftarrow$$

$$H_D = \frac{2.77+1.38}{5} = 0.83\text{ kN}\leftarrow$$

II. Moment distribution under arbitrary side sway Δ'

The fixed end moment due to a side sway Δ' is equal to :

$$M_{FAB} = -\frac{6EI\Delta'}{L^2} = -\frac{6EI\Delta'}{5^2} = -\frac{6\times100}{25} = -24\text{ kNm} = M_{FBA}$$

It is assumed that $EI\Delta' = 100\text{ kNm}^3$

The fixed end moment so obtained is of the same order as that in the case I. Similarly,

$$M_{FDC} = M_{FCD} = -24\text{ kNm}$$

The moment distribution for these fixed end moments is shown in Table 11.14b.

Table 11.14b

Joint		A	B		C		D
Member		AB	BA	BC	CB	CD	DC
DF		–	0.44	0.56	0.56	0.44	–
Cycle 1	FEM	– 24.0	– 24.0			– 24.0	– 24.0
	BAL		+10.56	+13.44	+13.44	+10.56	
Cycle 2	CO	+ 5.28		+ 6.72	+ 6.72		+ 5.28
	BAL		– 2.96	– 3.76	– 3.76	– 2.96	
Cycle 3	CO	– 1.48		– 1.88	– 1.88		– 1.48
	BAL		+ 0.83	+ 1.05	+ 1.05	+ 0.83	
Cycle 4	CO	+ 0.41		+ 0.52	+ 0.52		+ 0.41
	BAL		– 0.23	– 0.29	– 0.29	– 0.23	
Cycle 5	CO	– 0.11		– 0.15	– 0.15		– 0.11
	BAL		+ 0.07	+ 0.08	+ 0.08	+ 0.07	
Cycle 6	CO	+ 0.04		+ 0.04	+ 0.04		+ 0.04
	BAL		– 0.02	– 0.02	– 0.02	– 0.02	
Total moment M'		–19.86	–15.75	+15.75	+15.75	–15.75	–19.86
Corrected moment M'		–31.78	–25.20	+25.20	+25.20	–25.20	–31.78
M		–25.78	+10.90	–10.90	– 2.77	+ 2.77	+ 1.38
Net moment		–57.56	–14.30	+14.30	+22.43	–22.43	–30.40

The horizontal reactions H_A' and H_D' are given by

$$H_A' = \frac{-19.86-15.75}{5} = -7.122\text{ kN}\leftarrow = H_D'$$

The shear condition requires that the sum of all horizontal forces must be zero, that is,

$$(H_A + H_D) + k(H_A' + H_D') + 50 = 0$$

or, $$(-27.98+0.83) + k(-7.122-7.122) + 50 = 0 \quad \text{or,} \quad k = +1.60$$

The correct side sway is given by $\Delta = k\Delta' = k\dfrac{100}{EI} = \dfrac{160}{EI}$

The corrected moments are shown in Table 11.14b. These are the same values as obtained in Ex. 5.7 by the strain-energy method. O.K.

Example 11.15

Reanalyze the portal frame of Example 11.14 if it is subjected only to a horizontal load of 20 kN at B.

Solution

The frame will deflect in an anti-symmetric mode. The fixed end moments due to the external load are zero. Hence, the moment distribution will be carried out only for the arbitrary sway Δ' in one-half the frame shown in Fig.11.20.

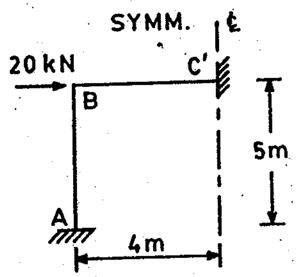


Fig. 11.20 One half frame due to anti-symmetry

(a) Fixed End Moments

$$M_{FAB} = -\frac{6EI\Delta'}{L^2} = -\frac{6\times 100}{5^2} = -24 \text{ kNm} = M_{FBA}$$

and $M_{FCD} = M_{FDC}$

(b) Distribution Factors

$$\text{Stiffness of column } AB = \frac{4EI}{L} = \frac{4EI}{5} = 0.8EI$$

$$\text{Stiffness of beam } BC = \frac{4EI}{L}$$

$$\text{Modified stiffness due to anti-symmetry} = 1.5\times\frac{4E(2I)}{8} = 1.5EI$$

$$\text{DF at B in the span } BA = \frac{0.8}{0.8+1.5} = 0.35$$

$$\text{span } BC = 0.65$$

The moment distribution is shown in Table 11.15. The shear condition may be written as,

$$k(H_A' + H_D') + 20 = 0$$

or, $2k\left(\frac{-17.8-15.6}{5}\right) + 20 = 0$, or, $k = 0.668$

The correct sway is equal to $\Delta = k\Delta' = 0.668 \times \frac{100}{EI} = \frac{66.8}{EI}$ m

The corrected moments are shown in Table 11.15.

Table 11.15

Joint		A	B	
Member		AB	BA	BC
DF			0.35	0.65
Cycle 1	FEM	− 24.00	− 24.00	
	BAL		+ 8.40	+ 15.60
Cycle 2	CO	+ 4.20		
	BAL	0		
Total moment		− 17.80	− 15.60	+ 15.60
Corrected moments		− 11.90	− 10.40	+ 10.40

The moments in the other half frame can be written by making use of anti-symmetry :

$$M_{CB} = M_{BC} = +10.40 \text{ kNm}$$
$$M_{CD} = M_{BA} = -10.40 \text{ kNm}$$
$$M_{DC} = M_{AB} = -11.90 \text{ kNm}$$

Example 11.16

Analyze the portal frame with inclined legs shown in Fig. 10.20a by the moment distribution method.

Solution

The frame is symmetrical about its center line with respect to geometry, but not with respect to the load. The beam deflects in an anti-symmetric mode. There are no fixed end moments due to the external load. Therefore, one-half frame will be analyzed for an arbitrary sway Δ' as shown in Fig. 11.21. The deflected shape was shown in Fig. 10.20b, Example 10.15.

(a) Fixed End Moments

$$M_{FAB} = -\frac{6EI\delta'}{L^2} = -\frac{6\times 100}{6^2} = -16.67 \text{ kNm}$$

$$M_{FBA} = -16.67 \text{ kNm}$$

Assuming, $EI\delta' = 100 \text{ kNm}^3$

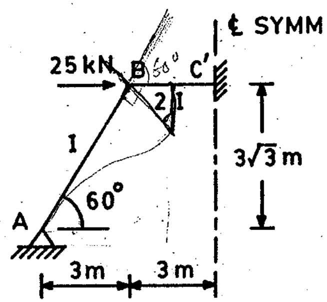


Fig. 11.21 One half frame due to anti-symmetry

$$M_{FBC} = \frac{6EI(B'B'' + C'C'')}{L^2} = \frac{6E(2I)(2\delta' \sin 30^0)}{6^2} = \frac{24EI\delta \sin 30^0}{6^2}$$

$$= \frac{24 \times 100 \times 0.5}{6^2} = +\ 33.34 \text{ kNm}$$

(b) Distribution Factors

$$\text{Stiffness of member AB} = \frac{3EI}{L} = 0.5\,EI$$

$$\text{Stiffness of member BC} = \frac{4EI}{L}$$

Modified stiffness of member BC due to anti-symmetry

$$= 1.5\left(\frac{4E \times 2I}{6}\right) = 1.5EI$$

$$\text{DF at B in the span BA} = \frac{0.5}{0.5+2} = 0.2$$

$$\text{span BC} = 0.8$$

The moment distribution is shown in Table 11.16.

The sway equation is

$$H_A + H_D + 25 = 0$$

or, $k[M_{BA} - (M_{BC} + M_{CB}) + M_{CD}] + 75\sqrt{3} = 0$, as discussed in Ex. 10.15.

or, $k[-13.34 - (13.34 + 13.34) - 13.34] + 75\sqrt{3} = 0$

or, $k = 2.436$

Table 11.16

Joint		A	B	
Member		AB	BA	BC
DF			0.20	0.80
	FEM	− 16.67	− 16.67	+ 33.34
	BAL	+ 16.67		
	CO		+ 8.34	
	BAL		0	
Cycle 1	Mod.FEM	0	− 8.34	+ 33.34
	BAL		− 5.00	− 20.00
Total moment		0	− 13.34	+ 13.34
Corrected moment		0	− 32.50	+ 32.50

The corrected sway is given by

$$\delta \cos 30° = k\,\delta' \cos 30° = 2.436 \times \frac{100}{EI} \times \frac{\sqrt{3}}{2} = \frac{210.96}{EI} \text{ m}$$

The corrected moments are shown in Table 11.16. The moments in the other half are :

$$M_{CB} = M_{BC} = +\ 32.50 \text{ kNm}$$

$$M_{CD} = M_{BA} = -\ 32.50 \text{ kNm}$$ O.K.

Example 11.17

Analyze the gable frame shown in Fig. 10.22a by the moment distribution method.

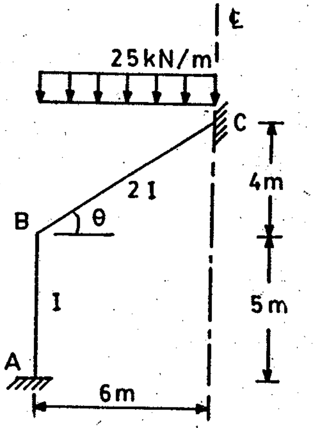


Fig. 11.22 One half frame due to symmetry

Solution

The frame is symmetrical about its centre line with respect to geometry and loading. There will be a symmetrical side sway. Its deflected shape and free body diagrams may be seen in Fig.10.22b, Ex.10.17. The moment distribution may be carried out for one - half the frame shown in Fig. 11.22 in two parts : for non-sway moments, and for sway moments.

I. Moment distribution due to applied loads

The frame is restrained against sway, and the fixed end moment are given by,

$$M_{FBC} = -\frac{wL^2}{12} = -\frac{25\times 6^2}{12} = -75 \text{ kNm}$$

$$M_{FCB} = +75 \text{ kNm}$$

Distribution Factors

Stiffness of member AB $= \dfrac{4EI}{L} = 0.8\,EI$

member BC $= \dfrac{4EI}{L} = \dfrac{4E(2I)}{2\sqrt{13}} = 1.11\,EI$

DF at B in span BA $= \dfrac{0.8}{0.8+1.11} = 0.42$

in span BC $= 0.58$

The moment distribution for these moments is shown in Table 11.17a.

Table 11.17a

Joint		A	B		C
Member		AB	BA	BC	CB
DF		–	0.42	0.58	–
Cycle 1	FEM			– 75.0	+ 75.00
	BAL		+ 31.5	+ 43.5	
Cycle 2	CO	+ 15.75			+ 21.75
	BAL	0			0
Total moment M		+ 15.75	+ 31.5	– 31.5	+ 96.75

II. Moment distribution due to arbitrary sway Δ'

(a) Fixed End Moments

$$M_{FAB} = \frac{6EI\Delta'}{5^2} = \frac{6\times 100}{5^2} = +24 \text{ kNm}$$

$M_{FBA} = +24$ kNm, (assuming $EI\Delta' = 100$ kNm³)

$$M_{FBC} = -\frac{6EI}{L^2_{BC}}(\Delta' \operatorname{cosec}\theta), \quad \tan\theta = \frac{4}{6}$$

$$= -\frac{6E(2I)\Delta'}{(2\sqrt{13})^2}\times\frac{2\sqrt{13}}{4} = -\frac{6}{4\sqrt{13}}\times EI\Delta'$$

$$= -0.416\times 100 = -41.60 \text{ kNm}$$

$$M_{FCB} = -\frac{6EI}{L^2_{BC}}\Delta' \operatorname{cosec}\theta = -41.60 \text{ kNm}$$

The moment distribution due to the arbitrary sway moments is shown in Table 11.17b.

Table 11.17b

Joint		A	B		C
Member		AB	BA	BC	CB
DF		–	0.42	0.58	–
Cycle 1	FEM	+ 24.0	+ 24.0	– 41.60	– 41.60
	BAL		+ 7.39	+ 10.21	
Cycle 2	CO	+ 3.70			+ 5.10
	BAL	0			
Total moment M'		+ 27.70	+ 31.39	– 31.39	– 36.50
Corrected moment M'		+ 114.84	+ 130.14	– 130.14	– 151.33
Moment M		+ 15.75	+ 31.50	– 31.50	+ 96.75
Net moment (M+M')		+ 130.59	+ 161.64	– 161.64	– 54.58

The sway equation is given by

$$H_A + H_E = 0 \qquad \text{(i)}$$

but $H_E = -H_A$, hence this leads to a trivial solution.

At joint B, the shear equation is

$$H_{BC} - H_{BA} = 0 \qquad \text{(ii)}$$

The horizontal forces H_{BC} and H_{BA} due to non - sway moments are given by (Refer Ex. 10.17) :

$$H_{BC} = \frac{M_{BC}+M_{CB}+450}{4} = 128.81 \text{ kN}\rightarrow$$

$$H_{BA} = \frac{M_{AB}+M_{BA}}{5} = \frac{15.75+31.5}{5} = 9.45 \text{ kN}\rightarrow$$

The horizontal forces due to the sway moments are given by :

$$H_{BC} = \frac{M_{BC} + M_{CB}}{4} = \frac{-31.39 - 36.50}{4} = -16.97 \text{ kN}$$

$$H_{BA} = \frac{M_{AB} + M_{BA}}{5} = \frac{27.70 + 31.39}{5} = 11.82 \text{ kN}$$

Substituting these values in the shear eq. (ii) gives

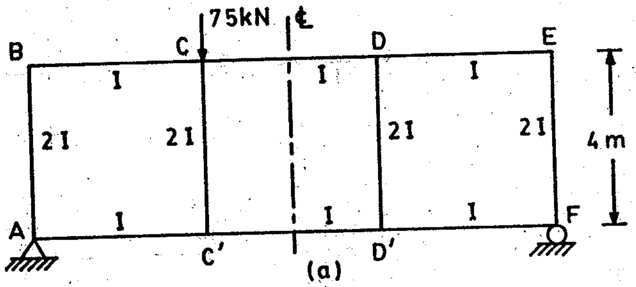
$$128.81 - k\,16.97 - 9.45 - k\,11.82 = 0$$

or, $\quad k = 4.146$

The corrected moments M' and the net moment (M+M') are shown in Table 11.17b. These are the same values as in Ex.10.17 by using the slope-deflection method.

Example 11.18

Analyze the Vierendeel girder shown in Fig. 11.23a by making use of symmetry and anti-symmetry.



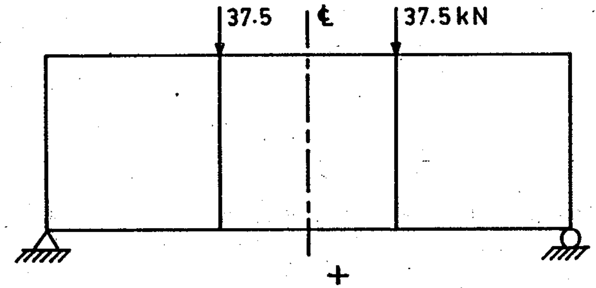
(a)

Fig. 11.23 Vierendeel girder

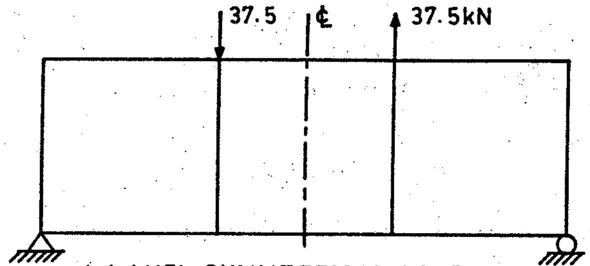
Solution

The Vierendeel girder or frame is symmetrical about the vertical axis but the loading is un-symmetrical. Let the load be broken into two separate systems, one a symmetrical system shown in Fig. 11.23b and another an anti-symmetrical system shown in Fig. 11.23c. Obviously the sum of these two systems is equal to the given loads. Therefore, according to the principle of superposition, the sum of the results for the two separate systems is equal to that for given loads. This approach reduces the computational efforts considerably.

The Vierendeel girder is a statically indeterminate structure but an approximate analysis can be adopted. It will become statically determinate if three hinges are introduced in each panel and the characteristic behaviour of the girder is maintained by placing the hinges at the mid-spans of the chord members and mid-height of the verticals. The Vierendeel girder becomes a statically determinate structure. However, such an analysis is not suitable for Vierendeel girder with :



(b) SYMMETRICAL LOAD



(c) ANTI-SYMMETRICAL LOAD

Fig. 11.23 Vierendeel girder contd.

(i) chords of widely different stiffness,

(ii) inclined members,

(iii) non-prismatic vertical members, and

(iv) loads applied away from node points.

Modified moment distribution methods have been developed for the analysis of Vierendeel girders. The substitute frame method can be used for the analysis of parallel chorded Vierendeel girders with chords of different stiffness in the panels, while the top and bottom chords of any panel are of the same section. The fixed end moments are calculated from the panel shears. The stiffness of the verticals is taken as six times of their stiffness (= I / L) and the distribution factors are determined. The carry over factors are taken as –1. The rest of the moment distribution procedure is as usual.

Case I : Symmetrical Loading Only one-half of the frame need be considered.

Member Stiffness

Member AB, $k_{AB} = 6 \times 2\,\frac{I}{4} = 3\,I = k_{CC'}$

$k_{BC} = \frac{I}{4} = 0.25\,I$

$k_{CD} = \frac{1}{2} \times \frac{I}{4} = 0.125\,I$

The stiffness of member CD is reduced by 50% because it is cut in two halves by the line of symmetry.

Distribution Factors

At joint B in span BA $= \frac{3I}{3I + 0.25I} = 0.92$

at joint B in span BC $= 0.08$

at joint C in span CB $= \frac{0.25\,I}{0.25\,I + 0.125\,I + 3\,I} = 0.074$

in span CC' $= \frac{3I}{3.375I} = 0.889$

in span CD $= 1 - 0.889 - 0.074 = 0.037$

Fixed End Moments

Reaction $R_A = 37.5$ kN, Shear in panel BC = 37.5 kN

$\therefore$ Fixed end moment $= -\,37.5 \times \frac{4}{2} \times \frac{1}{2} = -\,37.5$ kNm

Shear in panel CD = 0

$\therefore$ Fixed end moment = 0

The moment distribution procedure is shown in Table 11.18a. The moments in the remaining half frame have the same magnitude but opposite in sign.

Case II Anti - symmetrical loading

Member Stiffness

$k_{AB} = 3\,I$, $k_{BC} = 0.25\,I$

$k_{CD} = \frac{3}{2} \times \frac{I}{4} = 0.125\,I$

The stiffness of member CD is taken as 1.5 times because it is cut in two halves by the line of anti - symmetry

Table 11.18a

Joint		B		C		
Member		BA	BC	CB	CC'	CD
DF		0.92	0.08	0.074	0.889	0.037
Cycle 1	FEM		− 37.5	− 37.50		
	BAL	+ 34.5	+ 3.0	+ 2.78	+ 33.34	+ 1.38
Cycle 2	CO		− 2.78	− 3.00		
	BAL	+ 2.56	+ 0.22	+ 0.22	+ 2.67	+ 0.11
Total M		+ 37.06	− 37.06	− 37.5	+ 36.01	+ 1.49

Distribution Factors

At joint C in span CB $= \frac{0.25\,I}{0.25\,I + 3\,I + 0.375\,I} = 0.069$

in span CC' $= 0.827$

in span CD $= 0.104$

Fixed End Moments

Reaction $R_A = +\ 12.5$ kN ↑

Panel	BC	CD
Shear (kN)	12.5	− 25.0
FEM (kNm)	$-\,12.5 \times \frac{4}{2} \times \frac{1}{2} = -\,12.5$	$25 \times \frac{4}{2} \times \frac{1}{2} = 25$

The moment distribution procedure is shown in Table 11.18b. The moments in the other half have the same magnitude and the same sign.

Table 11.18b

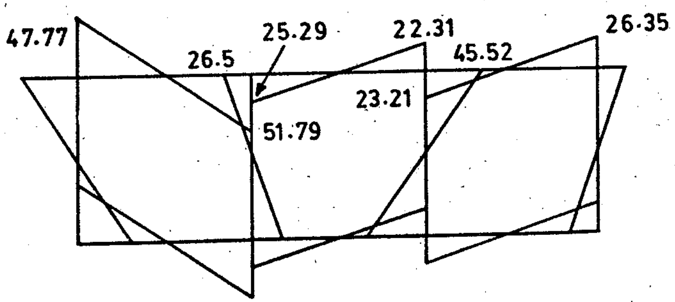
Joint		B		C		
Member		BA	BC	CB	CC'	CD
DF		0.92	0.08	0.069	0.827	0.104
Cycle 1	FEM		− 12.5	− 12.5		25.0
	BAL	+ 11.5	+ 1.0	− 0.86	− 10.34	− 1.30
Cycle 2	CO		+ 0.86	− 1.0		
	BAL	− 0.79	− 0.07	+ 0.069	+ 0.827	+0.104
Total M'		+ 10.71	− 10.71	− 14.29	− 9.51	+ 23.80
M		+ 37.06	− 37.06	− 37.50	+ 36.01	+ 1.49
Net (M' + M)		+ 47.77	− 47.77	− 51.79	+ 26.50	+ 25.29

The net moments in the half frame are computed as shown in Table 11.18c.

Table 11.18c

Member	DC	DD'	DE	ED	EF
(Table 11.18a) M	− 1.49	− 36.01	+ 37.50	+ 37.06	− 37.06
(Table 11.18b) M'	+ 23.80	− 9.51	− 14.29	− 10.71	+ 10.71
Net (M+M')	+ 22.31	− 45.52	+ 23.21	+ 26.35	− 26.35

The net bending moment diagram is shown in Fig.11.23d.



(d) NET BENDING MOMENT kNm

Fig. 11.23d Vierendeel girder contd.

11.9 COMMENTS ON THE MOMENT DISTRIBUTION METHOD

The moment distribution method is one of the most powerful tools for the analysis of statically indeterminate structures by hand. All the structures considered in this chapter consisted of prismatic members. The method is equally applicable to a structure consisting of curved members or non-prismatic members. The difficulty lies in the determination of fixed-end moments, stiffness and carry over factors for such members. For curved or non-prismatic members, these values can be determined using the column analogy method. Some additional comments are as follows :

1. It is better to take all calculations in the table to at least four places of decimal for better accuracy.
2. The moment distribution can be stopped at the end of any cycle (that is, after balancing the moments). It is desirable to stop it when the unbalanced moments become quite insignificant.
3. It should be ensured that the numerical sum of the balancing moments is exactly equal to that of the unbalanced moment in every cycle.
4. A consistent practice should be used while rounding -off the values. It is desirable to use nearest even values.
5. The check on the moment distribution works assuming that the fixed end moments as computed are correct . The check will work even if the fixed end moments or the distribution factors are taken wrong.

PROBLEMS

11.1 Analyze the beams in Figs. P3.4 and P3.5 by the moment distribution method. Draw shear force and bending moment diagrams.

11.2 Analyze the beams in Figs. P4.1 by the moment distribution method and draw elastic curve and the bending moment diagram.

11.3 Analyze the frames in Figs. P3.8a, b, c and d by the moment distribution method and also draw the elastic curves.

11.4 Analyze the frames in Figs. P5.4, P5.7 and P5.8 by the moment distribution method and draw the bending moment diagrams.

11.5 Analyze the frames in Figs. P6.3a and b by the moment distribution method and draw the bending moment diagrams.

11.6 Analyze the frames shown in Figs. P10.1, P10.2 and P10.3 of problem 10.5 if the column is pinned at support D in each case.

11.7 Analyze the box frame shown in Fig. P 6.4a by making use of symmetry.

11.8 Analyze the frame shown in Fig. P6.4b by making use of symmetry.

11.9 Analyze the frame shown in Fig. P6.4c using the moment distribution method.

11.10 Analyze the gable frames shown in Fig.P6.3c and d using the moment distribution method.

CHAPTER
twelve

DIRECT STIFFNESS METHOD-2D ELEMENTS

12.1 DEVELOPMENT OF STIFFNESS MATRICES

The concept of stiffness method was developed in section 1.3. The slope-deflection method and moment-distribution method fall under the system approach to stiffness method. The direct stiffness method falls under the member approach. This is a very computer friendly and the most powerful analytical tool developed so far. A right hand axes system as shown in Fig. 12.1 is used in this method which makes it quite easy to understand.

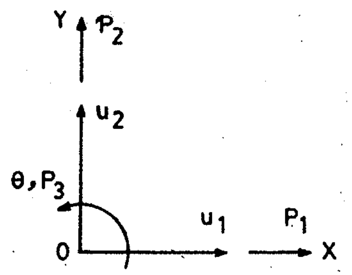


Fig. 12.1 Right hand axes system

TRUSS ELEMENT

Consider a pin-ended truss or a bar element i-j as shown in Fig. 12.2a. It can carry only axial force and undergo axial deformation. Only two coordinates are required to define its deflected shape i' - j' : u_1 and u_2. u_1 is the displacement of end i and u_2 is the displacement of end j. Let us define an orthogonal system of axis x-y such that the longitudinal axis of the truss element lies on the x-axis and y-axis is perpendicular to x-axis an shown in Fig. 12.2a. Let the axial force and axial displacement be positive in the direction of positive x-axis. The stiffness matrix of the truss element can be developed as follows :

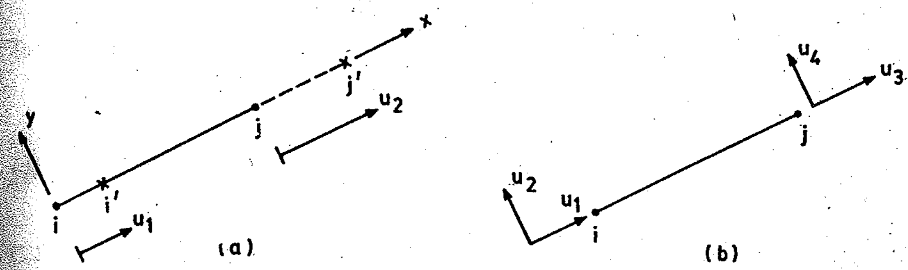


Fig. 12.2 Truss element in local axes

Case 1 ($u_1 \neq 0$, $u_2 = 0$)

Let the end i undergo a displacement equal to u_1 while the end j is restrained.

Strain $\epsilon = \dfrac{u_1}{L}$, L is member length

Stress $\sigma = E\epsilon = \dfrac{E u_1}{L}$

Force at end i required to produce u_1 displacement, $P_1 = A\sigma = \dfrac{AE}{L}u_1$

By equilibrium, the force at the other end j is equal to

$$P_2 = -P_1 = -\frac{AE}{L}u_1$$

Case 2 ($u_1 = 0$, $u_2 \neq 0$)

Now let us restrain the end i and impose a displacement equal to u_2 at the end j,

Strain $\epsilon = \dfrac{u_2}{L}$

Stress $\sigma = E\epsilon = \dfrac{E u_2}{L}$

Force at end j required to produce u_2 displacement, $P_2 = A\sigma = \dfrac{AE}{L}u_2$

By equilibrium,

$$P_1 = -P_2 = -\frac{AE}{L}u_2$$

Case 3 ($u_1 \neq 0$, $u_2 \neq 0$)

When both displacements are imposed, the resultant forces are :

$$P_1 = \frac{AE}{L}(u_1 - u_2)$$

and

$$P_2 = \frac{AE}{L}(-u_1 + u_2)$$

or
$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (12.1a)$$

or
$$P = K\Delta \qquad (12.1b)$$

where,
$$K_{2\times 2} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (12.2)$$

A = area of cross-section of the prismatic truss element

E = modulus of elasticity of the material

Case 4

Let us consider a general case where the ends of the truss undergo displacements u_1, u_2, u_3 and u_4 as shown in Fig. 12.2b. Only u_1 and u_3 will produce the axial force. Eq. 12.1a can be rewritten as :

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \qquad (12.3)$$

where, P_1 and P_2 are the forces at end i,

and P_3 and P_4 are the forces at end j,

BEAM ELEMENT

The stiffness matrix of a prismatic beam can be generated using any of the following methods :

1. Solution of basic differential equations
2. Any flexibility method
3. Conjugate beam method or moment-area method.

Let us use the conjugate beam method to generate the stiffness coefficients. Consider a prismatic beam shown in Fig. 12.3a. A 2-D beam is assumed to lie in the x-y plane. The translations u_1 and u_2 take place along x-axis and y-axis, respectively, while the rotation takes place about the z-axis. The degrees of freedom per joint are three. It has a total six degrees of freedom : four translations u_1, u_2 and u_4, u_5 and two rotations u_3 and u_6. The size of the stiffness matrix is 6 × 6. Let us impose each of the six displacements individually and compute the forces produced in the beam.

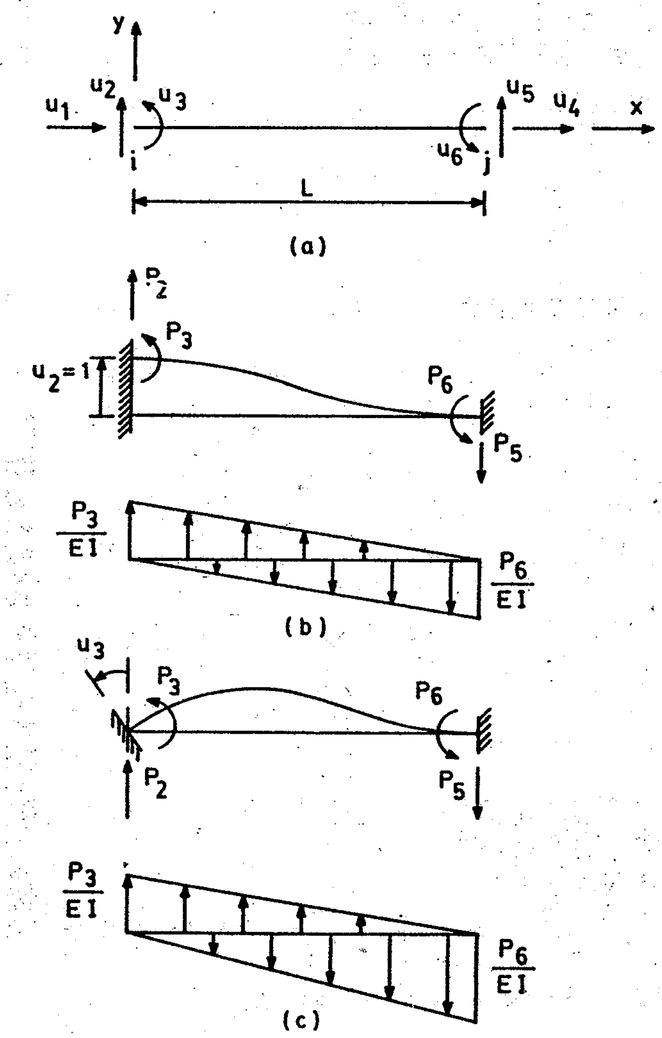


Fig. 12.3 Beam element - development of stiffness

Case 1 $u_1 \neq 0$

As derived in the case of a truss element

$$P_1 = \frac{AE}{L}u_1 \quad \text{and} \quad P_4 = -P_1 = -\frac{AE}{L}u_1 \qquad \text{(i)}$$

Therefore,
$$k_{11} = \frac{AE}{L} \quad \text{and} \quad k_{41} = -\frac{AE}{L} \qquad \text{(ii)}$$

Case 2 $u_2 \neq 0$

A displacement equal to u_2 is imposed along the positive y-axis while restraining all other displacements as shown in Fig. 12.3b. The conjugate beam and the M/EI loading are also shown in the same figure.

The displacement u_2 produces forces P_2, P_3, P_5 and P_6. Since the change of slope between the ends i and j are zero, the area of the M/EI diagram must be zero, that is, the total shear must be zero.

$$-\frac{1}{2}\frac{P_3 L}{EI} + \frac{1}{2}\frac{P_6 L}{EI} = 0, \qquad \text{or,} \qquad P_3 = P_6$$

Deflection at end i = moment of the M/EI loading between i and j, and it should be equal to u_2

or
$$\left[-\frac{P_3 L}{2EI}\times\frac{2L}{3} + \frac{P_6 L}{2EI}\times\frac{L}{3}\right] = -u_2$$

or
$$P_3 = \frac{6EI}{L^2}u_2 = P_6 \tag{iii}$$

By considering the equilibrium of the beam subjected to forces P_2, P_3, P_5 and P_6

$$P_2 = \frac{P_3+P_6}{L} = \frac{12EI}{L^3}u_2 \uparrow \tag{iv}$$

$$P_5 = -P_2 = -\frac{12EI}{L^3}u_2 \downarrow$$

The stiffness coefficients are :

$$k_{32} = \frac{6EI}{L^2} = k_{62}, \quad k_{22} = \frac{12EI}{L^3} \text{ and } k_{52} = -\frac{12EI}{L^3} \tag{v}$$

where k_{ij} = force at i due to a unit displacement at j

Case 3 $u_3 \neq 0$

A displacement (rotation) equal to u_3 is imposed about the z axis while restraining all other displacements as shown in Fig. 12.3c. The conjugate beam and the M/EI loading are also shown in the same figure. The rotation u_3 again produces forces P_2, P_3, P_5 and P_6. The deflection at end i with respect to end j is zero, that is,

Moment of M/EI loading about end i

$$= -\frac{P_3 L}{2EI}\times\frac{L}{3} + \frac{P_6 L}{2EI}\times\frac{2}{3}L = 0 \qquad \text{or} \qquad P_3 = 2P_6$$

Shear at end i is equal to slope at end i

or
$$\left(-\frac{P_3 L}{2EI}\times\frac{2}{3}L + \frac{P_6 L}{2EI}\times\frac{L}{3}\right)\frac{1}{L} = -\theta_i = -u_3 \text{ (clockwise rotation is + ve)}$$

or,
$$-\frac{P_3}{3} + \frac{P_6}{6} = -u_3\frac{EI}{L}$$

or,
$$P_3 = \frac{4EI}{L}u_3, \quad \text{or } k_{33} = \frac{4EI}{L}$$

$$P_6 = \frac{2EI}{L}u_3, \quad \text{or } k_{63} = \frac{2EI}{L} \tag{vi}$$

By considering the equilibrium of the beam subjected to forces P_2, P_3, P_5 and P_6.

$$P_2 = \frac{P_3+P_6}{2} = \frac{6EI}{L^2}u_3 \uparrow, \quad \text{or } k_{23} = \frac{6EI}{L^2}$$

and
$$P_5 = -P_2 = -\frac{6EI}{L^2}u_3 \downarrow, \quad \text{or } k_{53} = -\frac{6EI}{L^2} \tag{vii}$$

In the same manner, member forces due to u_4, u_5 and u_6 imposed individually at end j can be evaluated. The final values are as follows :

Due to u_4
$$P_4 = \frac{AE}{L}u_4, \quad \text{or } k_{44} = \frac{AE}{L}$$

$$P_1 = -P_4 = -\frac{AE}{L}u_4, \quad \text{or } k_{14} = -\frac{AE}{L} \tag{viii}$$

Due to u_5
$$P_3 = -\frac{6EI}{L^2}u_5, \quad \text{or } k_{35} = -\frac{6EI}{L^2}$$

$$P_6 = P_3, \quad \text{or } k_{65} = -\frac{6EI}{L^2} \tag{ix}$$

$$P_2 = \frac{P_3+P_6}{2} = -\frac{12EI}{L^3}u_5 \downarrow, \quad \text{or } k_{25} = -\frac{12EI}{L^3}$$

$$P_5 = -P_2 = \frac{12EI}{L^3}u_5 \uparrow, \quad \text{or } k_{55} = \frac{12EI}{L^3}$$

Due to u_6
$$P_3 = \frac{2EI}{L}u_6, \quad \text{or } k_{36} = \frac{2EI}{L}$$

$$P_6 = \frac{4EI}{L}u_6, \quad \text{or } k_{66} = \frac{4EI}{L}$$

$$P_2 = \frac{6EI}{L^2}u_6, \quad \text{or } k_{26} = \frac{6EI}{L^2} \tag{x}$$

$$P_5 = -\frac{6EI}{L^2}u_6\,, \quad \text{or} \quad k_{56} = -\frac{6EI}{L^2}$$

Equations (i) to (x) can be arranged in matrix form as follows :

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \quad (12.4)$$

or,

$$\begin{Bmatrix} \mathbf{P}_i \\ \mathbf{P}_j \end{Bmatrix}_{6\times 1} = \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} \\ \mathbf{K}_{ji} & \mathbf{K}_{jj} \end{bmatrix}_{6\times 6} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}_{6\times 1}$$

Each sub matrix is of size 3 × 3 and sub-vector is of size 3 × 1.

If axial deformations are ignored, the force-deformation relations can be written as :

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}_{4\times 1} = \begin{bmatrix} \frac{12EI}{L^3} & & \text{Symmetric} & \\ \frac{6EI}{L^2} & \frac{4EI}{L} & & \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_{4\times 4} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}_{4\times 1} \quad (12.6)$$

The corresponding degrees of freedom are shown in Fig. 12.4.

Fig. 12.4 Beam element d. o. f. without axial deformations

12.2 PROPERTIES OF STIFFNESS MATRICES

It is interesting to know some of the properties of the stiffness matrices developed in the previous section.

Equilibrium

The stiffness matrix represents a set of equations, relating member end displacements and member end forces in equilibrium. Thus, each column of the stiffness matrix must satisfy the equations of equilibrium. Consider column 5 of Eq. 12.5.

$$\begin{Bmatrix} P_{1i} \\ P_{2i} \\ P_{3i} \\ P_{4j} \\ P_{5j} \\ P_{6j} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\frac{12EI}{L^3} \\ -\frac{6EI}{L^2} \\ 0 \\ \frac{12EI}{L^3} \\ -\frac{6EI}{L^2} \end{Bmatrix} u_5$$

For static equilibrium :

$$\Sigma F_x = 0, \quad \Sigma F_y = 0\,, \quad \text{and} \quad \Sigma M = 0$$

On substituting the values of P_{1i} to P_{6j}

For $\Sigma F_x = 0, \quad P_{1i} + P_{4j} = 0$ OK

For $\Sigma F_y = 0, \quad P_{2i} + P_{5j} = 0$ OK

For $\Sigma M_i = 0, \quad P_{3i} + L\,P_{5j} + P_{6j} = 0$ OK

Symmetry

The stiffness matrix of any element or any structure is always symmetric about the diagonal, that is $k_{ij} = k_{ji}$. It is obvious in Eqs. 12.2, 12.3, 12.4 and 12.6. Symmetric property can be proved using the Maxwell's reciprocal theorem. This is a very useful property. In a computer, only the upper triangular matrix or only the lower triangular matrix need be stored. It saves considerable memory requirement and computer time.

Singularity

The stiffness matrix of an element is singular, that is, its inverse does not exist. This can be explained as follows with respect to Eq.12.4:

The first three rows represent the member forces at end i, while the last three rows represent the member forces at end j. Because of static equilibrium, for any specified set of displacement,

$$F_{xi} = -F_{xj}\,, \quad \text{and} \quad F_{yi} = -F_{yj}$$

It implies that the fifth row of a 6 × 6 stiffness matrix given by Eq.12.4 can be obtained from the second row by multiplying the third row by (−1). This means that rows 2 and 5 are linearly dependent. The same is true for rows 1 and 4. In fact, there are only three independent equations in a general 6 × 6 stiffness matrix.

Similarly, it can be shown that some of the columns are linearly dependent. In a 6 × 6 general stiffness matrix, three columns are linearly independent. This is due to the presence of rigid body displacements. Consider the rigid body motion of,

$$\Delta_{yi} = \Delta_{yj} = \Delta$$

with all other displacements zero. Substituting these in Eq. 12.4 gives,

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dfrac{12EI}{L^3} \\ \dfrac{6EI}{L^2} \\ 0 \\ -\dfrac{12EI}{L^3} \\ \dfrac{6EI}{L^2} \end{Bmatrix} \Delta + \begin{Bmatrix} 0 \\ -\dfrac{12EI}{L^3} \\ -\dfrac{6EI}{L^2} \\ 0 \\ \dfrac{12EI}{L^3} \\ -\dfrac{6EI}{L^2} \end{Bmatrix} \Delta = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Thus, equal displacements at the two ends, that is, rigid body displacements, result in zero force. Hence, columns 2 and 5 are linearly dependent,

or, $\{k\}_2 = -\{k\}_5$

Linear dependence between rows (or columns) of a matrix is a condition for singularity.

Rigid Body Displacement

If an element is subjected to a rigid body displacement or motion, no forces should develop at the ends of the element. Consider a beam element shown in Fig. 12.5 which has been displaced to a new position without any change in its length or relative rotation of its ends. The end displacements have the following relations :

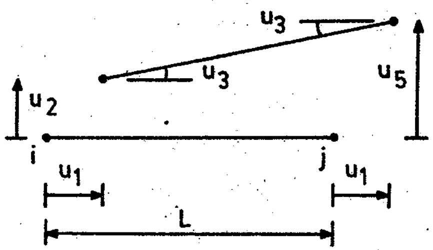


Fig. 12.5 Rigid body displacement

$$u_1 = u_4$$

$$u_3 = u_6 = \frac{u_5 - u_2}{L}$$

The displacement vector can be written as:

$$\Delta^T = \left\{ u_1 \quad u_2 \quad \frac{u_5 - u_2}{L} \quad u_1 \quad u_5 \quad \frac{u_5 - u_2}{L} \right\}$$

Let us substitute this displacement vector in Eq. 12.4 and calculate member force P_3 (say):

$$P_3 = 0 + \frac{6EI}{L^2} u_2 + \frac{4EI}{L}\left(\frac{u_5 - u_2}{L}\right) + 0 - \frac{6EI}{L^2} u_5 + \frac{2EI}{L}\left(\frac{u_5 - u_2}{L}\right) = 0$$

The same is true for any of the member end forces.

12.3 TRANSFORMATION OF COORDINATES

The force-deformation equation developed so for have been written in terms of local member forces and displacements. In practice, various members in a structure will be oriented differently and each member will have its own axes system. The direct stiffness method requires all stiffness relations to be written in terms of a common axis system called as *global axes*. We, therefore, require a relation between the local and global member end forces, and the local and global member end displacements. Since both forces and displacements are vector quantities, we can develop these relations for a general vector and then apply the result to either forces or displacements. Consider a vector OP shown in Fig. 12.6. The local axes is shown by x-y coordinate axes, and the global axes is shown by x'-y' coordinate axes. These two axes are inclined at an angle θ. The coordinates of point P are :

(x, y) in local coordinate system, and (x', y') in global coordinate system.

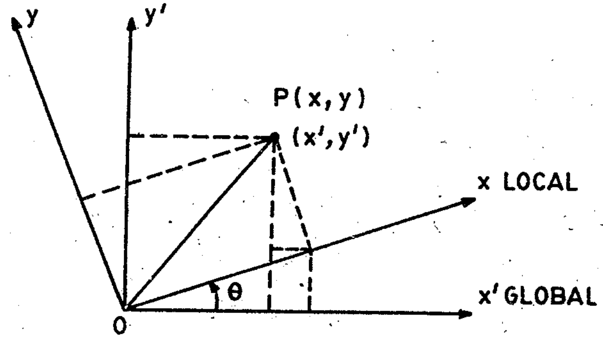


Fig. 12.6 Transformation of coordinate axes

By simple geometry, it can be seen that

$$x = x' \cos\theta + y' \sin\theta$$

and $$y = -x' \sin\theta + y' \cos\theta$$

In matrix notation

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix} \qquad (12.7a)$$

or $$\{\delta\} = [\lambda]\,\{\delta'\} \qquad (12.7b)$$

where, $$[\lambda]_{2\times 2} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad (12.7c)$$

Alternatively,

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

In matrix notation

$$\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \qquad (12.8a)$$

or $$\{\delta'\} = [\lambda]^T \{\delta\} \qquad (12.8b)$$

where, $[\lambda]$ = rotation matrix

λ is an orthogonal matrix, that is,

$$[\lambda][\lambda]^T = [I] \qquad \text{or,} \qquad [\lambda]^{-1} = [\lambda]^T$$

Truss Element

Let us consider a truss element shown in Fig. 12.7. u_1, u_2, u_3, and u_4 represent displacements along the local axes, while u_1', u_2', u_3', and u_4' represent displacements along the global axes x' - y'. The deformed shape of the truss element is shown in Fig. 12.8.

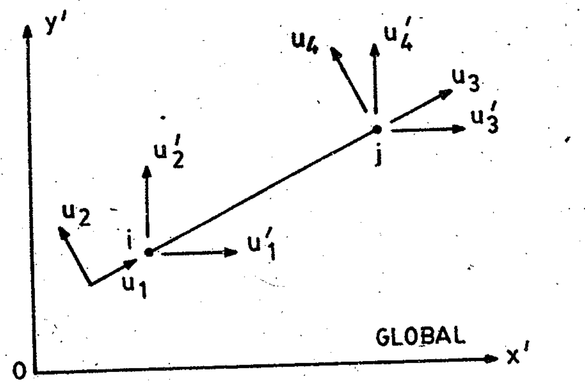


Fig. 12.7 Local and global displacements in a truss element

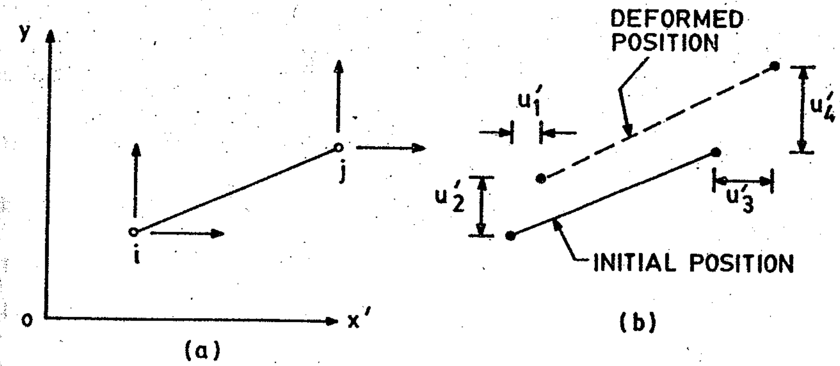


Fig. 12.8 Displaced position of a truss element

At end i,
$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}$$

at end j
$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} u_3' \\ u_4' \end{Bmatrix}$$

The displacements at the two ends can be considered simultaneously, and the relationship can be expressed as :

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{Bmatrix} \qquad (12.9)$$

or
$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \left[\begin{array}{c:c} \lambda & 0 \\ \hdashline 0 & \lambda \end{array}\right] \begin{Bmatrix} \delta_i' \\ \delta_j' \end{Bmatrix} \qquad (12.10a)$$

or
$$\{\Delta_m\} = [R]\,\{\Delta_m'\} \qquad (12.10b)$$

where, δ_i = member deformation vector at end i in local system
δ_j = member deformation vector at end j in local system
Δ_m = member deformation vector in local system
R = rotation transformation matrix for the complete element

$$= \left[\begin{array}{c:c} \lambda & 0 \\ \hdashline 0 & \lambda \end{array}\right] \qquad (12.10c)$$

The prime indicates the respective vectors in global system.

A similar expression may be written for the corresponding load vectors at each end of the element.

$$\begin{Bmatrix} P_i \\ \hdashline P_j \end{Bmatrix} = \left[\begin{array}{c:c} \lambda & 0 \\ \hdashline 0 & \lambda \end{array}\right] \begin{Bmatrix} P_i' \\ \hdashline P_j' \end{Bmatrix} \tag{12.10d}$$

$$\text{or,} \quad \{P_m\} = [R]\{P_m'\} \qquad \text{or,} \qquad \mathbf{P_m} = \mathbf{R\,P_m'} \tag{12.10e}$$

The force-displacement relation (Eq. 12.1b) can also be written in local and global systems,

$$\{P_m\} = [K_m]\{\Delta_m\} \quad \text{in local system} \tag{12.11a}$$

$$\{P_m'\} = [K_m'].\{\Delta_m'\} \quad \text{in global system} \tag{12.11b}$$

Let us substitute Eqs. 12.10 b and Eq. 12.10 e in Eq. 12.11a

$$\begin{aligned} [R]\{P_m'\} &= [K_m][R]\{\Delta_m'\} \\ \{P_m'\} &= [R]^{-1}[K_m][R]\{\Delta_m'\} \\ \{P_m'\} &= [R]^T[K_m][R]\{\Delta_m'\} \end{aligned} \tag{12.11c}$$

because $[R]^{-1} = [R]^T$ being orthogonal matrix.

Comparing Eqs. 12.11b and 12.11c,

$$[K_m'] = [R]^T[K_m][R] \quad \text{or,} \quad \mathbf{K_m'} = \mathbf{R^T K_m R} \tag{12.12}$$

Thus, member stiffness in global coordinates can be obtained from the corresponding stiffness in local coordinates using the triple matrix product given by Eq. 12.12. The transformation of end displacements from local to global coordinates is given by Eq. 12.10b, and the transformation of end forces or loads is given by Eq. 12.10d. These are the most significant relations which form the backbone of the direct stiffness method.

Beam Element

Consider a beam element shown in Fig. 12.9a. It shows the six degrees of freedom in global system. Its displaced position is shown in Fig. 12.9b. It is obvious that out of the

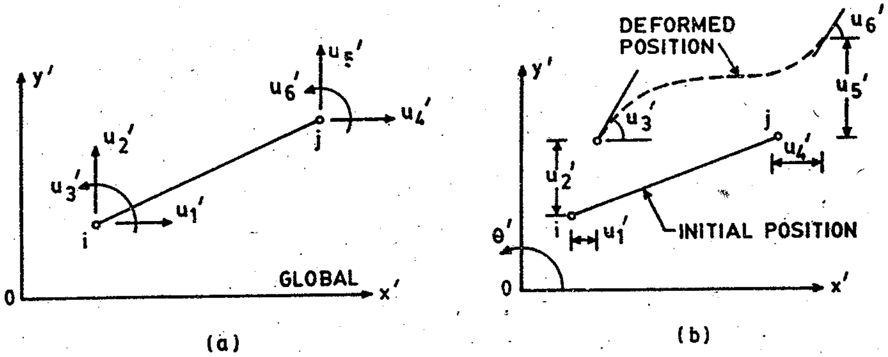


Fig. 12.9 Displaced position of a beam element

three displacements u_1', u_2' and u_3' at end i, only the first two translational components need be transformed. The local z axis and the global z'-axis coincide. Hence the rotational components u_3 and u_3' need no transformation. Thus, the rotation matrix λ can be written as :

$$\lambda_{3\times3} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.13a}$$

and complete rotation matrix for the beam element is given by

$$\mathbf{R}_{6\times6} = \left[\begin{array}{ccc:ccc} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{12.13b}$$

$$\text{or,} \quad \mathbf{R} = \left[\begin{array}{c:c} \lambda & 0 \\ \hdashline 0 & \lambda \end{array}\right]_{6\times6} \tag{12.13c}$$

Eqs. 12.10b, 12.10d and 12.12 derived earlier for a truss element are also valid for a beam element except that the size of these vectors or matrices will be 6×1 or 6×6 instead of 4×1 or 4×4.

12.4 ELEMENT LOAD VECTOR

An important characteristic of the stiffness methods including the slope-deflection method and the moment-distribution method is that there must be a one to one correspondence between loads and displacements. The loads acting across the span must be replaced by an equivalent nodal loading which satisfies two criteria. First, the equivalent loading must cause the same structural response as the original loading, and second, it must be definable using the reference coordinate system. The equivalent nodal loads are opposite to the fixed end forces as shown in Fig. 12.10. The fixed end forces restrain the joint displacements and thus represent the reactions. The equivalent nodal loads represent the actions. The fixed end forces can be conveniently calculated using the column analogy method.

In the slope deflection and the moment distribution methods, clockwise moment was treated as positive. In the direct stiffness method, anti-clockwise moment is taken as positive. The positive forces or actions act in the direction of the positive axes. Thus, for the beam shown in Fig. 12.10a, the element load vector for the equivalent loading shown in Fig. 12.10c can be written as :

$$\{P_m\}^T_{1\times6} = -\{P_e\}^T_{1\times6} = \{0 \quad -\frac{wL}{2} \quad -\frac{wL^2}{12} \quad 0 \quad -\frac{wL}{2} \quad \frac{wL^2}{12}\}_{1\times6}$$

The fixed end forces for typical loadings are shown in Appendix C.

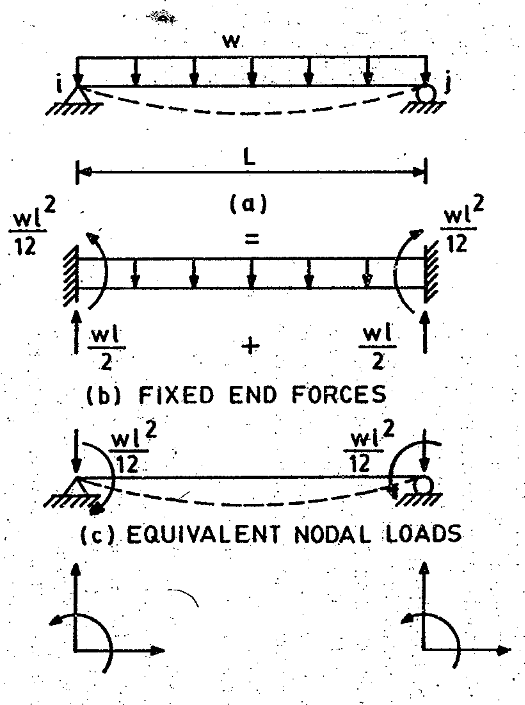


Fig. 12.10 Equivalent nodal loads

12.5 ASSEMBLY OF GLOBAL MATRICES

STIFFNESS MATRIX

For a given structure, stiffness matrix of each member can be computed in local axes system. The stiffness matrix in global axes system can be computed by making use of Eq. 12.12. The next step is to generate the stiffness matrix of the complete structure. This is also called *global stiffness matrix* of the structure. It basically means that the contribution of stiffness of all members meeting at a given joint must be added together and this process must be continued for each joint or node in the structure. Thus, there is a need to determine the locations where the elements of a member stiffness matrix are to be placed in the global stiffness matrix of the structure.

Let us consider a pin-jointed frame shown in Fig. 12.11a. The joints or nodes are numbered sequentially from 1 to 3. Similarly, the members are also numbered sequentially. The degree of freedom per node is 2, hence the total degree of freedom of the frame is equal to

= no of nodes × d.o.f. per node = $3 \times 2 = 6$.

Hence, size of global stiffness matrix of the structure is 6×6. The locations where the contribution of member stiffness matrices are to be placed depend upon the manner of node numbering. The locations can be determined knowing the member connectivities, that is, node numbers at its two ends. The global stiffness matrix of each member is 4×4 and is shown in Figs. 12.11b, c and d

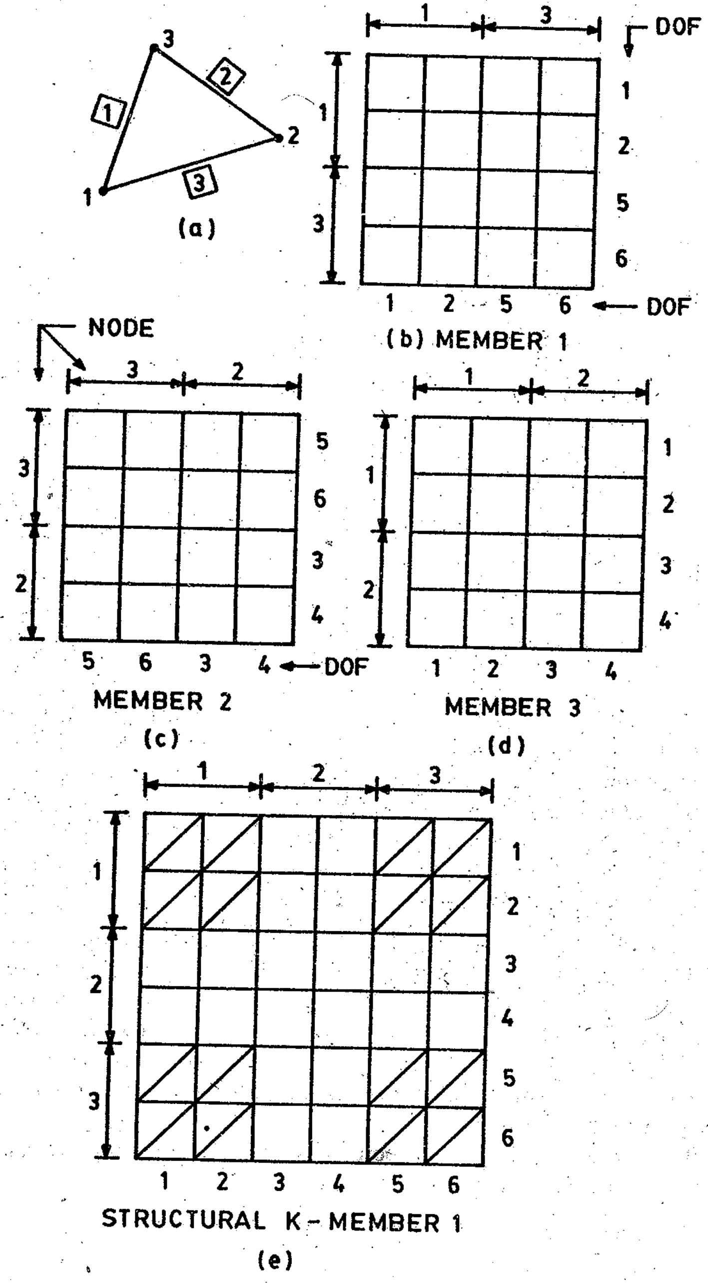


Fig. 12.11 Assembly of K - pin-jointed truss

Member	Connectivity	
	i	j
1	1	3
2	3	2
3	1	2

where i = node number at any one end of a member
j = node number at the other end of the same member

The location vector of member 1 is given as :

$$I2 = 2\times i = 2\times 1 = 2,\quad J2 = 2\times j = 2\times 3 = 6$$
$$I1 = I2 - 1 = 1,\quad J1 = J2 - 1 = 5$$
$$\{LV\}^T = \{I1 \quad I2 \quad J1 \quad J2\} = \{1 \quad 2 \quad 5 \quad 6\}$$

The location vector of member 2 is given as :

$$I2 = 2\times i = 2\times 3 = 6,\quad J2 = 2\times j = 2\times 2 = 4$$
$$I1 = I2 - 1 = 5,\quad J1 = J2 - 1 = 3$$
$$\{LV\}^T = \{5 \quad 6 \quad 3 \quad 4\}$$

The location vector of member 3 is given as :

$$I2 = 2\times i = 2\times 1 = 2,\quad J2 = 2\times j = 2\times 2 = 4$$
$$I1 = I2 - 1 = 1,\quad J1 = J2 - 1 = 3$$
$$\{LV\}^T = \{1 \quad 2 \quad 3 \quad 4\}$$

In Fig. 12.11b, the first two rows and columns represent the contribution of the node i (node 1), the last two rows and columns represent the contribution of the node j (node 3). The first row and first column represent the contribution of global x- d.o.f. (d.o.f. 1), and the second row and second column represent the contribution of global y-d.o.f. (d.o.f. 2) at node 1. Similarly, the third row and third column represent the contribution of global x-d.o.f. (d.o.f. 5), and the fourth row and fourth column represent the contribution of global y-d.o.f. (d.o.f. 6) at node 3.

Similarly, the contributions from each d.o.f. of members 2 and 3 are indicated in Figs. 12.11c and d. The structural stiffness matrix is shown in Fig. 12.11e. The first two rows and columns represent the contribution of node 1, the next two rows and columns represent the contribution of node 2, and the last two rows and columns represent the contribution of node 3. The corresponding nodal d.o.f. are also indicated along the row and the column. Thus, each element of each member matrix will go to the corresponding location in the structure matrix. These elements are added algebraically in the global stiffness matrix. The shaded portions in Fig. 12.11e show the locations where the contributions of member 1 are to be placed.

The process of assembling the global stiffness matrix can be easily carried out by visual inspection. Once this process is clearly understood, it is convenient to write a subroutine to compute the location vector LV or location matrix LM and assemble the global stiffness matrix knowing the global stiffness matrix of each element. A conceptual

subroutine is shown in Fig. 12.12. The array ST represents member stiffness matrix which is computed in subroutine TRUSS. The array STIF represents the structural stiffness matrix. The DIMENSION statements are not shown for the sake of clarity.

ASSEMBLY OF P AND K FOR TRUSS ELEMENTS

FORTRAN

```
C    NEL = total number of truss elements
C    NODI = node number at end i
C    LM = location vector (4x1)
C    PE = element load vector (4x1)
C    P = structure load vector
C    ST = element stiffness matrix (4x4)
C    STIF = structure stiffness matrix
C
     DO 100 M = 1, NEL
      CALL TRUSS (M)
      LM(2) = 2 * NODI (M)
      LM(4) = 2 * NODJ (M)
      LM(1) = LM (2) - 1
      LM(3) = LM (4) - 1
       DO 200 I = 1, 4
        II = LM (I)
        P(I I) = P(I I) + PE (I)
         DO 200 J = 1, 4
          J J = LM (J)
          STIF (I I, J J) = STIF (I I, J J) + ST (I, J)
200   CONTINUE
100  CONTINUE
```

C

```
for {m = 1; m <= nel ; m ++ }
{       TRUSS (m) ;
lm[2] = 2* nodi [m] ;
lm[4] = 2* nodj [m] ;
lm[1] = lm[2] - 1;
lm[3] = lm[4] - 1;
 for (i = 1; i <= 4 ; i ++)
{       ii = lm[i] ;
        p[ii] += pe [i] ;
  for ( j = 1 ; j <= 4 ; j ++ )
{          jj = lm[j] ;
           stif[ii] [jj] += st[i] [j] ;
}}}
```

Fig. 12.12 Program segments of assembly of K and P for a plane truss

Now let us consider a rigid jointed assembly shown in Fig. 12.13a. There are three nodes and two members. The d.o.f. per node is 3. Hence, the total d.o.f. of the structure is 3 × 3 = 9. The stiffness matrix of member 1 is shown in Fig. 12.13b. The location vector of member 1 is given as :

$$I3 = 3\times i = 3\times 1 = 3,\quad J3 = 3\times j = 3\times 3 = 9$$
$$I2 = I3 - 1 = 2,\quad J2 = J3 - 1 = 8$$
$$I1 = I3 - 2 = 1,\quad J1 = J3 - 2 = 7$$
$$\{LV\}^T = \{1 \quad 2 \quad 3 \quad 7 \quad 8 \quad 9\}$$

The first three rows and first three columns of the matrix correspond to node i while the last three rows and the last three columns correspond to node j. The nodal d.o.f. are also marked along the rows and columns as shown in Fig. 12.13b. The structural stiffness matrix in Fig. 12.13c shows the locations where the contribution of member 1 will be located. Thus, the procedure of assembly of global stiffens matrix of the rigid-jointed frame structure is the same as that of the pin-jointed truss. The program segment shown in Fig. 12.12 can be easily used for a framed structure by changing the indices appropriately.

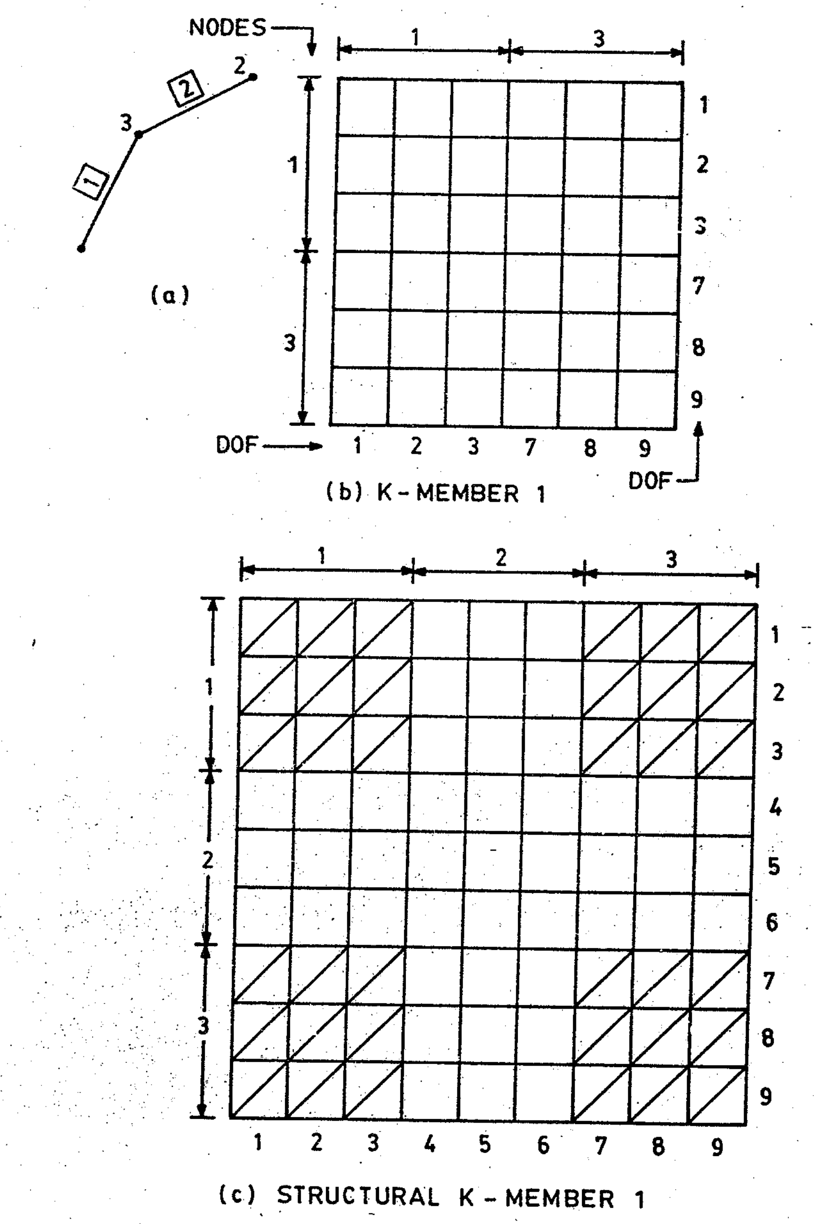


Fig. 12.13 Assembly of K - Rigid -jointed frame

Load Vector

The element global load vectors can also be added to generate the structural load vector in the same manner as the stiffness matrix. The location vector computed earlier provides one to one correspondence between the element load vector and the structure load vector. The loads applied directly at the nodes must be resolved or input along the global axes. The nodal loads are read directly into the global load vector of the structure while the element loads are assembled.

$$\{P\} = \{P_o\} + \{P_m\} \tag{12.14a}$$

$$= \{P_o\} - \{P_e\} \tag{12.14b}$$

where, $\{P_o\}$ = loads applied directly at the nodes

$\{P_m\}$ = fixed end forces due to the loads applied on the members

Consider a truss assembly shown in Fig. 12.14a. There are 4 nodes. The structure load vector will be of 8 × 1 size. The equivalent nodal forces are shown in Fig. 12.14b. The global load vector can be assembled as follows :

$$\{P\}_{8\times1} = \begin{array}{c} \left\{\begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \end{array}\right\} \end{array} \begin{array}{c} \\ 1 \\ \\ 2 \\ \\ 3 \\ \\ 4 \end{array} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ W_1 \\ W_2 \\ 0 \\ -W_3 \end{array}\right\}$$

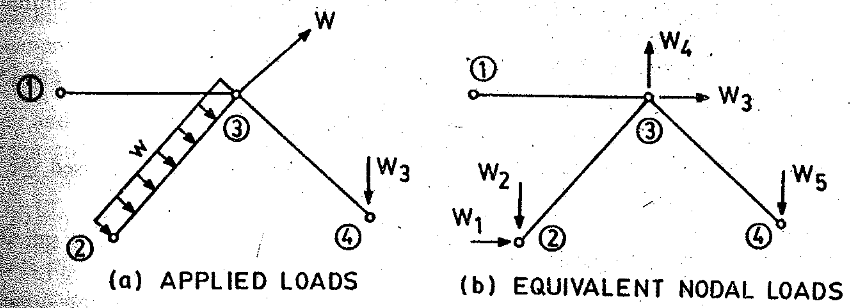
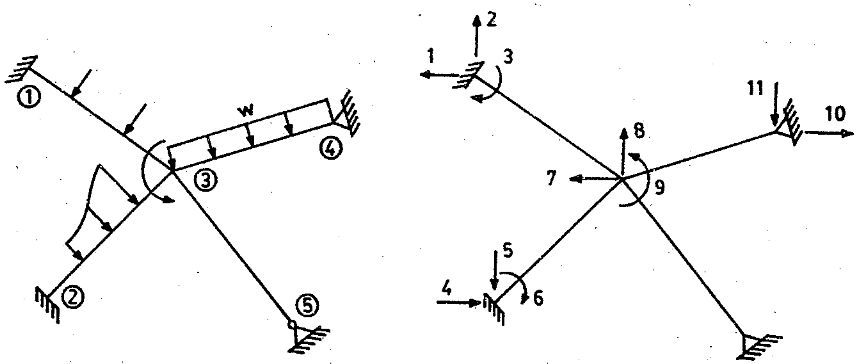


Fig. 12.14 Equivalent load vector - pin jointed truss

The program segment shown in Fig. 12.12 assembles the global load vector {P} from the element load vectors $\{PE\}_{4\times1}$.

Similarly, consider a rigid jointed plane frame assembly shown in Fig. 12.15a. The equivalent nodal loads are opposite of the fixed end forces. The element load vector (6 × 1) can be transformed from local system to global system using Eq. 12.10d. The net nodal loads are shown in Fig. 12.15b. The structural load vector can be assembled as follows :

$$\mathbf{P}^T_{15\times 1} = \{\overset{1}{-P_1 \;\; P_2 \;\; -P_3} \,\vdots\, \overset{2}{P_4 \;\; -P_5 \;\; -P_6} \,\vdots\, \overset{3}{-P_7 \;\; P_8 \;\; P_9} \,\vdots\, \overset{4}{P_{10} \;\; -P_{11} \;\; 0} \,\vdots\, \overset{5}{0 \;\; 0 \;\; 0}\}$$



(a) APPLIED LOADS (b) EQUIVALENT NODAL LOADS

Fig. 12.15 Equivalent load vector - rigid jointed frame

The program segment shown in Fig. 12.12 can be easily modified to assemble the element load vector generated in the element subroutines into the structural load vector.

The final assembled structural stiffness matrix, and load vector are related similar to Eq. 12.11.

$$\mathbf{P}_{n\times 1} = \mathbf{K}_{n\times n}\, \Delta_{n\times 1} \qquad (12.15)$$

where n = total structural d.o.f.

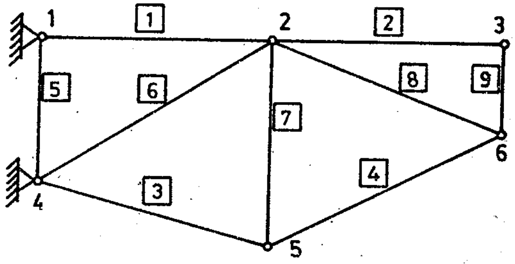
The properties of structural stiffness matrix are identical to those discussed for the member stiffness matrix. In other words, the above equation is singular.

EFFECT OF NODE NUMBERING

The *shape* of the structural stiffness matrix depends considerably on the order in which the joints or the nodes of the structure are numbered. Let us consider a plane truss shown in Fig. 12.16a.

D.O.F. per node = 2, total nodes = 6

size of structure stiffness matrix = 6 × 2 = 12



(a)

Fig. 12.16 a Node numbering in a truss

Let the member stiffness matrix in global system looks like

$$\mathbf{K} = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}_{4\times 4}$$

where x indicates an element of the structure matrix.

The structural stiffness matrix can be assembled making use of the location vectors as explained before and is shown in Fig. 12.16b. The empty cells in the matrix show zero values. Two x's in a cell show that two members meeting at the node under consideration are contributing , whereas, five x's in a cell show that five members meeting at node 2 are contributing to the nodal stiffness matrix. All the non-zero elements of the structural stiffness matrix are located within the marked area. The structural matrix is said to be a *banded matrix*. The *half band width* is defined as the maximum number of non-zero off-diagonal elements in the structure matrix plus one for the diagonal term. Thus, due to symmetry, only the upper triangular matrix need be stored in a computer algorithm.

The node numbering of a structure is done such that the band width is minimum for optimum computer solution. This becomes even more important for a large structure where the number of nodes may be 5000 or much more. A banded matrix is shown in Fig. 12.17. Let us renumber the truss of Fig. 12.16 as shown in Fig. 12.18a. The numbering is done width-wise. The structural stiffness matrix can be assembled as shown in Fig. 12.18b. The half or semi-band width is 8 against 10 obtained earlier. In a equation solver, such as, the Choleskey method only upper triangular matrix need be stored as shown in Fig. 12.18c. The elements on the diagonal are shifted to the first column of the respective rows. The storage requirement in Fig. 12.18b is n × n where n is total d.o.f. of the structure. The storage requirement in Fig. 12.18c is n × m where m is the semi-band width. The saving is enormous in a large problem. A sparse matrix may some time lead to an ill-conditioned set of equations and the solution may be incorrect.

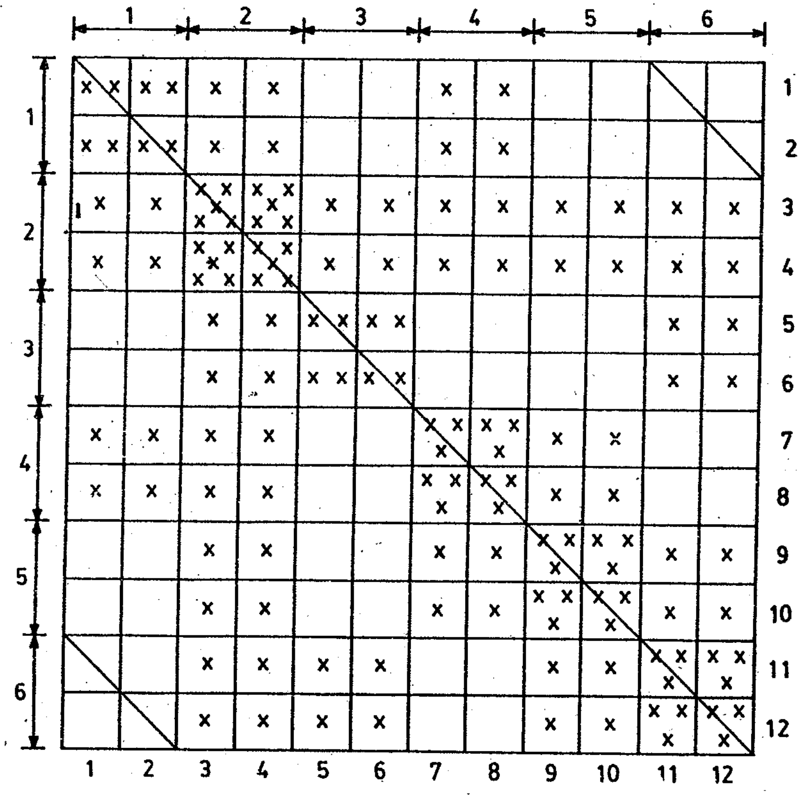


Fig. 12.16b Stiffness matrix of a truss

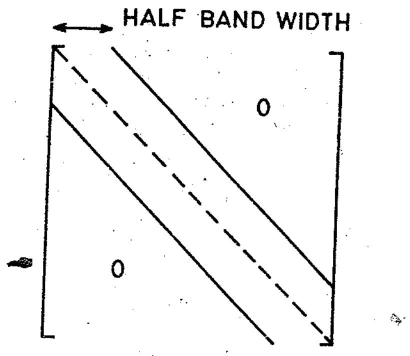
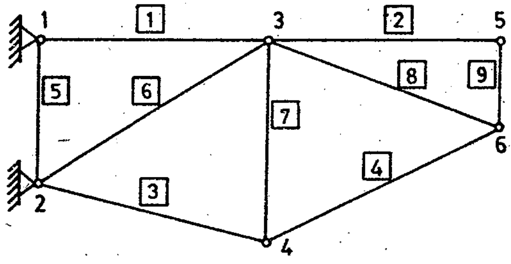
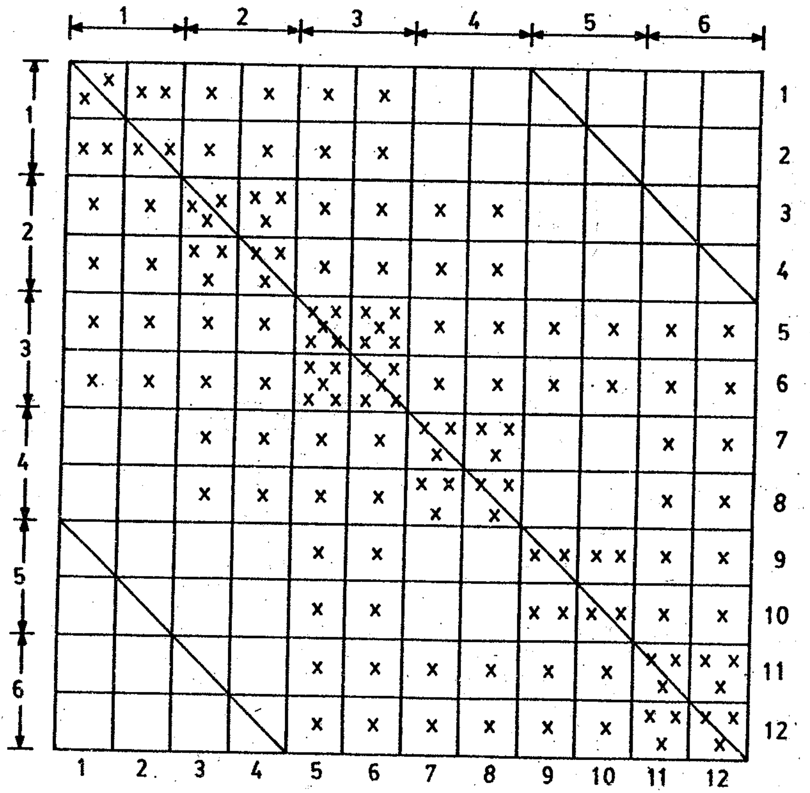


Fig. 12.17 Banded stiffness matrix

The program segment shown in Fig. 12.12 can be very easily modified to skip the elements of the lower triangular matrix and shift the diagonal elements to the first column.



(a)



(b) Stiffness matrix

Fig. 12.18 Node numbering in a truss

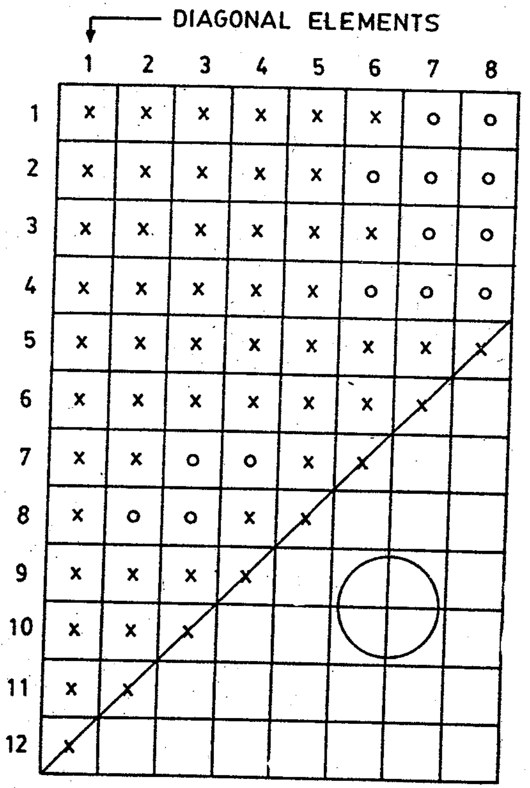


Fig. 12.18c Stiffness matrix

The program segment shown in Fig. 12.19 stores the **K** matrix in the form shown in Fig. 12.18c.

ASSEMBLY OF BANDED K AND P FOR A TRUSS ELEMENT

FORTRAN

```fortran
      DO 100 M = 1, NEL
       CALL TRUSS (M)
        LM(2) = 2 * NODI (M)
        LM(4) = 2 * NODJ (M)
        LM(3) = LM (4) - 1
         DO 200 I = 1, 4
          I I = LM (I)
          P(I I) = P(I I) + PE (I)
           DO 200 J = 1, 4
            I F ( LM (J). LT. I I ) GO TO 200
            J J = LM (J) - I I + 1
      STIF (I I , J J) = STIF (I I , J J) + ST (I, J)
200   CONTINUE
100   CONTINUE
```

C

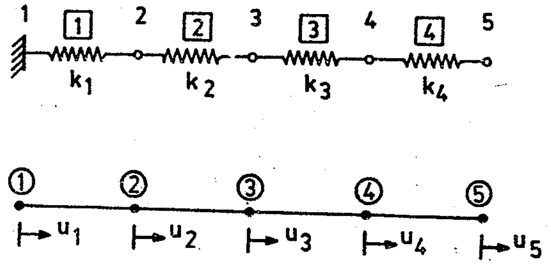
```c
for  {m = 1 ; m <= nel ; m ++
          TRUSS (m) ;
lm[2] = 2 * nodi [m] ;
lm[4] = 2 * nodj [m] ;
lm[1] = lm[2] - 1 ;
lm[3] = lm[4] - 1 ;
for {i = 1 ; i <= 4 ; i ++
{        ii = lm[i] ;
         p [ii] += pe[i] ;
for (j = 1 ; j <= 4 ; j ++ )
{        i f (lm[j] < ii )       {
         goto CONTINUE ; }
         JJ = LM[J] - II + 1 ;
         stif[ii][jj] += st[i][j]
CONTINUE: {  printf ("BYE") ;
          }
}}}
```

Fig. 12.19 Program segment of assembly of K in U.T.M and banded form

12.6 ILLUSTRATIVE EXAMPLES

Example 12.1

For the four spring system shown in Fig. 12.20a, construct the structural stiffness matrix.



(b)

Fig. 12.20

Solution

In this case, the local and global axes coincides. The stiffness matrix of a spring is given as :

$$\mathbf{K} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

The nodal d.o.f. are shown in Fig. 12.20b. The stiffness matrices of each of the four springs are as follows:

$$\mathbf{K}'_1 = k_1 \begin{array}{c} \begin{matrix} 1 & \;\; 2 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \end{matrix}, \quad \mathbf{K}'_2 = k_2 \begin{array}{c} \begin{matrix} 2 & \;\; 3 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{matrix} 2 \\ 3 \end{matrix}$$

$$\mathbf{K}'_3 = k_3 \begin{array}{c} \begin{matrix} 3 & \;\; 4 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{matrix} 3 \\ 4 \end{matrix}, \quad \mathbf{K}'_4 = k_4 \begin{array}{c} \begin{matrix} 4 & \;\; 5 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{matrix} 4 \\ 5 \end{matrix}$$

System K

The stiffness matrix of the structure or the system can be assembled with the help of location vectors as shown along with the member stiffnesses. There are 5 nodes, and d.o.f. per node is 1.

$\therefore$ Size of structural stiffness matrix = 5×5

$$[K] = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} k_1 & -k_1 & & 0 & \\ -k_1 & k_1+k_2 & -k_2 & & \\ & -k_2 & k_2+k_3 & -k_3 & \\ & & -k_3 & k_3+k_4 & -k_4 \\ & 0 & & -k_4 & k_4 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

Four important points may be noted :

1. The structural stiffness matrix is symmetric and banded. The semi-band width is 2.
2. Each degree of freedom carries the contribution from all the members meeting at that node. Thus, locations (2, 2), (3, 3) (4, 4) carry the contributions from the two springs meeting at the corresponding nodes.
3. The elements of the member stiffness matrix are added algebraically in the system stiffness matrix at the cells identified by the location vectors.
4. System matrix **K** is singular, that is, its inverse does not exist because of the presence of the rigid body displacements.

Example 12.2

Prepare a template for generating global stiffness matrix of a 2-D truss element on a LOTUS worksheet.

Solution

The following data and matrices are required to prepare a LOTUS template :

Length = L, Area = A, Modulus of elasticity = E

Inclination of member at node i with the global x-axis = θ degree
Stiffness matrix in local coordinates is given as

$$\mathbf{K} = \begin{bmatrix} \frac{AE}{L} & 0 & -\frac{AE}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{AE}{L} & 0 & \frac{AE}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rotation matrix **R** is given as

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Let us enter the data and store various matrices in the LOTUS worksheet in the cells as indicated.

Member no.,	cell B2
L	cell B3
A	cell B4
E	cell B5
θ in degree	cell B6
θ in radian	cell D6
K	cell B 8.E11
R	cell B13.E16
$\mathbf{R}^T$	cell B18.E21
Product $\mathbf{R}^T$ **K**	cell B23.E26
Product $\mathbf{R}^T$ **K R**	cell B28.E31

The computational steps are as follows :

Step 1 Enter the value of L = 300 cm in cell B3, A = 15 cm² in cell B4, E = 12000 kN/cm² in cell B5, and θ = 30° in cell B6. Only the numerical values are to be entered as shown in Fig. 12.21.

Step 2 In cell D6, enter the formula : + B6 * @ PI/180
to compute the value of θ in radian.

Step 3 In cell B8, enter the formula : + B4 * B5/B3
In cell D8, enter : – B8
In cell B10, enter : – B8
In cell D10, enter : + B8
In the remaining 12 cells of B8 to E11, enter : 0 (zero)

Step 4 In cell B13, enter the formula : @ COS (D6)
In cell C13, enter the formula : @ SIN (D6)
In cells B14 and D16, enter : – C13
In cells C14, D15 and E16, enter : + B13
In cell E15, enter : + C13
In the remaining 8 cells of B13 to E16, enter : 0

Step 5 In cell B18, C19, D20 and E21, enter : + B13
In cells C18 and E20, enter : – C13
In cells B19 and D21, enter : + C13
In the remaining 8 cells of B18 to E21, enter : 0

Step 6 The data matrices **K**, **R** and $\mathbf{R}^T$ are now ready. Let us compute $\mathbf{R}^T\mathbf{K}$ using the DATA, MATRIX MULTIPLICATION Command :
/D M M
- range of first matrix : B18.E21
range of second matrix : B8.E11
range of output matrix : B23.E26
This gives $\mathbf{R}^T$ **K.**
Now let us compute $\mathbf{R}^T$ **KR**, again using the D M M command :

/D M M
range of first matrix : B23.E26
range of second matrix : B13.E16
range of output matrix : B28.E31
This gives global stiffness matrix $\mathbf{K}' = \mathbf{R}^T \mathbf{K}\mathbf{R}$.

Step 7 For other truss members, cells A2 to E31 can be copied to A33 to E62 locations using the COPY command, and data can be changed at appropriate locations. The formulae need not be typed again. And so on.

The output is shown as follows :

	A	B	C	D	E
1		2-D Truss	element	no. 1	
2	L=	300			
3	A=	15			
4	E=	12000			
5					
6	Theta =	30	Theta (r)=	0.523598	
7					
8		600	0	– 600	0
9	[K]	0	0	0	0
10		– 600	0	600	0
11		0	0	0	0
12					
13		0.866025	0.5	0	0
14	[R]	– 0.5	0.866025	0	0
15		0	0	0.866025	0.5
16		0	0	– 0.5	0.866025
17					
18		0.866025	– 0.5	0	0
19	[R] t	0.5	0	0	0
20		0	0	0.866025	– 0.5
21		0	0	– 0.5	0.866025
22					
23		519.6152	0	–519.615	0
24	Rt*K	300	0	– 300	0
25		–519.615	0	519.6152	0
26		– 300	0	300	0
27					
28		450	259.8076	– 450	– 259.807
29		259.8076	150	– 259.807	–150
30	Rt*K*R	– 450	–259.807	450	259.8076
31		– 259.807	–150	259.8076	150
32					

Fig. 12.21 sample LOTUS template

Example 12.3

Prepare a template for generating global stiffness matrix of a 2-D beam element on a LOTUS worksheet.

Solution

The following data and matrices are required to prepare a LOTUS template :

Length = L, Area = A, Moment of inertia = I, Modulus of elasticity = E

Inclination of member at node i with the global x-axis = θ degree
Stiffness matrix in local coordinates is given as

$$\mathbf{K} = \left[\begin{array}{ccc:ccc} \frac{AE}{L} & & & & \$ & \\ 0 & \frac{12EI}{L^3} & & & & \\ & \frac{6EI}{L^2} & \frac{AE}{L} & & & \\ \hdashline -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & & \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array}\right]$$

The rotation matrix R is given as

$$\mathbf{R} = \left[\begin{array}{c:c} \lambda & 0 \\ \hdashline 0 & \lambda \end{array}\right]_{6\times 6} \quad \text{where } \lambda = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let us enter the data and store various matrices in the LOTUS worksheet in the cells as indicated.

Member no.,	cell B2
L	cell B3
A	cell B4
I	cell B5
E	cell B6
θ in degree	cell B7
θ in radian	cell D7
K	cells B9. G14

R	cells B16.G21
$\mathbf{R}^T$	cells B23.G28
Product $\mathbf{R}^T\,\mathbf{K}$	cells B30.G35
Product $\mathbf{R}^T\,\mathbf{K}\,\mathbf{R}$	cells B37.G42

The computational steps are as follows :

Step 1 Enter the value of L = 400 cm in cell B3, A = 55 cm^2 in cell B4, I = 8600 cm^4 in cell B5, E = 20000 kN/cm^2 in cell B6, and θ = 270° in cell B7. Only the numerical values are to be entered as shown in Fig. 12.22.

Step 2 In cell D7, enter the formula : + B7 * @ PI/180
to compute the value of θ in radians (= 4.712388).

Step 3 In cell B9, enter the formula : + B4 * B6/B3
In cell C10, enter the formula : + 12 * B6 * B5/B3^3
In cell C11, enter the formula : + 6 * B6 * B5/B3^2
In cell D11, enter the formula : + 4 * B6 * B5/B3
In cell D14, enter the formula : + 0.5 B4 * D11

Now copy these values with appropriate signs in the remaining cells of B to G14. Elsewhere enter a zero value. This gives **K** in local coordinates.

Step 4 In cell B16, enter the formula : @ COS (D7)
In cell C16, enter the formula : @ SIN (D7)
In cells C17, E19 and F20, enter : + B16
In cell F19, enter : + C16
In cells B17, and E20, enter : – C16
In cells D18 and G21, enter : 1
In the remaining 26 cells of B16 to G21, enter : 0
This gives **R**

Step 5 In cells B23, C24, E26 and F27, enter : + B16
In cells C23 and F26, enter : – C16
In cells B24 and E27, enter : + C16
In cells D25 and G28, enter : 1
In the remaining 26 cells of B23 to G28, enter : 0
This gives $\mathbf{R}^T$

Step 6 The data matrices **K**, **R** and $\mathbf{R}^T$ are now ready. Let us compute $\mathbf{R}^T\,\mathbf{K}$ using the DATA, MATRIX MULTIPLICATION command :
/D M M
range of first matrix : B23.G28
range of second matrix : B9.G14
range of output matrix : B30.G35
Now let us compute $\mathbf{R}^T\,\mathbf{K}\mathbf{R}$, again using the D M M command :
/D M M
range of first matrix : B30.G35
range of second matrix : B16.G21

range of output matrix : B37.G42
This gives global stiffness matrix $\mathbf{K}' = \mathbf{R}^T\,\mathbf{K}\mathbf{R}$.

Step 7 For other beam members, cells A2 to G42 can be copied to A44 to G84 using the COPY command,
And so on.

The output is shown as follows :

	A	B	C	D	E	F	G
1	Member 1						
2							
3	L =	400					
4	A =	55					
5	I =	8600					
6	E =	20000					
7	theta =	270	theta (r)=	4.712388			
8							
9		2750	0	0	– 2750	0	0
10		0	32.25	6450	0	– 32.25	6450
11		0	6450	1720000	0	– 6450	860000
12	[K1]	– 2750	0	0	2750	0	0
13		0	– 32.25	– 6450	0	32.25	– 6450
14		0	6450	860000	0	– 6450	1720000
15							
16		–1.8E–16	1	0	0	0	0
17		1	–1.8E–16	0	0	0	0
18	[R]	0	0	1	0	0	0
19		0	0	0	–1.8E–16	–1	0
20		0	0	0	1	–1.8E–16	0
21		0	0	0	0	0	1
22							
23		–1.8E–16	– 1	0	0	0	0
24		– 1	–1.8E–16	0	0	0	0
25	[R] t	0	0	1	0	0	0
26		0	0	0	–1.8E–16	1	0
27		0	0	0	– 1	–1.8E–16	0
28		0	0	0	0	0	1
29							
30		–5.1E–13	32.25	6450	5.1E–13	–32.25	6450
31		–2750	–5.9E–15	–1.2E-12	2750	5.9E–15	–1.2E–12
32	R t * K	0	6450	1720000	0	– 6450	860000
33		5.1E–13	– 32.25	– 6450	–5.1E–13	32.25	– 6450
34		2750	5.9E–15	1.2E–12	–2750	–5.9E–15	1.2E–12
35		0	6450	860000	0	– 6450	1720000
36							
37		32.25	5.0E–13	6450	– 32.25	–5.0E–13	6450
38	Rt*K*R	5.0E–13	2750	–1.2E–12	–5.0E–13	–2750	–1.2E–12
39		6450	–1.2E–12	1720000	– 6450	1.2E–12	860000
40		– 32.25	–5.0E–13	– 6450	32.25	5.0E–13	– 6450
41		–5.0E–13	– 2750	1.2E–12	5.0E–13	2750	1.2E–12
42		6450	–1.2E–12	860000	– 6450	1.2E–12	1720000

Fig. 12.22 sample LOTUS template

12.7 BOUNDARY CONDITIONS

It has been shown in sec. 12.2 that the stiffness matrix of a member is singular and it contains rigid body displacements. The same is the situation of the structure stiffness matrix. The rigid body displacement can be eliminated from the stiffness matrix by introducing appropriate boundary conditions. A structure may consist of roller supports, hinge supports or fixed supports. The purpose of introducing boundary conditions is to restrain the d.o.f. corresponding to these supports. Sometimes, we like to specify certain displacements and wish to compute the corresponding forces produced in the structure. Yielding or settlement of supports is another commonly encountered situation. The boundary conditions can be specified in different manners.

1. *Zero Displacements* - Deleting rows and columns

The effect of inactive d.o.f. can be eliminated by deleting the corresponding rows and columns from the structural stiffness matrix, load vector and displacement vector. **K**, Δ and **P** must be compacted and linear simultaneous equations can be solved for the unknown displacement vector. This method is very convenient for hand computations but is very cumbersome to program. The compacted force-displacement relationship is written as :

$$\mathbf{P_r} = \mathbf{K_r}\,\mathbf{\Delta_r} \qquad (12.16)$$

2. *Zero Displacements* - Number all support inactive d.o.f. at the end

The unrestrain nodes or free nodes are also called as the *active nodes* and the corresponding d.o.f. are called *active d.o.f.* The constraint nodes are also called as *inactive nodes* and the corresponding d.o.f. are called *inactive d.o.f.* The force-deformation equations (Eq. 12.15) can be rearranged so that all the active d.o.f. are arranged together and all the inactive d.o.f. are arranged at the end.

$$\mathbf{P} = \mathbf{K}\,\mathbf{\Delta} \qquad (12.15)$$

or,

$$\begin{Bmatrix} P_1 \\ \hline P_2 \end{Bmatrix} = \left[\begin{array}{c:c} K_{11} & K_{12} \\ \hdashline K_{21} & K_{22} \end{array}\right] \begin{Bmatrix} \Delta_1 \\ \hline \Delta_2 \end{Bmatrix} \qquad (12.17)$$

where, $\mathbf{P_1}$ = known load vector
$\mathbf{P_2}$ = unknown load vector corresponding to support reactions
$\mathbf{\Delta_1}$ = unknown displacement vector corresponding to active d.o.f.
$\mathbf{\Delta_2}$ = known or zero displacement vector corresponding to active d.o.f. or supports.

Eq. 12.17 can be rewritten as :

$$\{P_1\} = [K_{11}]\{\Delta_1\} + [K_{12}]\{\Delta_2\} \qquad (12.17a)$$

and

$$\{P_2\} = [K_{21}]\{\Delta_1\} + [K_{22}]\{\Delta_2\} \qquad (12.17b)$$

Eq. 12.17a gives the unknown displacements,

$$\{\Delta_1\} = [K_{11}]^{-1}\{P_1\} - [K_{11}]^{-1}[K_{12}]\{\Delta_2\} \qquad (12.18a)$$

Knowing $\{\Delta_1\}$, unknown load vector $\{P_2\}$ can be computed using Eq. 12.17b.

If $\{\Delta_2\} = \mathbf{0}$

$\{P_1\} = [K_{11}]\{\Delta_1\}$ (or, $\mathbf{P_r} = \mathbf{K_r}\,\mathbf{\Delta_r}$)

or, $$\{\Delta_1\} = [K_{11}]^{-1}\{P_1\} \qquad (12.18b)$$

and $$\{P_2\} = [K_{21}]\{\Delta_1\} \qquad (12.18c)$$

In general, a load vector consists of two components :

(a) loads applied directly at the nodes, P_o
(b) loads applied on the members, that is, equivalent nodal loads, P_e.

At active nodes

$$\{P_1^*\} = \{P_{o1}\} - \{P_{e1}\} \qquad (12.19a)$$

At inactive nodes, the load vector further consists of two components :

(a) loads applied directly at the nodes or through the members connected to such nodes, that is,

$$\{P_2^*\} = \{P_{o2}\} - \{P_{e2}\} \qquad (12.19b)$$

(b) reactions from the active degrees of freedom, that is,

$$\{P_2\} = [K_{21}]\{\Delta_1\} + [K_{22}]\{\Delta_2\} \qquad (12.17b)$$

The support reactions corresponding to the load vector $\{P_2^*\}$ are equal and opposite, that is, $-\{P_2^*\}$. The net support reactions are given by

$$\{P_s\} = \{P_2\} - \{P_2^*\} \qquad (12.20)$$

The limitation of this method is that it is not convenient to number the nodes in the above manner. Moreover, the partitioning shown in Eq. 12.17 is cumbersome to program and complicates the computation of member forces.

3. *Specified Displacements*

Let us consider the following set of equations :

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} \qquad (12.21a)$$

Suppose δ_2 has a known support displacement value equal to δ_2^*.
Eq. 12.21 a can be modified to read :

$$\begin{bmatrix} K_{11} & 0 & K_{13} \\ 0 & 1 & 0 \\ K_{31} & 0 & K_{33} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{Bmatrix} P_1 - K_{12}\,\delta_1^* \\ \delta_2^* \\ P_3 - K_{32}\,\delta_2^* \end{Bmatrix} \qquad (12.21b)$$

Modifying the **K** and **P** as shown in Eq. 12.21b effectively removes the i th row and column of the **K**, Δ and **P**. All the quantities on the right hand side are known. The joint displacements can be found which yield $\delta_2 = \delta_2^*$.

Program STAP

A Structural Analysis Program (STAP -3D) based on the theory developed in Chapters 12 and 13 will be presented in Chapter 14. It has a unique feature to introduce zero displacements as well as specified displacements.

Zero Displacements

An array named ID (identity) is generated as the nodal data alongwith the boundary conditions are read. A boundary condition has to be specified along each of the six degrees of freedom per node or joint. A restrained degree of freedom is specified as 1, while an unrestrained degree of freedom is specified as zero. Thus, the array ID stores the node number and the information about each of its six degrees of freedom. All restrained d.o.f. are stored as zero, while the unrestrained d.o.f. are numbered sequentially from 1. Thus, it gives the total number of active d.o.f. and therefore, the total number of active equations. The program segment shown in Fig. 12.23 establishes a relation between an active nodal d.o.f. and equation numbers.

	FORTRAN	C
C		neq = 0 ;
C	NEQ = equation number	for (n = 1 ; n <= numnp ; n ++) {
C	NUMNP = total number of nodes	for (n = 1 ; n <= ndof ; n ++) {
C	NDOF = total d.o.f. per node	if (id[n][i] > 0 {
	NEQ = 0	goto LABEL 1 ; }
	DO 50 N = 1, NUMNP	neq + = 1 ;
	DO 60 N = 1, NDOF	id[n][i] = neq ;
	IF(ID(N,I).GT.0) GO TO 70	goto LABEL 2 ;
	NEQ = NEQ + 1	LABEL 1 : {
	ID(N,I) = NEQ	id[n][i] = 0 ;
	GO TO 60	}
70	ID (N,I) = 0	LABEL 2 : {
60	CONTINUE	for (i = 1 ; i <=ndof ; i ++ {
	WRITE (*,*) (N, ID(N,I), I = 1, NDOF)	printf ("% f" , id[n][i] ; }
50	CONTINUE	}
		}}

Fig. 12.23 Relationship between active d.o.f. and equation numbers

This information is very useful while assembling the global structural stiffness matrix and the load vector. The program is instructed to skip the contribution of a particular d.o.f. in STIF and P if it is restrained. This is done through the statement

IF (LM(I) . LE . 0) GO TO 200

The relevant program segment is shown in Fig.12.24.

FORTRAN	C
DO 100 M = 1, NEL	for (m = 1 ; m <= nel ; m ++
CALL TRUSS (M)	{ TRUSS (m) ;
I = NODI (M)	i = nodi [m] ;
J = NODJ (M)	j = nodj [m] ;
LM (1) = ID (I, 1)	lm[1] = id[i][1] ;
LM (2) = ID (I, 2)	lm[2] = id[i][2] ;
LM (3) = ID (J, 1)	lm[3] = id[j][1] ;
LM (4) = ID (J, 2)	lm[4] = id[j][2] ;
DO 200 I = 1, 4	for (i = 1 ; i <= 4 ; i ++)
IF (LM(I).LE.0) GO TO 200	{ if (lm[i] <= 0) {
I I = LM (I)	goto CONTINUE1 ; }
P(I I) = P(I I) + PE (I)	ii = lm[i] ;
DO 200 J = 1, 4	p[ii] += pe[i];
I F (LM (J). LT. I I) GO TO 200	for (j = 1 ; j <= 4 ; j ++)
J J = LM (J) - I I + 1	{ if (lm[j] < ii) {
STIF (I I , J J) = STIF (I I , J J) + ST (I, J)	goto CONTINUE2 ; }
200 CONTINUE	jj = lm[j] – ii + 1 ;
100 CONTINUE	stif[ii][jj] += st[i][j] ;
	CONTINUE1 : {
	goto xy (50,20) ;
	printf("Press any key") ;
	getch( ) ;
	}
	CONTINUE2 : {
	printf("BYE") ;
	}
	}}}

Fig. 12.24. Program segment of assembly of K in U.T.M. and banded force with boundary conditions

Specified Displacements

The specified displacements at the given nodes are imposed on the structure through the use of boundary elements. A boundary element is simply a spring. It may be a translational spring or a rotational spring. Its force-deformation relation is given by

$$p = k\,\delta \quad \text{for translational springs} \tag{12.22a}$$

or

$$m = k\,\theta \quad \text{for rotational springs} \tag{12.22b}$$

The stiffness of the spring is usually specified to be a very high value say 10^{10}. At a given node as many boundary elements or springs may be specified as the d.o.f. of the node. The force-deformation equations are modified similar to Eq. 12.21b so that the desired displacements are imposed.

The reaction in the spring gives the force required to produce the specified displacement.

12.8 SUPPORT REACTIONS

Support reactions can be evaluated by making use of Eqs. 12.20 which is a some what complicated procedure. The easier way is to again use boundary elements. In case of rigid supports, the displacement is specified to be zero. The boundary elements are placed in the direction of the support d.o.f. An infinite spring stiffness leads to zero displacement. The reaction in the spring gives the support reactions along the spring direction.

12.9 INCLINED ROLLER SUPPORT

The displacement of a joint must be described in terms of independent components of displacement with respect to the global axes, otherwise the set of equations

$\{P_r\} = [K_r]\{\Delta_r\}$ would be singular.

For a roller support, the independent components of the displacement are the displacements normal and tangential to the surface guiding the roller. If this surface is oriented parallel to one of the axes as shown in Fig. 12.25a then it does not pose any problem. If the surface is inclined with respect to the global axes as shown in Fig. 12.25b, then the displacement components are related to each other and the matrix becomes singular.

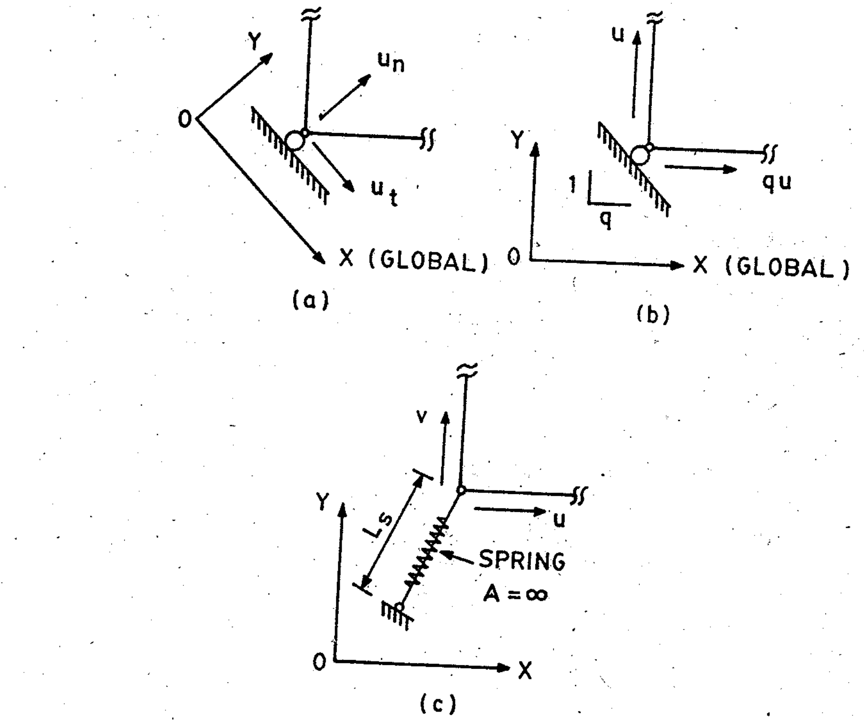


Fig. 12.25 Inclined roller supports

To over come this problem, the roller support is replaced by a boundary element. It should offer the same restraint as the roller support. Thus, the spring should have infinite axial stiffness normal to the inclined surface. It would offer no resistance normal to its own longitudinal axis. The length of the spring is of the same order as other members in the structure.

If the joint of the structure is rigid, the spring should have infinite axial area and moment of inertia. If the joint of the structures is pinned, only its area need be infinite as shown in Fig. 12.25c. Examples 14.2 and 14.4 explain the use of the boundary elements.

12.10 SUMMARY OF DIRECT STIFFNESS METHOD

The principles involved in performing the direct stiffness matrix analyses have been explained above. A flow diagram showing the inportant tasks to be accomplished by a stiffness program is shown in Fig. 12.26. For those interested in the details of programming a source listing for the analyses of 3-D framed structures is available on a floppy and its salient features are discussed in Chapter 14. The basic steps are summarized in the following :

Step 1 Idealize the structure and establish global axes.

Step 2 Number the nodes so as to obtain a minimum band width.

Step 3 Identify the type of members, that is truss element, beam element or boundary elements. Number the members in each category.

Step 4 Compile the basic data : nodal coordinates, material and member geometrical properties, member connectivity (i and j nodes).

Step 5 Calculate element stiffness matrix in local coordinates $\mathbf{k}_m$ and element load vector $\mathbf{P}_m$.

Step 6 Calculate rotation matrix λ and $\mathbf{R}$, and get global member stiffness and member load vector.

$$\mathbf{R} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad (12.10c)$$

$$\mathbf{K}_m' = \mathbf{R}^T \mathbf{K}_m \mathbf{R} \quad (12.12)$$

$$\mathbf{P}_m = \mathbf{R}\,\mathbf{P}_m' \quad (12.10e)$$

Step 7 Assemble the structural stiffness matrix and structural load vector. Superimpose nodal forces, if any.

Step 8 Introduce boundary conditions to eliminate rigid body d.o.f. and take care of specified displacements, if any.

Step 9 Solve the system of linear simultaneous equations and get the unknown nodal displacements.

$$\mathbf{P}_r = \mathbf{K}_r\,\Delta_r \quad (12.16)$$

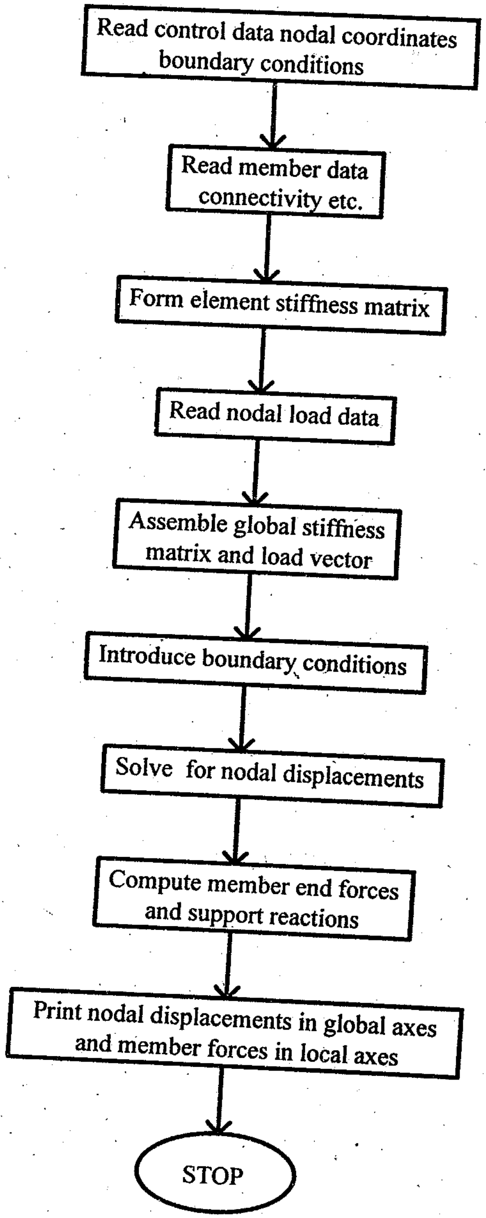


Fig. 12.26 Flow chart for a general purpose stiffness matrix program

Step 10 Extract element end displacements and transform into local coordinates

$$\Delta_m = R\ \Delta_m' \tag{12.10b}$$

Step 11 Solve for member end forces using

$$P_m^* = P_m + P_e \tag{12.23a}$$

or,

$$P_m^* = K_m\ \Delta_m + P_e \tag{12.23b}$$

These equations may be written in the local or global coordinates as appropriate.

Step 12 Superimpose the solution of kinematically indeterminate structure to get span moments.

Step 13 Evaluate the support reactions by considering the static equilibrium of boundary nodes or by using boundary elements.

$$P_s = P_2 - P_2^* \tag{12.20}$$

Step 14 Print out joint displacements and member forces.

12.11 ILLUSTRATIVE EXAMPLES

Example 12.4

Analyze the three bar assembly using the stiffness matrix method as shown in Fig. 12.27a.

Solution

Let the origin of the global axes be at node 4. The nodal d.o.f. are shown in Fig. 12.27b.

Nodal data

Node	x (cm)	y (cm)
1	– 173.2	300
2	0	300
3	300	300
4	0	0

Member data

Member	Connectivity		Geometric properties		Material properties	Direction cosine	
	I	J	A cm²	L cm	E kN/ cm²	C	S
1	4	1	6	346.42	8000	– 0.50	0.866
2	4	2	6	300	20000	0	1
3	4	3	6	424.20	8000	0.707	0.707

$C = \cos\theta$, $S = \sin\theta$

Member stiffness matrix in global system is given by

$$K_m' = R^T\ K_m\ R \tag{12.12}$$

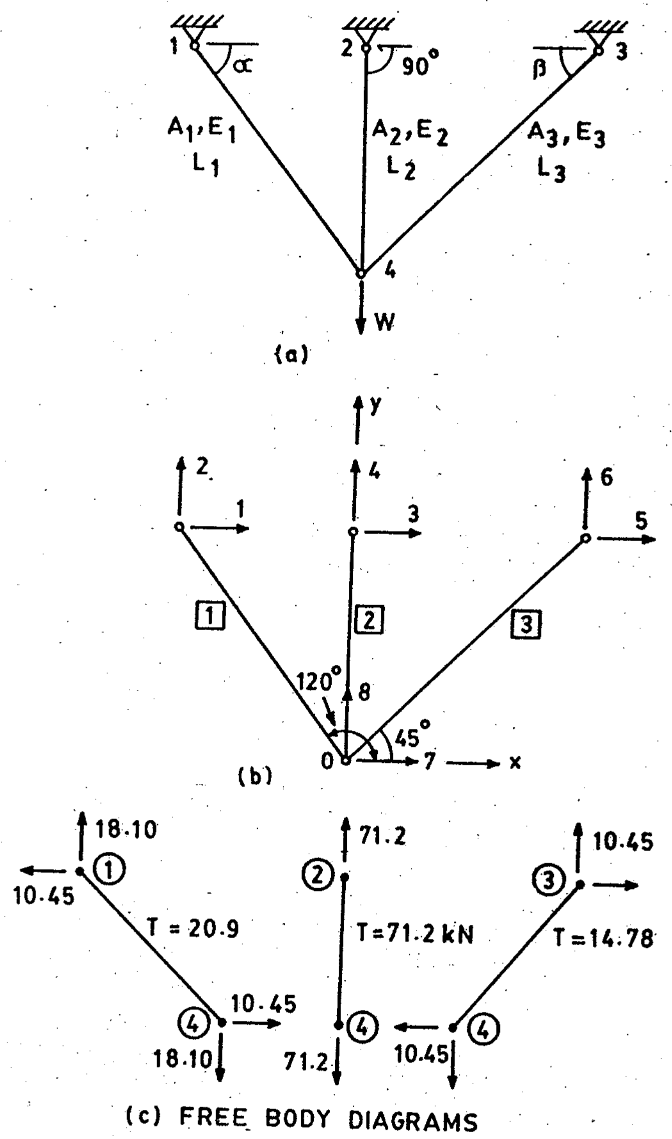


Fig. 12.27

or,
$$\mathbf{R} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\lambda = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix}$$

$$\mathbf{K}_m' = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ & \text{Symm.} & & S^2 \end{bmatrix}$$

$$\mathbf{K}_{m\,1}' = \begin{array}{c} \begin{array}{cccc} 7 & 8 & 1 & 2 \end{array} \\ \begin{bmatrix} 34.64 & -60 & -34.64 & 60 \\ & 103.92 & 60 & -103.92 \\ & & 34.64 & -60 \\ & \$ & & 103.92 \end{bmatrix} \end{array} \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array}$$

$$\mathbf{K}_{m\,2}' = \begin{array}{c} \begin{array}{cccc} 7 & 8 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 400 & 0 & -400 \\ & & 0 & 0 \\ & \$ & & 400 \end{bmatrix} \end{array} \begin{array}{c} 7 \\ 8 \\ 3 \\ 4 \end{array} \qquad \mathbf{K}_{m\,3}' = \begin{array}{c} \begin{array}{cccc} 7 & 8 & 5 & 6 \end{array} \\ \begin{bmatrix} 56.575 & 56.575 & -56.575 & -56.575 \\ & 56.575 & -56.575 & -56.575 \\ & & 56.575 & 56.575 \\ & \$ & & 56.575 \end{bmatrix} \end{array} \begin{array}{c} 7 \\ 8 \\ 5 \\ 6 \end{array}$$

There are a total of four nodes, and the structural d.o.f. $= 4 \times 2 = 8$. The stiffness matrix can be assembled since the location vectors are indicated on the member stiffness matrices.

$$\mathbf{K} = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 34.64 & -60 & & & & & -34.64 & 60 \\ & 103.92 & & & & & 60 & -103.92 \\ & & 0 & 0 & & & 0 & 0 \\ & & & 400 & & & 0 & -400 \\ & & & & 56.575 & 56.575 & -56.575 & -56.575 \\ & & & & & 56.575 & -56.575 & -56.575 \\ & & & & & & (34.64+0+56.575) & (-60+0+56.575) \\ & & & \text{Symmetric} & & & & (103.92+400+56.575) \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

Boundary conditions

$$u_1 = 0 = u_2 = u_3 = u_4 = u_5 = u_6$$

The structural stiffness matrix reduces to 2×2 by eliminating the first six rows and columns.

Load vector It is a 8×1 vector

$$\{P\} = \{P_o\} + \{P_m\} \quad (12.14a)$$

$$P^T = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -P\}_{1\times 8}$$

Eliminating the first six elements corresponding to the boundary conditions:

$$P_r = \begin{Bmatrix} 0 \\ -P \end{Bmatrix}_{2\times 1} = \begin{Bmatrix} 0 \\ -100 \end{Bmatrix}$$

The force-deformation relation gives $P_r = K_r \, \Delta_r$

or, $$\begin{Bmatrix} 0 \\ -100 \end{Bmatrix} = \begin{bmatrix} 91.215 & -3.425 \\ -3.425 & 560.495 \end{bmatrix} \begin{Bmatrix} u_7 \\ u_8 \end{Bmatrix} \quad \text{or,} \quad \begin{Bmatrix} u_7 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} -0.0067 \\ -0.178 \end{Bmatrix}$$

The member forces can be determined by using the respective force-deformation relations : (Fig. 12.27c)

$$\{P_m'\} = [K_m'] \{\Delta_m'\} \quad (12.11b)$$

Force in member 1

$$\begin{Bmatrix} P_7 \\ P_8 \\ P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} 34.64 & -60 & -34.64 & 60 \\ & 103.92 & 60 & -103.92 \\ & & 34.64 & -60 \\ & \$ & & 103.92 \end{bmatrix} \begin{Bmatrix} -0.0067 \\ -0.1780 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 10.45 \\ -18.10 \\ -10.45 \\ 18.10 \end{Bmatrix}$$

Force in member 2

$$\begin{Bmatrix} P_7 \\ P_8 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 400 & 0 & -400 \\ & & 0 & 0 \\ & \$ & & 400 \end{bmatrix} \begin{Bmatrix} -0.0067 \\ -0.1780 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -71.2 \\ 0 \\ 71.2 \end{Bmatrix}$$

Force in member 3

$$\begin{Bmatrix} P_7 \\ P_8 \\ P_5 \\ P_6 \end{Bmatrix} = 56.575 \begin{bmatrix} 1 & 1 & -1 & -1 \\ & 1 & -1 & -1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \begin{Bmatrix} -0.0067 \\ -0.1780 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -10.45 \\ -10.45 \\ 10.45 \\ 10.45 \end{Bmatrix}$$

Example 12.5

Analyze the pin-jointed truss shown in Fig. 12.28a using the stiffness matrix method. Take Area of diagonal members = $10\sqrt{2}$ cm² and that of the rest = 10 cm², and E = constant.

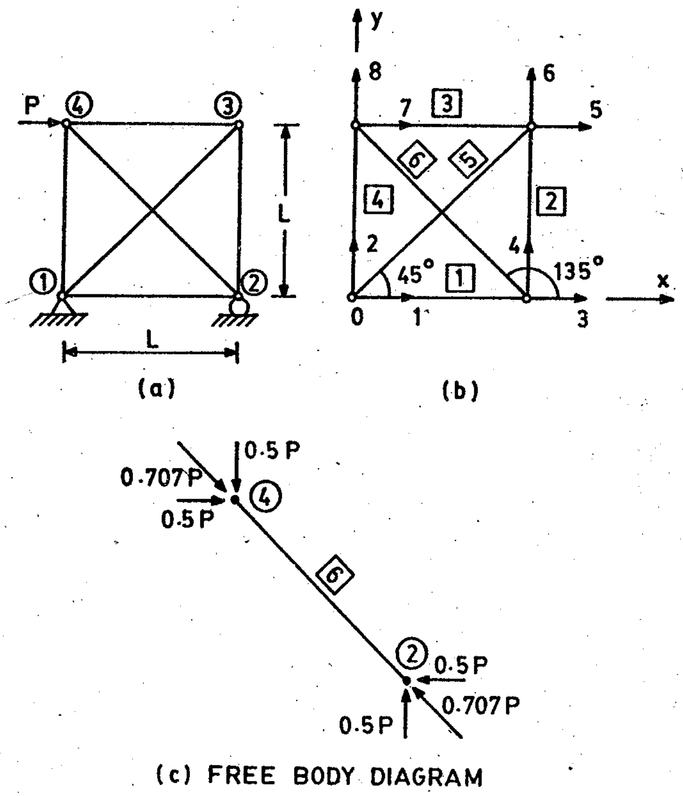


Fig. 12.28

Solution

Let the origin of the global axes be at node 1. The nodal degrees of freedom are shown in Fig. 12.28b.

Nodal data

Node	x (cm)	y (cm)
1	0	0
2	L	0
3	L	L
4	0	L

Member data

Member	Connectivity		Geometric properties		Material properties	Direction cosines		
	I	J	A cm^2	L cm	E, kN/cm^2	θ	$\cos\theta$	$\sin\theta$
1	1	2	10	L	E	0°	1	0
2	2	3	10	L	E	90°	0	1
3	4	3	10	L	E	0°	1	0
4	1	4	10	L	E	90°	0	1
5	1	3	$10\sqrt{2}$	$L\sqrt{2}$	E	45°	$1/\sqrt{2}$	$1/\sqrt{2}$
6	2	4	$10\sqrt{2}$	$L\sqrt{2}$	E	135°	$-1/\sqrt{2}$	$1/\sqrt{2}$

Global stiffness of a member is given by $\mathbf{K}_m' = \mathbf{R}^T\ \mathbf{K}_m\ \mathbf{R}$

$$\mathbf{K}_m' = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ & \text{Symm.} & & S^2 \end{bmatrix}_{4\times 4}$$

$$\mathbf{K}'_{m1} = \frac{10E}{L}\begin{array}{c}\begin{array}{cccc}1&2&3&4\end{array}\\ \left[\begin{array}{cccc}1&0&-1&0\\ &0&0&0\\ &&1&0\\ &&&0\end{array}\right]\end{array}\begin{array}{c}1\\2\\3\\4\end{array} \qquad \mathbf{K}'_{m2} = \frac{10E}{L}\begin{array}{c}\begin{array}{cccc}3&4&5&6\end{array}\\ \left[\begin{array}{cccc}0&0&0&0\\ &1&0&-1\\ &&0&0\\ &&&1\end{array}\right]\end{array}\begin{array}{c}3\\4\\5\\6\end{array}$$

$$\mathbf{K}'_{m3} = \frac{10E}{L}\begin{array}{c}\begin{array}{cccc}7&8&5&6\end{array}\\ \left[\begin{array}{cccc}1&0&-1&0\\ &0&0&0\\ &&1&0\\ &&&0\end{array}\right]\end{array}\begin{array}{c}7\\8\\5\\6\end{array} \qquad \mathbf{K}'_{m4} = \frac{10E}{L}\begin{array}{c}\begin{array}{cccc}1&2&7&8\end{array}\\ \left[\begin{array}{cccc}0&0&0&0\\ &1&0&-1\\ &&0&0\\ &&&1\end{array}\right]\end{array}\begin{array}{c}1\\2\\7\\8\end{array}$$

$$\mathbf{K}'_{m5} = \frac{10\sqrt{2}E}{L\sqrt{2}}\begin{array}{c}\begin{array}{cccc}1&2&5&6\end{array}\\ \left[\begin{array}{cccc}0.5&0.5&-0.5&-0.5\\ &0.5&-0.5&-0.5\\ &&0.5&0.5\\ &&&0.5\end{array}\right]\end{array}\begin{array}{c}1\\2\\5\\6\end{array} \qquad \mathbf{K}_{m6} = \frac{10\sqrt{2}E}{L\sqrt{2}}\begin{array}{c}\begin{array}{cccc}3&4&7&8\end{array}\\ \left[\begin{array}{cccc}0.5&-0.5&-0.5&0.5\\ &0.5&0.5&-0.5\\ &&0.5&-0.5\\ &\text{Symm.}&&0.5\end{array}\right]\end{array}\begin{array}{c}3\\4\\7\\8\end{array}$$

There are a total of four nodes, and therefore, the structural d.o.f. $= 4 \times 2 = 8$. The stiffness matrix can be assembled since the location vectors are indicated on the member

global stiffness matrices.

$$\mathbf{K} = \frac{10E}{L} \times$$

$$\begin{array}{c}\begin{array}{cccccccc}1&2&3&4&5&6&7&8\end{array}\\ \left[\begin{array}{cccccccc}
1+0+0.5 & 0+0+0.5 & -1 & 0 & -0.5 & -0.5 & 0 & 0\\
 & 0+1+0.5 & 0 & 0 & -0.5 & -0.5 & 0 & -1\\
 & & 1+0+0.5 & 0+0-0.5 & 0 & 0 & -0.5 & 0.5\\
 & & & 0+1+0.5 & 0 & -1 & 0.5 & 0.5\\
 & & & & (0+1+0.5) & (0+0+0.5) & -1 & 0\\
 & & & & & (1+0+0.5) & 0 & 0\\
 & & & & & & (1+0+0.5) & (0+0-0.5\\
 & & & & & & & (0+1+0.5)
\end{array}\right]\end{array}\begin{array}{c}1\\2\\3\\4\\5\\6\\7\\8\end{array}$$

Boundary conditions

$$u_1 = 0 = u_2 = u_4$$

Eliminating rows and columns corresponding to these d.o.f., that is, 1, 2 and 4, the structural global stiffness matrix reduces to 5×5.

Load vector It is a 8×1 vector

$$\mathbf{P}^T = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad P \quad 0\}$$

Eliminating, the 1, 2 and 4 row elements corresponding to the boundary conditions:

$$\mathbf{P}_r^T = \{0 \quad 0 \quad 0 \quad P \quad 0\}$$

The force-deformation relation gives

$$\mathbf{P}_r = \mathbf{K}_r\ \Delta_r \tag{12.16}$$

or,

$$\begin{Bmatrix}0\\0\\0\\P\\0\end{Bmatrix} = \frac{10E}{L}\begin{array}{c}\begin{array}{ccccc}3&5&6&7&8\end{array}\\ \left[\begin{array}{ccccc}1.5&0&0&-0.5&0.5\\ &1.5&0.5&-1&0\\ &&1.5&0&0\\ &&&1.5&-0.5\\ &\$&&&1.5\end{array}\right]\end{array}\begin{Bmatrix}u_3\\u_5\\u_6\\u_7\\u_8\end{Bmatrix}$$

This system of simultaneous equations can be solved using the Gauss elimination method.

$$\begin{Bmatrix}u_3\\u_5\\u_6\\u_7\\u_8\end{Bmatrix} = \frac{PL}{10E}\begin{Bmatrix}0.5\\1.5\\-0.5\\2.0\\0.5\end{Bmatrix}$$

The member forces can be determined using the corresponding force-deformation relations.

Force in member 6 (Fig. 12.28c)

$$\{P_m{}^*\} = \{P_m\} + \{P_e\} \tag{12.23a}$$

or, $$\{P_m{}^*\} = [K'_m]\{\Delta'_m\} + \{0\}$$

$$\begin{Bmatrix} P_3 \\ P_4 \\ P_7 \\ P_8 \end{Bmatrix} = \frac{10E}{L}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ & 0.5 & 0.5 & -0.5 \\ & & 0.5 & -0.5 \\ & & & 0.5 \end{bmatrix}\begin{Bmatrix} 0.5 \\ 0 \\ 2.0 \\ 0.5 \end{Bmatrix}\frac{PL}{10E} = P\begin{Bmatrix} -0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{Bmatrix}$$

The free body diagram of member 6 is shown in Fig. 12.28c. There is a net compressive force equal to 0.707P.

Example 12.6

Analyze the simply supported beam carrying a uniform load by the direct stiffness method shown in Fig. 12.29a. Neglect axial deformations.

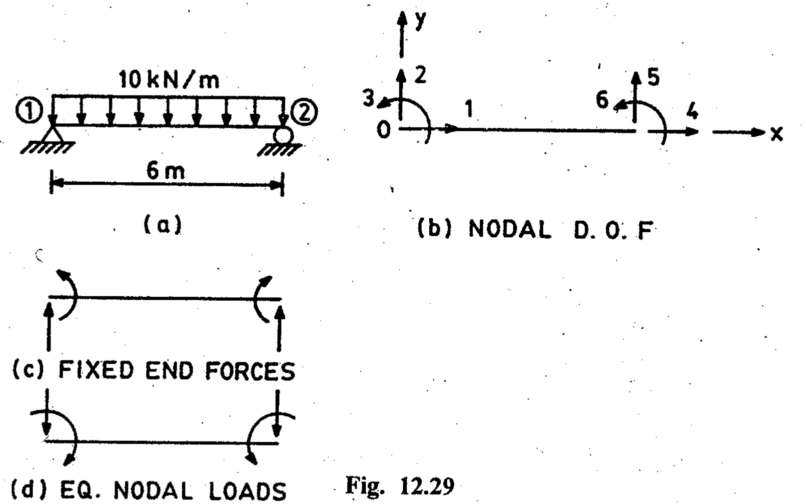


Fig. 12.29

Solution

Let the origin of the global axis be at node 1. The nodal degrees of freedom are shown in Fig. 12.29b. Since, there is only one member, its **K** in local coordinates coincides with that of the global coordinates. **K** is a 4 × 4 matrix since axial deformations are to be ignored, that is, $u_1 = u_4 = 0$

$$K_m' = K_m = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & \$ & & 4L^2 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 2 & 3 & 5 & 6 \end{array} \\ \begin{bmatrix} 800 & 2400 & -800 & 2400 \\ & 9600 & -2400 & 4800 \\ & & 800 & -2400 \\ & & & 9600 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \\ 5 \\ 6 \end{array}$$

This is also the structural stiffness matrix.

Boundary conditions

$$u_2 = 0 = u_5$$

The structural stiffness matrix reduces to 2 × 2 by eliminating the first and third rows and columns.

$$K_r = \begin{bmatrix} 9600 & 4800 \\ 4800 & 9600 \end{bmatrix}$$

Load vector

$$P = \begin{Bmatrix} P_2 \\ P_3 \\ P_5 \\ P_6 \end{Bmatrix} = \{P_o\} - \{P_e\} \tag{12.14b}$$

There is no load acting directly at the nodes. Therefore $\mathbf{P_o = 0}$.

The equivalent nodal loads due to the uniform load acting on the beam are shown in Fig. 12.29c, d.

$$P_e^T = \left\{\frac{wL}{2} \quad \frac{wL^2}{12} \quad \frac{wL}{2} \quad -\frac{wL^2}{12}\right\} = \{30 \quad 30 \quad 30 \quad -30\}$$

Reduced load vector in view of the boundary conditions,

$$P_r = \begin{Bmatrix} -30 \\ 30 \end{Bmatrix}$$

The force-deformation relation for the structure gives, $P_r = K_r\,\Delta_r$

or $$\begin{Bmatrix} -30 \\ 30 \end{Bmatrix} = \begin{bmatrix} 9600 & 4800 \\ 4800 & 9600 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_6 \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u_3 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} -0.00625 \\ 0.00625 \end{Bmatrix}$$

Exact slopes at the ends are given by

$$\theta = \frac{wL^3}{24EI} = \frac{10 \times 6^3}{24 \times 14400} = 0.00625 \text{ rad} \qquad \text{O.K.}$$

Support reactions : Let us rewrite the member equilibrium relation :

$$P_m' = K'_m \, \Delta'_m$$

$$\text{or} \quad P_m' = \begin{array}{c} \begin{array}{cccc} 3 & 6 & 2 & 5 \end{array} \\ \left[\begin{array}{cc:cc} 9600 & 4800 & 2400 & -2400 \\ 4800 & 9600 & 2400 & -2400 \\ \hdashline 2400 & 2400 & 800 & -800 \\ -2400 & -2400 & -800 & 800 \end{array}\right] \end{array} \begin{Bmatrix} -0.00625 \\ 0.00625 \\ -0 \\ 0 \end{Bmatrix} \begin{array}{c} 3 \\ 6 \\ 2 \\ 5 \end{array}$$

$$P_s = P_2 - P^*_2 \qquad (12.20)$$

$$= K_{21}\,\Delta_1 - P_{02} - P_{e2}$$

$$\begin{Bmatrix} R_2 \\ R_4 \end{Bmatrix} = \begin{bmatrix} 2400 & 2400 \\ -2400 & -2400 \end{bmatrix} \begin{Bmatrix} -0.00625 \\ 0.00625 \end{Bmatrix} - 0 + \begin{Bmatrix} 30 \\ 30 \end{Bmatrix} = \begin{Bmatrix} 30 \\ 30 \end{Bmatrix} \text{kN}$$

Member end forces

$$\{P_m{}^*\} = \{P'_m\} + \{P'_e\} \text{ in local or global coordinates}$$

$$\{P_m{}^*\} = [K'_m]\,\{\Delta'_m\} + \{P'_e\} \qquad (12.23b)$$

$$P_m{}^* = \begin{bmatrix} 800 & 2400 & -800 & 2400 \\ & 9600 & -2400 & 4800 \\ & & 800 & -2400 \\ & & & 9600 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.00625 \\ 0 \\ 0.00625 \end{Bmatrix} + \begin{Bmatrix} 30 \\ 30 \\ 30 \\ -30 \end{Bmatrix} = \begin{Bmatrix} 30 \\ 0 \\ 30 \\ 0 \end{Bmatrix} \text{kN}$$

Example 12.7

Analyze the non-prismatic beam subjected to a uniform load as shown in Fig.12.30a. Neglect axial deformations.

Solution

Let us assume a node at the point where moment of inertia changes. The d.o.f. are shown in Fig. 12.30b. The origin of the global axes is taken at node 1.

Nodal data

Node	x(m)	y(m)
1	0	0
2	4	0
3	7	0

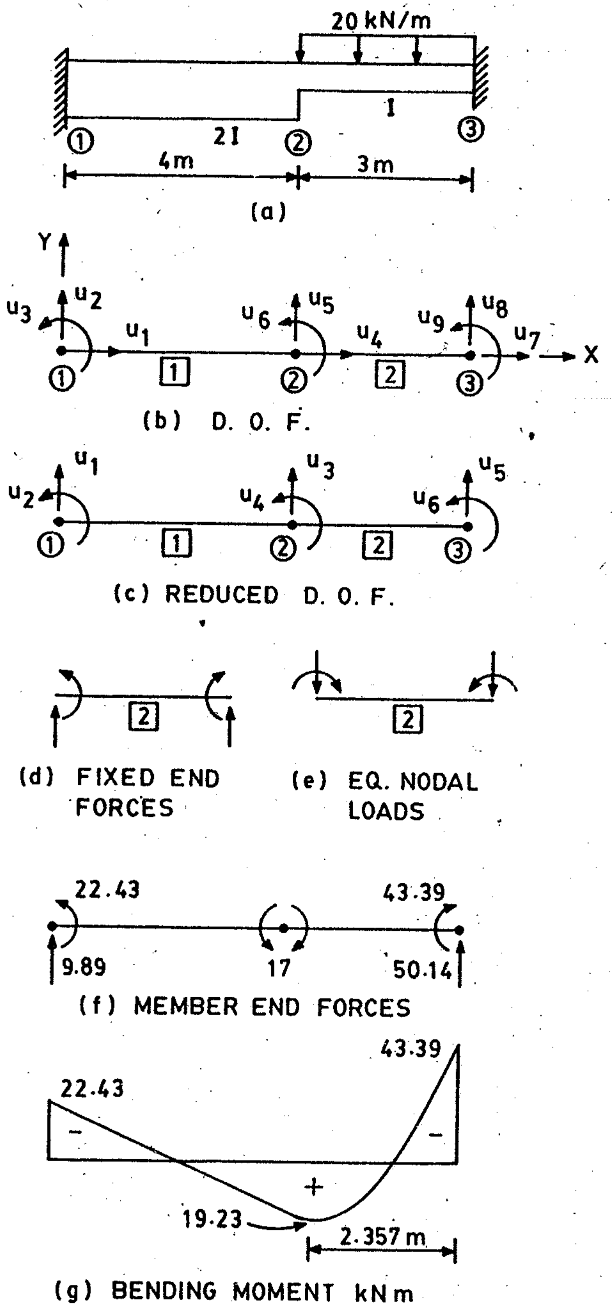


Fig. 12.30

Member data

Member	Connectivity		Geometric properties		Material properties	Direction cosine	
	I	J	A m^2	L m	E kN/m^2	C	S
1	1	2	A	4	E	1	0
2	2	3	A	3	E	1	0

Since the local and global axes coincides, and the axial deformations are ignored, the d.o.f. may be reduced as shown in Fig. 12.30c. The member stiffness matrix can be written as follows :

$$\mathbf{K'_m} = \mathbf{K_m} = \frac{EI}{L^3}\begin{bmatrix} 12 & & & \$ \\ 6L & 4L^2 & & \\ -12 & -6L & 12 & \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$\mathbf{K_{m1}} = EI \begin{array}{c} \begin{matrix} 1 & \quad 2 & \quad 3 & \quad 4 \end{matrix} \\ \begin{bmatrix} 0.375 & & & \$ \\ 0.75 & 2 & & \\ -0.375 & -0.75 & 0.375 & \\ 0.75 & 1 & -0.75 & 2 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$\mathbf{K_{m2}} = EI \begin{array}{c} \begin{matrix} 3 & \quad 4 & \quad 5 & \quad 6 \end{matrix} \\ \begin{bmatrix} 0.445 & & & \$ \\ 0.667 & 1.334 & & \\ -0.445 & -0.667 & 0.445 & \\ 0.667 & 0.667 & -0.667 & 1.334 \end{bmatrix} \end{array} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

The system stiffness matrix can be assembled as follows :

$$\mathbf{K'} = EI \begin{array}{c} \begin{matrix} 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 \end{matrix} \\ \begin{bmatrix} 0.375 & & & & & \$ \\ 0.75 & 20 & & & & \\ -0.375 & -0.75 & 0.375+0.445 & & & \\ -0.75 & 1 & -0.75+0.667 & 2+1.334 & & \\ & & -0.445 & -0.667 & 0.445 & \\ & & 0.667 & 0.667 & -0.667 & 1.334 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

Boundary conditions

$$u_1 = 0 = u_2 \quad \text{and} \quad u_5 = 0 = u_6$$

The 1, 2, 5 and 6 rows and columns are to be deleted to obtain the reduced $\mathbf{K}_{r\,2\times 2}$

$$K_r = EI \begin{array}{c} \begin{matrix} 3 & \quad 4 \end{matrix} \\ \begin{bmatrix} 0.82 & -0.083 \\ -0.083 & 3.334 \end{bmatrix} \end{array} \begin{matrix} 3 \\ 4 \end{matrix}$$

Load vector

There is no load on member 1. The equivalent nodal loads due to uniform load on member 2 are shown in Fig.12.30 d and e.

$$\mathbf{P} = \mathbf{P_o} - \mathbf{P_e}$$

There is no load acting directly at the nodes, therefore, $\mathbf{P_o} = \mathbf{0}$

or, $$\mathbf{P^T_e} = \left\{0 \quad 0 \quad \frac{wL}{2} \quad \frac{wL^2}{12} \quad \frac{wL}{2} \quad -\frac{wL^2}{12}\right\} = \{0 \quad 0 \quad 30 \quad 15 \quad 30 \quad -15\}$$

Reduced load vector

$$\mathbf{P_r} = \mathbf{P_o} - \mathbf{P_e} = \mathbf{0} - \begin{Bmatrix} 30 \\ 15 \end{Bmatrix} = \begin{Bmatrix} -30 \\ -15 \end{Bmatrix}$$

The force-deformation relation for the structure gives, $\mathbf{P_r} = \mathbf{K_r}\,\boldsymbol{\Delta}_r$

or, $$\begin{Bmatrix} -30 \\ -15 \end{Bmatrix} = EI \begin{bmatrix} 0.82 & -0.083 \\ -0.083 & 3.334 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} \quad \text{or,} \quad \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{(-1)}{EI} \begin{Bmatrix} 37.134 \\ 5.424 \end{Bmatrix}$$

Member forces

The member forces can be determined by using the corresponding force-deformation relations

$$\{P_m{}^*\} = \{P_m\} + \{P_e\} \tag{12.23a}$$

Member 1

$$\{P_m{}^*\}_1 = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}_1 = \begin{bmatrix} 0.375 & & & \$ \\ 0.75 & 2 & & \\ -0.375 & -0.75 & 0.375 & \\ 0.75 & 1 & -0.75 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -37.134 \\ -5.424 \end{Bmatrix} + \{0\} = \begin{Bmatrix} 9.86 \\ 22.43 \\ -9.86 \\ 17.00 \end{Bmatrix}$$

Similarlys, forces in member 2 can be determined.

Member 2

$$\{P_m{}^*\} = [K_m]\{\Delta_m\} + \{P_e\}$$

$$\{P_m{}^*\}_2 = \begin{Bmatrix} -20.14 \\ -32.0 \\ 20.14 \\ -28.39 \end{Bmatrix} + \begin{Bmatrix} 30 \\ 15 \\ 30 \\ -15 \end{Bmatrix} = \begin{Bmatrix} 9.86 \\ -17.0 \\ 50.14 \\ -43.39 \end{Bmatrix}$$

The net member forces are shown in Fig.12.30 f, g. The forces at node 2 are in equilibrium.

Example 12.8

Analyze the propped cantilever beam shown in Fig.12.31a by the direct stiffness method.

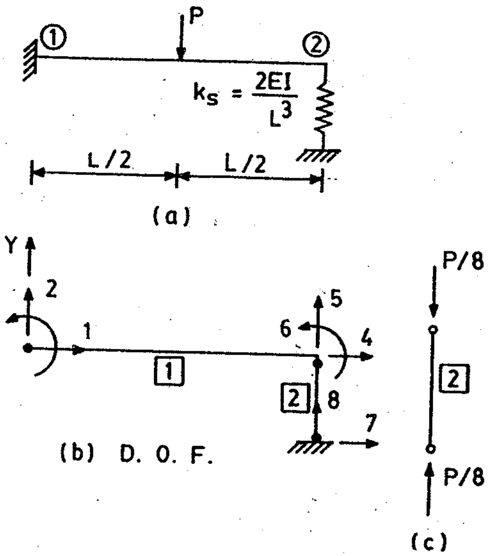


Fig. 12.31

Solution

There are two types of elements in this structure: beam element and truss element. The nodal d.o.f. are shown in Fig.12.31b. Let the origin of the global axis be at node 1.

Nodal data

Node	x(m)	y(m)
1	0	0
2	L	0
3	L	– L

Member data

Member	Type	Connectivity		Geometric properties		Direction cosine	
		I	J	AE	EI	C	S
1	beam	1	2	–	EI	1	0
2	truss	3	2	$\frac{2EI}{L^3}$	–	0	1

Stiffness matrix in global system

$$K'_{m\,1} = \begin{array}{c} \\ \left[\begin{array}{cccccc} \frac{AE}{L} & & & & & \$ \\ 0 & \frac{12EI}{L^3} & & & & \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & & & \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & & \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array}\right] \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$K'_{m\,2} = \frac{AE}{L} \begin{array}{cc} \begin{array}{cccc} 7 & 8 & 4 & 5 \end{array} & \\ \left[\begin{array}{cccc} 0 & & & \$ \\ 0 & 1 & & \\ 0 & 0 & 0 & \\ 0 & -1 & 0 & 1 \end{array}\right] & \begin{array}{c} 7 \\ 8 \\ 4 \\ 5 \end{array} \end{array} \quad \text{where,} \quad \frac{AE}{L} = \frac{2EI}{L^3}$$

The structural stiffness matrix can be assembled as follows :

$$
\mathbf{K} = \begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\left[\begin{array}{cccccccc}
\frac{AE}{L} & & & & & & & \\
0 & \frac{12EI}{L^3} & & & & & & \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & & & & & \\
-\frac{AE}{L} & 0 & 0 & \left(\frac{AE}{L}+0\right) & & & & \\
0 & \frac{-12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \left(\frac{12EI}{L^3}+\frac{2EI}{L^3}\right) & & & \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} & & \\
 & & & 0 & 0 & & 0 & 0 \\
 & & & 0 & -\frac{2EI}{L^3} & & 0 & \frac{2EI}{L^3}
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
$$

Boundary conditions

$$u_1 = 0 = u_2 = u_3 \quad \text{and} \quad u_7 = 0 = u_8$$

If axial deformation is ignored, $u_4 = 0$. The corresponding rows and columns can be eliminated.

$$
\mathbf{K}_r = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cc} 5 & 6 \end{array} \\ \begin{bmatrix} 14 & -6L \\ -6L & 4L^2 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \end{array}
$$

Load vector

$$\{P\} = \{P_o\} - \{P_e\}$$

There is no load directly at the nodes, hence $\mathbf{P_o} = \mathbf{0}$

Fixed end forces due to the point load are as follows :

$$\mathbf{P_e}^T = \left\{0 \quad \frac{P}{2} \quad \frac{PL}{8} \quad 0 \quad \frac{P}{2} \quad -\frac{PL}{8} \quad 0 \quad 0\right\}$$

The reduced load vector due to the boundary conditions is

$$\mathbf{P_r} = \begin{Bmatrix} -\frac{P}{2} \\ \frac{PL}{8} \end{Bmatrix}$$

The structural force-deformation relation gives, $\mathbf{P_r} = \mathbf{K_r}\,\Delta_r$

or, $$\begin{Bmatrix} -\frac{P}{2} \\ \frac{PL}{8} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 14 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_5 \\ u_6 \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u_5 \\ u_6 \end{Bmatrix} = -\frac{PL^2}{16EI}\begin{Bmatrix} L \\ 1 \end{Bmatrix}$$

Force in truss element

$$\{P^*_m\} = [K_m]\{\Delta_m\} + \{P_e\}$$

$$\begin{Bmatrix} P_7 \\ P_8 \\ P_4 \\ P_5 \end{Bmatrix} = \frac{2EI}{L^3}\begin{bmatrix} 0 & & & \$ \\ 0 & 1 & & \\ 0 & 0 & 0 & \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_5 \end{Bmatrix} + 0 = \begin{Bmatrix} 0 \\ \frac{P}{8} \\ 0 \\ -\frac{P}{8} \end{Bmatrix}$$

The spring carries a compressive force equal to P/8 as shown in Fig. 12.31c.

Example 12.9

Analyze the continuous beam shown in Fig. 12.32a by the direct stiffness method. Neglect axial deformations.

Solution

Let us consider the global axis at node 1. The d.o.f. are shown in Fig. 12.32b.

Nodal data

Node	x	y
	m	m
1	0	0
2	4	0
3	9	0
4	11	0

Element data

Member	Connectivity		Geometric properties		Material properties
	I	J	I	L(m)	E
1	1	2	I	4	E
2	2	3	2I	5	E
3	3	4	I	2	E

Member stiffness matrix is given by:

$$\mathbf{K'_m} = \frac{EI}{L^3}\begin{bmatrix} 12 & & & \$ \\ 6L & 4L^2 & & \\ -12 & -6L & 12 & \\ 6L & 2L^2 & -6L & \end{bmatrix}$$

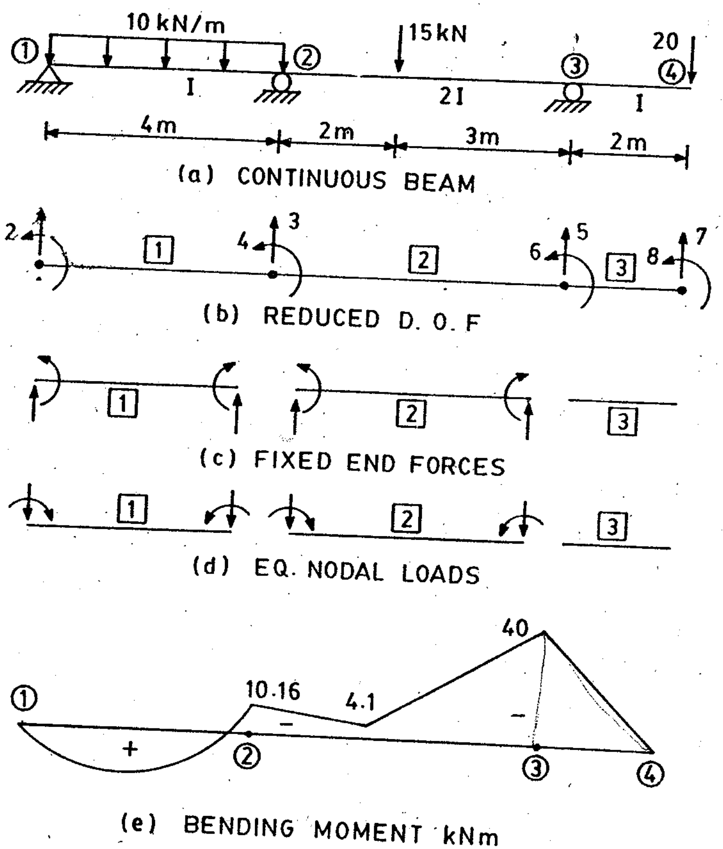


Fig. 12.32

$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\mathbf{K}_{m\,1} = EI & \left[\begin{array}{cccc} 0.1875 & & \$ & \\ 0.375 & 1.0 & & \\ -0.1875 & -0.375 & 0.1875 & \\ 0.375 & 0.50 & -0.375 & 1.0 \end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
$$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\
\mathbf{K}_{m\,2} = EI & \left[\begin{array}{cccc} 0.192 & & \$ & \\ 0.48 & 1.6 & & \\ -0.192 & -0.48 & 0.192 & \\ 0.48 & 0.80 & -0.48 & 1.6 \end{array}\right]
\end{array}
\begin{array}{c} \\ 3 \\ 4, \\ 5 \\ 6 \end{array}
\quad
\begin{array}{c c}
 & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\
\mathbf{K}_{m\,3} = EI & \left[\begin{array}{cccc} 1.5 & & \$ & \\ 1.5 & 2.0 & & \\ -1.5 & -1.5 & 1.5 & \\ 1.5 & 1.0 & -1.5 & 2.0 \end{array}\right]
\end{array}
\begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
$$

Assembly of global K

$$
\begin{array}{c c}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
= EI & \left[\begin{array}{cccccccc}
0.1875 & & & & & & & \\
0.375 & 1.0 & & & & & & \\
-0.1875 & -0.375 & 0.1875+0.192 & & & & & \\
0.375 & 0.50 & -0.375+0.48 & 1+1.6 & & & & \\
 & & -0.192 & -0.48 & 0.192+1.5 & & & \\
 & & 0.48 & 0.80 & -0.48+1.5 & 1.6+2.0 & & \\
 & & & & -1.5 & -1.5 & 1.5 & \\
 & & & & 1.5 & 1.0 & -1.5 & 2.0
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
$$

Boundary conditions

$$u_1 = 0 = u_3 = u_5$$

The reduced stiffness matrix is given by

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 2 & 4 & 6 & 7 & 8 \end{array} \\
\mathbf{K}_r = EI & \left[\begin{array}{ccccc}
1.0 & & & \$ & \\
0.5 & 2.6 & & & \\
0 & 0.80 & 3.6 & & \\
 & 0 & -1.5 & 1.5 & \\
 & & 1.0 & -1.5 & 2.0
\end{array}\right]
\end{array}
\begin{array}{c} \\ 2 \\ 4 \\ 6 \\ 7 \\ 8 \end{array}
$$

Its semi-band width is 3.

Load vector

The fixed end forces and equivalent nodal loads are shown in Figs. 12.32 c and d.

$$\mathbf{P} = \mathbf{P}_0 - \mathbf{P}_e$$

Only a 20kN load is applied directly at joint 4.

$$\therefore \quad \mathbf{P}_0^T = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -20 \quad 0\}$$

Fixed end forces on member 1 are :

$$P_1 = P_3 = \frac{wL}{2} = \frac{10\times4}{2} = 20 \text{ kN}$$

and $$P_2 = -P_4 = +\frac{wL^2}{12} = +\frac{10\times4^2}{12} = +13.34 \text{ kNm} \quad \therefore \mathbf{P}^1_e = \begin{Bmatrix} 20 \\ 13.34 \\ 20 \\ -13.34 \end{Bmatrix}$$

On Member 2

$$P_3 = \frac{Pb}{L},\ P_4 = \frac{Pab^2}{L^2},\ P_5 = \frac{Pa}{L},\ P_6 = -\frac{Pa^2 b}{L^2}$$

or $$P_3 = \frac{15\times 3}{5} = 9\text{ kN},\ P_4 = \frac{15\times 2\times 3^2}{5^2} = +10.8\text{ kNm},\ P_5 = \frac{15\times 2}{5} = 6\text{ kN}$$

$$P_6 = -\frac{15\times 2^2\times 3}{5^2} = -7.2\text{ kNm} \qquad \therefore \quad P_e^2 = \begin{Bmatrix} 9 \\ 10.8 \\ 6 \\ -7.2 \end{Bmatrix}$$

The fixed end force vector is given by

$$\mathbf{P_e} = \begin{Bmatrix} 20 \\ 13.34 \\ 20+9 \\ -13.34+10.8 \\ 6+0 \\ -7.2+0 \\ 0 \\ 0 \end{Bmatrix} = \begin{array}{l} \begin{Bmatrix} 20 \\ 13.34 \\ 29.0 \\ -2.54 \\ 6.0 \\ -7.2 \\ 0 \\ 0 \end{Bmatrix} \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

The reduced load vector is given by

$$\mathbf{P_r} = \mathbf{P_o} - \mathbf{P_e} = \begin{Bmatrix} -13.34 \\ 2.54 \\ 7.2 \\ -20.0 \\ 0 \end{Bmatrix}_{5\times 1} \begin{array}{l} 2 \\ 4 \\ 6 \\ 7 \\ 8 \end{array}$$

Solution Vector

The joint displacements can be determined using the system force-deformation relations.

$$\mathbf{P_r} = \mathbf{K_r}\,\Delta_r$$

or $$\begin{Bmatrix} -13.34 \\ 2.54 \\ 7.2 \\ -20.0 \\ 0 \end{Bmatrix} = EI \begin{bmatrix} 1.0 & & & & \$ \\ 0.5 & 2.6 & & & \\ 0 & 0.8 & 3.6 & & \\ & 0 & -1.5 & 1.5 & \\ & & 1.0 & -1.52 & \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}$$

or, $$\begin{Bmatrix} u_2 \\ u_4 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \frac{1}{EI} \begin{Bmatrix} -19.90 \\ 13.13 \\ -27.07 \\ -107.47 \\ -67.06 \end{Bmatrix}$$

Member forces

These can be determined by writing the member force deformation relations,

$$\{P^*_m\} = \{P_m\} + \{P_e\}$$
$$= [K_m]\{\Delta_m\} + \{P_e\}$$

Member 1

$$P^*_1 = \begin{bmatrix} 0.1875 & & & \$ \\ 0.375 & 1.0 & & \\ -0.1875 & -0.375 & 0.1875 & \\ 0.375 & 0.50 & -0.375 & 1.0 \end{bmatrix} \begin{Bmatrix} 0 \\ -19.9 \\ 0 \\ 13.13 \end{Bmatrix} + \begin{Bmatrix} 20 \\ 13.34 \\ 20 \\ -13.34 \end{Bmatrix} = \begin{Bmatrix} 17.46 \\ 0 \\ 22.54 \\ -10.16 \end{Bmatrix}$$

Member 2

$$P^*_2 = \begin{bmatrix} 0.192 & & & \$ \\ 0.48 & 1.6 & & \\ -0.192 & -0.48 & 0.192 & \\ 0.48 & 0.80 & -0.48 & 1.6 \end{bmatrix} \begin{Bmatrix} 0 \\ 13.13 \\ 0 \\ -27.07 \end{Bmatrix} + \begin{Bmatrix} 9 \\ 10.8 \\ 6 \\ -7.2 \end{Bmatrix} = \begin{Bmatrix} 2.31 \\ 10.15 \\ 12.69 \\ -40 \end{Bmatrix}$$

Member 3

$$P^*_3 = \begin{bmatrix} 1.5 & & & \$ \\ 1.5 & 2.0 & & \\ -1.5 & -1.5 & 1.5 & \\ 1.5 & 1.0 & -1.5 & 2.0 \end{bmatrix} \begin{Bmatrix} 0 \\ -27.07 \\ -107.47 \\ -67.06 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 20 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 40 \\ 0 \\ 0 \end{Bmatrix}$$

The bending moment diagram is shown in Fig. 12.32e.

Example 12.10

Analyze the inclined frame shown in Fig. 12.33 a using the direct stiffness method.
AE = 8000 kNm², EI = 20000 kNm².

Solution

Let the global axis be at node 1. The d.o.f. are shown in Fig. 12.33b.

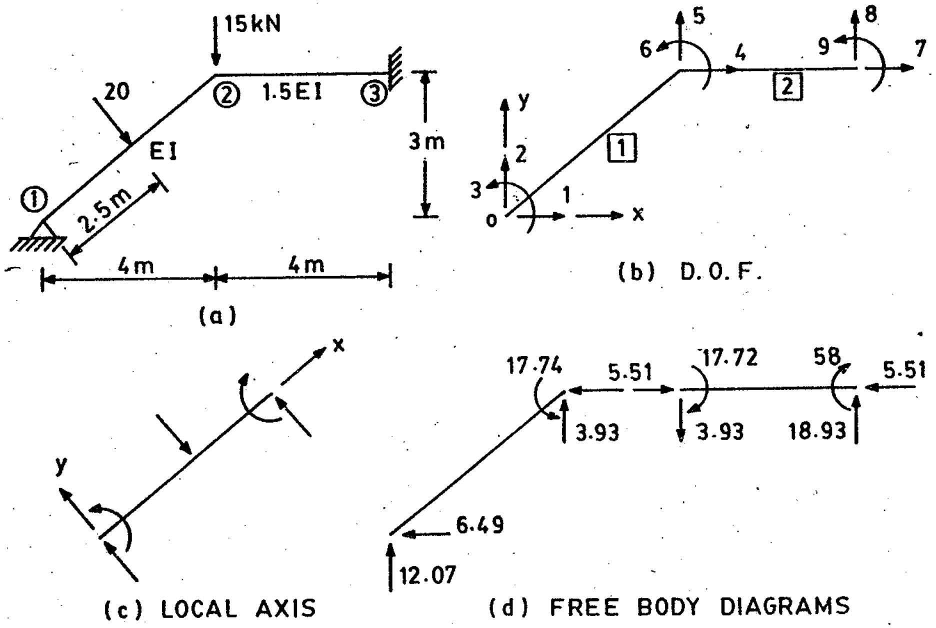


Fig. 12.33

Nodal data

Node	x (m)	y (m)
1	0	0
2	4	3
3	8	3

Member data

Member	Connectivity I	Connectivity J	Geometric properties AE kNm^2	Geometric properties EI kNm^2	Geometric properties L m	Direction cosines C	Direction cosines S
1	1	2	8000	20000	5	0.8	0.6
2	2	3	8000	30000	4	1.0	0

Member stiffness in global coordinates is given by

$$\mathbf{K'}_m = \mathbf{R}^T \mathbf{K}_m \mathbf{R} \tag{12.12}$$

where K_m and R, are given by Eqs. 12.4 and 12.13b. Substituting the member data in Eq. 12.12 gives

$$\mathbf{K'}_{m1} = \begin{array}{cc} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} & \\ \begin{bmatrix} 1715.20 & & & & & \$ \\ -153.60 & 1804.80 & & & & \\ -2880 & 3840 & 16000 & & & \\ -1715.20 & +153.60 & 2880 & 1715.20 & & \\ 153.60 & -1804.80 & -3840 & -153.60 & 1804.80 & \\ -2880 & 3840 & 8000 & 2880 & -3840 & 16000 \end{bmatrix} & \begin{matrix} 1\\2\\3\\4\\5\\6 \end{matrix} \end{array}$$

$$\mathbf{K'}_{m2} = \begin{array}{cc} \begin{matrix} 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} & \\ \begin{bmatrix} 2000 & & & & & \$ \\ 0 & 8437.50 & & & & \\ 0 & 16875 & 45000 & & & \\ -2000 & 0 & 0 & 20000 & & \\ 0 & -8437.5 & -16875 & 0 & 8437.5 & \\ 0 & 16875 & 22500 & 0 & -16875 & 45000 \end{bmatrix} & \begin{matrix} 4\\5\\6\\7\\8\\9 \end{matrix} \end{array}$$

The system stiffness matrix can be assembled with the help of location vectors.

Boundary conditions

$$u_1 = 0 = u_2 \quad \text{and} \quad u_7 = 0 = u_8 = u_9$$

The corresponding rows and columns must be deleted in the global system matrix. It reduces to a 4 × 4 matrix.

$$\mathbf{K}_r = \begin{array}{cc} \begin{matrix} 3 & 4 & 5 & 6 \end{matrix} & \\ \begin{bmatrix} 16000 & & & \$ \\ 2880 & 1715.20+2000 & & \\ -3840 & -153.60 & 1804.80+8437.50 & \\ 8000 & 2880 & -3840+16875 & 16000+45000 \end{bmatrix} & \begin{matrix} 3\\4\\5\\6 \end{matrix} \end{array}$$

$$= \begin{bmatrix} 16000 & & & \$ \\ 2880 & 3715.20 & & \\ -3840 & -153.60 & 10242.3 & \\ 8000 & 2880 & 13035 & 61000 \end{bmatrix} \begin{matrix} 3\\4\\5\\6 \end{matrix}$$

Load vector

$$\mathbf{P} = \mathbf{P}_0 - \mathbf{P}_e$$

Member 1

Fixed end forces in local coordinates Fig 12.33c

$$P_e^T = \{P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5 \quad P_6\}$$

where, $P_1 = 0 = P_4$, $\quad P_2 = P_5 = \dfrac{P}{2} = 10 \text{ kN}$

$$P_3 = +\frac{PL}{8} = +\frac{20 \times 5}{8} = +12.5 \text{ kNm} = -P_6$$

This local vector can be transformed in global coordinate system.

$$\begin{Bmatrix} P_i \\ \hdashline P_j \end{Bmatrix} = \begin{bmatrix} \lambda & \vdots & 0 \\ \hdashline 0 & \vdots & \lambda \end{bmatrix} \begin{Bmatrix} P'_i \\ \hdashline P'_j \end{Bmatrix} \quad \text{or,} \quad \begin{Bmatrix} P'_i \\ \hdashline P'_j \end{Bmatrix} = \begin{bmatrix} \lambda & \vdots & 0 \\ \hdashline 0 & \vdots & \lambda \end{bmatrix} \begin{Bmatrix} P_i \\ \hdashline P_j \end{Bmatrix}$$

$$P' = \left[\begin{array}{ccc:ccc} 0.8 & -0.6 & 0 & & & \\ 0.6 & 0.8 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hdashline & & & 0.8 & -0.6 & 0 \\ & 0 & & 0.6 & 0.8 & 0 \\ & & & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} 0 \\ 10 \\ 12.5 \\ \hdashline 0 \\ 10 \\ -12.5 \end{Bmatrix} = \begin{Bmatrix} -0.6 \\ 8.0 \\ 12.5 \\ \hdashline -0.6 \\ 8.0 \\ -12.5 \end{Bmatrix}$$

Member 2

There is no load on the member, $\therefore \mathbf{P_e = 0}$

Therefore, global load vector is given by

$$P' = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \hdashline 0 \\ -15 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -6.0 \\ 8.0 \\ 12.5 \\ \hdashline -6.0 \\ 8.0 \\ -12.5 \\ \hdashline 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 6.0 \\ -8.0 \\ -12.5 \\ \hdashline 6.0 \\ -23.0 \\ 12.5 \\ \hdashline 0 \\ 0 \\ 0 \end{Bmatrix}_{9\times1}$$

$\therefore$ Reduced load vector in view of the boundary conditions is given as :

$$P_r^T = \{-12.5 \quad 6.0 \quad -23.0 \quad 12.5\}_{1\times4}$$

Solution vector

$$P_r = K_r \Delta_r \quad \text{or,} \quad \Delta_r = K_r^{-1} \Delta_r$$

$$\Delta_r = \{-0.00356 \quad 0.00275 \quad -0.00580 \quad 0.00178\}$$

Member end forces (Fig. 12.33d)

The member end forces are given by :

$$\{P_m^*\} = \{P_m\} + \{P_e\} = [K_m]\{\Delta_m\} + \{P_e\}$$

Member 1

$$P_1^* = \begin{Bmatrix} -0.49 \\ 4.07 \\ -12.50 \\ 0.49 \\ -4.07 \\ 30.24 \end{Bmatrix} + \begin{Bmatrix} -6.0 \\ 8.0 \\ 12.5 \\ -6.0 \\ 8.0 \\ -12.5 \end{Bmatrix} = \begin{Bmatrix} -6.49 \\ 12.07 \\ 0 \\ -5.51 \\ 3.93 \\ 17.74 \end{Bmatrix}$$

Member 2

$$P_2^* = \begin{Bmatrix} 5.50 \\ -18.93 \\ -17.72 \\ -5.51 \\ 18.93 \\ -58.0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 15 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 5.50 \\ -3.93 \\ -17.72 \\ -5.51 \\ 18.93 \\ -58.0 \end{Bmatrix}$$

Example 12.11

Analyze the portal frame shown in Fig.12.34a by the direct stiffness method neglecting the axial deformations. Take $E = 200$ GPa, $I = 300 \times 10^{-6}$ m^4, $A = 100 \times 10^{-4}$ m^2.

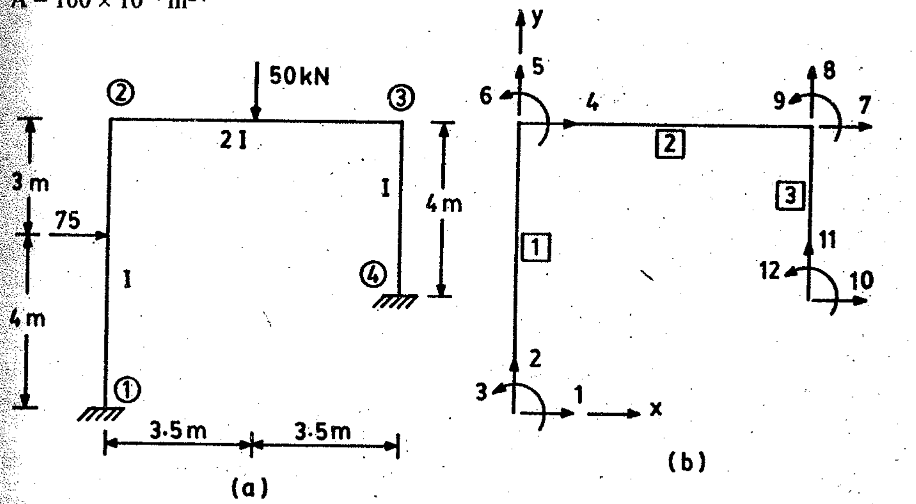


Fig. 12.34

Solution

The global axes are taken at node 1. The d.o.f. and member numbers are shown in Fig.12.34b.

Nodal data

Node	x(m)	y(m)
1	0	0
2	0	7
3	7	7
4	7	3

Element data

Member	Connectivity		Geometric property			Material property	Direction cosine	
	I	J	L (m)	A(m²) × 10^{-4}	I(m⁴) × 10^{-6}	E(kN/m²) × 10^6	C	S
1	1	2	7	100	300	200	0	1
2	2	3	7	100	600	200	1	0
3	4	3	4	100	300	200	0	1

Element stiffness matrices in global coordinates are given by

$$\mathbf{K'}_{6x6} = \mathbf{R^T\,K_m\,R}$$

$$\text{where} \quad \mathbf{R} = \begin{bmatrix} \lambda & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & \lambda \end{bmatrix}_{6\times 6}$$

The element stiffness matrices can be easily generated on a spread sheet using LOTUS software on a PC.

$$
\begin{array}{rcl}
 & \begin{matrix} 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 \end{matrix} & \\
\mathbf{K'}_{m\,1} = & \begin{bmatrix} 2099 & & & & \$ & \\ 0 & 285714 & & & & \\ -7347 & 0 & 34286 & & & \\ -2099 & 0 & 7347 & 2099 & & \\ 0 & -285714 & 0 & 0 & 285714 & \\ -7347 & 0 & 17143 & 7347 & 0 & 34286 \end{bmatrix} & \begin{matrix} 1\\2\\3\\4\\5\\6 \end{matrix}
\end{array}
$$

$$
\begin{array}{rcl}
 & \begin{matrix} 4 & \quad 5 & \quad 6 & \quad 7 & \quad 8 & \quad 9 \end{matrix} & \\
\mathbf{K'}_{m\,2} = & \begin{bmatrix} 285714 & & & & \$ & \\ 0 & 4198 & & & & \\ 0 & 14694 & 68571 & & & \\ -285714 & 0 & 0 & 285714 & & \\ 0 & -4198 & -14694 & 0 & 4198 & \\ 0 & 14694 & 34286 & 0 & -14694 & 68571 \end{bmatrix} & \begin{matrix} 4\\5\\6\\7\\8\\9 \end{matrix}
\end{array}
$$

$$
\begin{array}{rcl}
 & \begin{matrix} 10 & \quad 11 & \quad 12 & \quad 7 & \quad 8 & \quad 9 \end{matrix} & \\
\mathbf{K'}_{m\,3} = & \begin{bmatrix} 11250 & & & & \$ & \\ 0 & 500000 & & & & \\ -22500 & 0 & 60000 & & & \\ -11250 & 0 & 22500 & 11250 & & \\ 0 & -500000 & 0 & 0 & 500000 & \\ -22500 & 0 & 30000 & 22500 & 0 & 60000 \end{bmatrix} & \begin{matrix} 10\\11\\12\\7\\8\\9 \end{matrix}
\end{array}
$$

Boundary conditions

(I) ***Considering axial deformations***

$$u_1 = 0 = u_2 = u_3 \quad \text{and} \quad u_{10} = 0 = u_{11} = u_{12}$$

The system stiffness matrix can be assembled ignoring the rows and columns corresponding to the above d.o.f. It reduces to a 6 × 6 matrix as shown below :

$$
\begin{array}{rcl}
 & \begin{matrix} 4 & \quad 5 & \quad 6 & \quad 7 & \quad 8 & \quad 9 \end{matrix} & \\
\mathbf{K'}_r = & \begin{bmatrix} 2099+285714 & & & & \$ & \\ 0 & 285714+4198 & & & & \\ 7347 & 14694 & 34286+68571 & & & \\ -285714 & 0 & 0 & 285714+11250 & & \\ 0 & -4198 & -14694 & 0 & 4198+500000 & \\ 0 & 14694 & 34286 & 22500 & -14694 & 68571+60000 \end{bmatrix} & \begin{matrix} 4\\5\\6\\7\\8\\9_{6\times 6} \end{matrix}
\end{array}
$$

$$
\text{or,} \quad \mathbf{K}_r = \begin{bmatrix} 287813 & & & & & \$ \\ 0 & 289912 & & & & \\ 7347 & 14694 & 102857 & & & \\ -285714 & 0 & 0 & 296964 & & \\ 0 & -4198 & -14694 & 0 & 504198 & \\ 0 & 14694 & 34286 & 22500 & -14694 & 128571 \end{bmatrix}
$$

(II) ***Ignoring axial deformations***

The member stiffness matrix reduces to a 4 × 4 matrix as given by Eq.12.6. The same result can be obtained by using the following boundary conditions and dropping the contribution of AE/L of elements 1, 2 and 3.

$$u_1 = 0 = u_2 = u_3$$

$$u_{10} = 0 = u_{11} = u_{12}$$

$$u_2 = u_5 = 0; \quad u_8 = u_{11} = 0$$

$$\text{and} \quad u_4 = u_7 = \text{lateral sway}$$

The system stiffness matrix reduces to a 3 × 3 matrix as shown below.

$$\mathbf{K}_r = \begin{array}{c} \begin{array}{ccc} 4 & 6 & 9 \end{array} \\ \begin{bmatrix} 2099+11250 & & \$ \\ 7347 & 34286+68571 & \\ 22500 & 34286 & 68571+60000 \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 6 \\ 9 \end{array} = \begin{bmatrix} 13349 & & \$ \\ 7347 & 102857 & \\ 22500 & 34286 & 128571 \end{bmatrix}_{3\times 3}$$

Load vector

$$\mathbf{P} = \mathbf{P}_0 - \mathbf{P}_e$$

Member 1

$$p_1 = -\frac{75\times 3}{7} = -32.1 \text{ kN}, \quad p_3 = +\frac{75\times 4\times 3^2}{7^2} = +55.1 \text{ kNm}$$

$$p_4 = -42.9 \text{ kN}, \quad p_6 = -\frac{75\times 4^2\times 3}{7^2} = -73.4 \text{ kNm}$$

Member 2

$$p_5 = +25 = p_8, \quad p_6 = +\frac{50\times 7}{8} = +43.75 \text{ kNm}, \quad p_9 = -43.75 \text{ kNm}$$

The load vectors were calculated in global coordinates. The system load vector can be assembled as follows :

$$\mathbf{P'}^T = (-)\{-32.1 \;\; 0 \;\; 55.1 \;\; -42.9 \;\; 25(-73.4+43.75) \;\; 0 \;\; 25 \;\; -43.75 \;\; 0 \;\; 0 \;\; 0\}_{1\times 12}$$

Since $\mathbf{P}_o$ is zero

(i) If axial deformations are considered, the reduced load vector can be written as :

$$\mathbf{P'}^T = \{42.9 \;\; -25 \;\; 29.65 \;\; 0 \;\; -25 \;\; 43.75\}_{1\times 6}$$

(ii) If axial deformations are ignored, the reduced system load vector can be written as:

$$\mathbf{P'}^T = \{42.9 \;\; 29.65 \;\; 43.75\}_{1\times 3}.$$

Solution vectors

I. If axial deformations are considered II. If axial deformations are ignored

$$\Delta_r = \begin{Bmatrix} 0.0038 \\ -0.000076 \\ 0.00013 \\ 0.0037 \\ -0.000056 \\ -0.00034 \end{Bmatrix} \qquad \Delta_r = \begin{Bmatrix} 0.0037 \\ 0.00014 \\ -0.00035 \end{Bmatrix} \begin{array}{c} 4 \\ 6 \\ 9 \end{array}$$

Member end forces

Member end forces can be determined using the relation

$$\mathbf{P}^*_m = \mathbf{P}_m + \mathbf{P}_e$$

This calculation can be easily done using LOTUS, and the forces are as follows :

$$\mathbf{P}^*_1 = \begin{Bmatrix} -8.93 \\ 21.71 \\ 30.15 \\ 8.93 \\ -21.71 \\ 32.37 \end{Bmatrix} + \begin{Bmatrix} -32.1 \\ 0 \\ 55.1 \\ -42.9 \\ 0 \\ -73.4 \end{Bmatrix} = \begin{Bmatrix} -41.0 \\ 21.71 \\ 85.25 \\ -34.0 \\ -21.71 \\ -41.0 \end{Bmatrix}$$

$$\mathbf{P}^*_2 = \begin{Bmatrix} 34.0 \\ -3.17 \\ -3.0 \\ -34.0 \\ 3.17 \\ -19.2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 25 \\ 43.75 \\ 0 \\ 25 \\ -43.75 \end{Bmatrix} = \begin{Bmatrix} 34.0 \\ 21.83 \\ 40.75 \\ -34.0 \\ 28.17 \\ -62.95 \end{Bmatrix} \quad \text{and} \quad \mathbf{P}^*_3 = \begin{Bmatrix} -34 \\ 28 \\ 73 \\ 34 \\ -28 \\ 62.85 \end{Bmatrix}$$

Example 12.12

Analyze the portal frame whose support 4 sinks by 10 mm and rotates by 0.004 rad, clockwise, as shown in Fig. 12.35 by the direct stiffness method.
V_4 = 10 mm, θ_4 = 0.004 rad. clockwise, E = 200 GPa, I = 300 × 10^{-6} m^4.

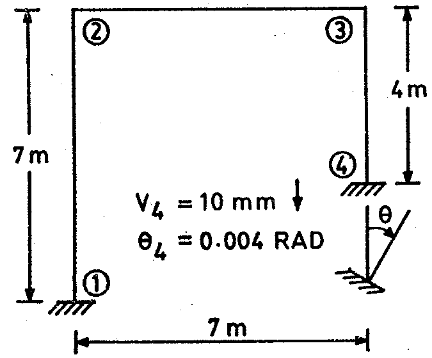


Fig. 12.35

Solution

The stiffness matrices of the frame members can be developed as in Example 12.11. The nodal d.o.f. are shown in Fig.12.34b. The boundary conditions are as follows :

$$u_1 = 0 = u_2 = u_3$$
$$u_{10} = 0, \quad u_{11} = -0.01 \text{ m}, \quad u_{12} = -0.004 \text{ radian}$$

This will lead to a 8 × 8 stiffness matrix.

If axial deformations are ignored, the boundary conditions are as follows :

$$
\begin{aligned}
u_1 &= 0 = u_2 = u_3\\
u_2 &= u_5 = 0\\
u_8 &= u_{11} = -0.01 = u_{11}^* \quad \text{say}\\
u_4 &= u_7\\
u_{10} &= 0,\; u_{12} = -0.004 = u_{12}^* \quad \text{say}
\end{aligned}
$$

This leads to a 5 × 5 stiffness matrix as shown below :

$$
\mathbf{K_r} = \begin{array}{c}
\begin{array}{ccccc} 4 & 6 & 9 & 11 & 12 \end{array}\\
\begin{bmatrix}
2099+11250 & & & \$ & \\
7347 & 34286+68571 & & & \\
22500 & 34286 & 68571+60000 & & \\
0 & -14694 & -14694 & 1{,}000{,}000+4198 & \\
22500 & 0 & 30000 & 0 & 60000
\end{bmatrix}
\end{array}
\begin{array}{c} 4\\ 6\\ 9\\ 11\\ 12 \end{array}
$$

The corresponding force-deformation relation can be written as follows :

$$
\begin{Bmatrix} 0\\0\\0\\0\\0 \end{Bmatrix} =
\begin{bmatrix}
13349 & & & \$ & \\
7347 & 102857 & & & \\
22500 & 34286 & 128571 & & \\
0 & -14694 & -14694 & 1{,}004{,}198 & \\
22500 & 0 & 30000 & 0 & 60000
\end{bmatrix}
\begin{Bmatrix} u_4\\u_6\\u_9\\u_{11}\\u_{12} \end{Bmatrix}
$$

Since the displacements u_{11} and u_{12} are known, the above equations can be rearranged as follows :

$$
\begin{Bmatrix} -22500u_{12}^*\\ 14694u_{11}^*\\ (14694u_{11}^*\\ -30000u_{12}^*)\\ u_{11}^*\\ u_{12}^* \end{Bmatrix} =
\begin{bmatrix}
13349 & & & \$ & \\
7347 & 102857 & & & \\
22500 & 34286 & 128571 & & \\
0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{Bmatrix} u_4\\u_6\\u_9\\u_{11}\\u_{12} \end{Bmatrix}
$$

or,

$$
\begin{Bmatrix} 90.0\\ -146.94\\ -26.94\\ -0.01\\ -0.004 \end{Bmatrix} =
\begin{bmatrix}
13349 & & & \$ & \\
7347 & 102857 & & & \\
22500 & 34286 & 128571 & & \\
0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{Bmatrix} u_4\\u_6\\u_9\\u_{11}\\u_{12} \end{Bmatrix}
$$

or,

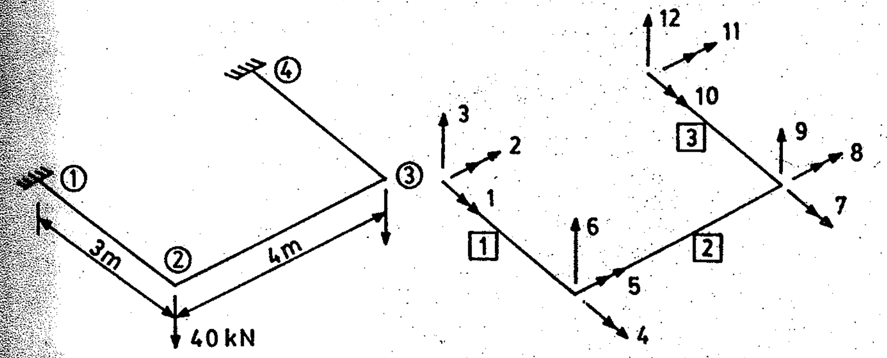
$$
\begin{Bmatrix} u_4\\u_6\\u_9\\u_{11}\\u_{12} \end{Bmatrix} =
\begin{Bmatrix} 0.0103\\ -0.001639\\ -0.00157\\ -0.01\\ -0.004 \end{Bmatrix}
$$

This is the same result as obtained by the slope deflection method, Ex.10.11.

Example 12.13

Analyze the bracket shown in Fig.12.36a by the direct stiffness method.

Members 1, 3 : $I = 8600\ \text{cm}^4$, $J = 4300\ \text{cm}^4$
Member 2 : $I = 5100\ \text{cm}^4$, $J = 2500\ \text{cm}^4$
$E = 20000\ \text{kN/cm}^2$, $\nu = 0.30$



(a) BRACKET (b) D.O.F.

Fig. 12.36

Solution

This bracket is similar to a grid structure in which the loads act perpendicular to the plane of the structure. The stiffness matrix of a grid element and its transformation matrix are given by Eqs.13.3 and 12.13. Let us consider the global axis at node 1 and the d.o.f. are shown in Fig. 12.36b.

Nodal data

Node	x(cm)	y(cm)
1	0	0
2	300	0
3	300	400
4	0	400

Element data

Member	Connectivity		Geometric property			Material property		Inclination with x-axis
	I	J	J cm^4	I cm^4	L cm	E kN/cm^2	ν	θ
1	1	2	4300	8600	300	20000	0.3	0°
2	2	3	2500	5100	400	20000	0.3	90°
3	4	3	4300	8600	300	20000	0.3	0°

The member stiffnesses in global coordinates can be written using Eqs.12.12 as follows :

$$\mathbf{K}_1 = \begin{array}{cc} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} & \\ \begin{bmatrix} 110370 & & & & \$ & \\ 0 & 2293334 & & & & \\ 0 & -11467 & 76.45 & & & \\ -110370 & 0 & 0 & 110370 & & \\ 0 & 1146667 & -11467 & 0 & 2293334 & \\ 0 & 11467 & -76.45 & 0 & 11467 & 76.45 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

$$\mathbf{K}_2 = \begin{array}{cc} \begin{matrix} 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} & \\ \begin{bmatrix} 1020000 & & & & \$ & \\ 0 & 48125 & & & & \\ 3825 & 0 & 19 & & & \\ 510000 & 0 & 3825 & 1020000 & & \\ 0 & -48125 & 0 & 0 & 48125 & \\ -3825 & 0 & -19 & -3825 & 0 & 19 \end{bmatrix} & \begin{matrix} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix} \end{array}$$

$$\mathbf{K}_3 = \begin{array}{cc} \begin{matrix} 10 & 11 & 12 & 7 & 8 & 9 \end{matrix} & \\ \begin{bmatrix} 110370 & & & & \$ & \\ 0 & 2293334 & & & & \\ 0 & -11467 & 76.45 & & & \\ -110370 & 0 & 0 & 110370 & & \\ 0 & 1146667 & -11467 & 0 & 2293334 & \\ 0 & 11467 & -76.45 & 0 & 11467 & 76.45 \end{bmatrix} & \begin{matrix} 10 \\ 11 \\ 12 \\ 7 \\ 8 \\ 9 \end{matrix} \end{array}$$

Boundary conditions

The boundary conditions are as follows :

$$u_1 = 0 = u_2 = u_3 \quad \text{and} \quad u_{10} = 0 = u_{11} = u_{12}$$

The reduced system stiffness matrix can be assembled as shown below :

$$\mathbf{K}_r = \begin{array}{cc} \begin{matrix} 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} & \\ \begin{bmatrix} \begin{matrix}1020000+110370\\=1130370\end{matrix} & & & & \$. & \\ 0 & \begin{matrix}48125+2293334\\=2341460\end{matrix} & & & & \\ 3825 & 0 & \begin{matrix}19.+76.45\\=95.45\end{matrix} & & & \\ 510000 & 0 & 3825 & \begin{matrix}1020000+110370\\=1130370\end{matrix} & & \\ 0 & -48125 & 0 & 0 & \begin{matrix}48125+2293334\\=2341460\end{matrix} & \\ -3825 & 0 & -19 & -3825 & 11467 & \begin{matrix}19+76.45\\=95.45\end{matrix} \end{bmatrix} & \begin{matrix} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix} \end{array}$$

Load vector

Since there is only one nodal load, the reduced global load vector can be written as :

$$\mathbf{P}_r^T = \{0 \quad 0 \quad -40 \quad 0 \quad 0 \quad 0\}_{1\times6}$$

Solution vector

The solution vector can be obtained from the equation : $\mathbf{P}_r = \mathbf{K}_r\,\Delta_r$

A suitable solution algorithm in double precision may be employed because there are some very large numbers in $\mathbf{K}_r$.

or, $\Delta^T_r = \{\,0.0039 \quad 0.0093 \quad -1.89 \quad 0.0039 \quad 0.0012 \quad -0.20\}$

Member end forces

$$\mathbf{P}^*_m = \mathbf{P}_m + \mathbf{P}_e$$

Member 1

$$\begin{array}{rccccccc} & & 1 & 2 & 3 & 4 & 5 & 6 \\ \mathbf{P}_1^{*T} & = \{ & -434.5 & -11026 & 38.06 & 434.5 & -390 & -38.06\} \end{array}$$

Member 2

$$\begin{array}{rccccccc} & & 4 & 5 & 6 & 7 & 8 & 9 \\ \mathbf{P}_2^{*T} & = \{ & -434.5 & 389.0 & -1.94 & -434.5 & -389.0 & 1.94\} \end{array}$$

Member 3

$$\begin{array}{rccccccc} & & 10 & 11 & 12 & 7 & 8 & 9 \\ \mathbf{P}_3^{*T} & = \{ & -434.5 & -977 & 1.94 & 434.5 & 389.0 & -1.94\} \end{array}$$

Example 12.14

A portal frame has two pin-ended diagonal members as shown in Fig. 12.37a. Analyze the member forces by using the direct stiffness method.

	Column	Beam	Brace
A cm^2	55	40	10
I cm^4	8600	3400	
E kN/cm^2	20000	20000	20000

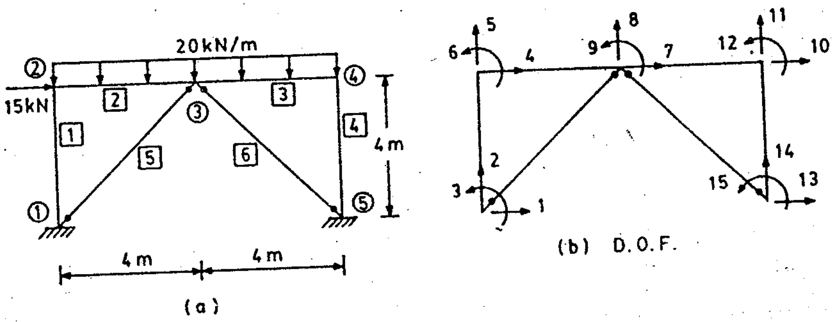


Fig. 12.37

Solution

There are two types of members used in this frame : beam type and truss type. They can be joined at a single node since their translational d.o.f. are compatible. The d.o.f. are shown in Fig.12.37 b. The center of the global axis is taken at node 2.

Nodal data

Node	x(cm)	y(cm)
1	0	0
2	400	0
3	800	0
4	0	− 400
5	800	− 400

Member data

Member	Connectivity		Geometric property			Material property	Inclination with x-axis
Beam type	I	J	A cm^2	I cm^4	L cm	E kN/cm^2	θ
1	1	2	55	8600	400	20000	90°
2	2	3	40	3400	400	20000	0°
3	3	4	40	3400	400	20000	0°
4	4	5	55	8600	400	20000	270°
Truss type							
1	1	3	10	-	565.7	20000	45°
2	3	5	10	-	565.7	20000	315°

The member stiffness matrices in global coordinates can be developed using LOTUS spread sheet.

Stiffness matrix for beam elements

$$
\mathbf{K}_1 = \begin{array}{cccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & \\
\hline
32 & & & & \$ & & 1 \\
0 & 2750 & & & & & 2 \\
-6450 & 0 & 1720000 & & & & 3 \\
-32 & 0 & 6450 & 32 & & & 4 \\
0 & -2750 & 0 & 0 & 2750 & & 5 \\
-6450 & 0 & 860000 & 6450 & 0 & 1720000 & 6
\end{array}
$$

$$
\mathbf{K}_2 = \begin{array}{cccccc|c}
4 & 5 & 6 & 7 & 8 & 9 & \\
\hline
2000 & & & & \$ & & 4 \\
0 & 12.75 & & & & & 5 \\
0 & 2550 & 680000 & & & & 6 \\
-2000 & 0 & 0 & 2000 & & & 7 \\
0 & -12.75 & -2550 & 0 & 12.75 & & 8 \\
0 & 2550 & 340000 & 0 & -2550 & 680000 & 9
\end{array}
$$

$$
\mathbf{K}_3 = \begin{array}{cccccc|c}
7 & 8 & 9 & 10 & 11 & 12 & \\
\hline
2000 & & & & \$ & & 7 \\
0 & 12.75 & & & & & 8 \\
0 & 2550 & 680000 & & & & 9 \\
-2000 & 0 & 0 & 2000 & & & 10 \\
0 & -12.75 & -2550 & 0 & 12.75 & & 11 \\
0 & 2550 & 340000 & 0 & -2550 & 680000 & 12
\end{array}
$$

$$
\mathbf{K}_4 = \begin{array}{cccccc|c}
10 & 11 & 12 & 13 & 14 & 15 & \\
\hline
32 & & & & \$ & & 10 \\
0 & 2750 & & & & & 11 \\
6450 & 0 & 1720000 & & & & 12 \\
-32 & 0 & -6450 & 32 & & & 13 \\
0 & -2750 & 0 & 0 & 2750 & & 14 \\
6450 & 0 & 860000 & -6450 & 0 & 1720000 & 15
\end{array}
$$

Stiffness matrix for truss elements

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \\
\mathbf{K}_1 = & \left[\begin{array}{cccc} 177 & & & \$ \\ 177 & 177 & & \\ -177 & -177 & 177 & \\ -177 & -177 & 177 & 177 \end{array}\right]\begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \end{array}
\end{array}
,\quad
\begin{array}{cc}
 & \begin{array}{cccc} 7 & 8 & 13 & 14 \end{array} \\
\mathbf{K}_2 = & \left[\begin{array}{cccc} 177 & & & \$ \\ -177 & 177 & & \\ -177 & 177 & 177 & \\ 177 & -177 & -177 & 177 \end{array}\right]\begin{array}{c} 7 \\ 8 \\ 13 \\ 14 \end{array}
\end{array}
$$

The system stiffness matrix can now be assembled with the help of location vectors. It is a 15×15 matrix.

Boundary conditions

If axial deformations are ignored, the boundary conditions are as follows :

$$
\begin{aligned}
u_1 &= 0 = u_2 = u_3 \\
u_2 &= u_5 = 0 \\
u_4 &= u_7 = u_{10} \\
u_{10} &= 0 = u_{11} = u_{12} \\
u_{13} &= 0 = u_{14} = u_{15} \\
u_{14} &= u_{11} = 0
\end{aligned}
$$

The system K reduces to a 5×5 matrix as shown below :

$$
\begin{array}{cc}
 & \begin{array}{ccccc} 4 & 6 & 8 & 9 & 12 \end{array} \\
\mathbf{K}_r = & \left[\begin{array}{ccccc}
\begin{array}{c}32+32+\\177+177\end{array} & & & & \\
6450 & \begin{array}{c}1720000+\\680000\end{array} & & & \\
177-177 & -2550 & \begin{array}{c}12.75+177+\\177+12.75\end{array} & & \\
0 & 340000 & -2550+2550 & \begin{array}{c}680000+\\680000\end{array} & \\
6450 & 0 & 2550 & 340000 & \begin{array}{c}1720000+\\680000\end{array}
\end{array}\right]\begin{array}{c} 4 \\ 6 \\ 8 \\ 9 \\ 12 \end{array}
\end{array}
$$

or
$$
\mathbf{K}_r = \left[\begin{array}{ccccc}
418 & & \$ & & \\
6450 & 240,0000 & & & \\
0 & -2550 & 380 & & \\
0 & 340000 & 0 & 1360000 & \\
6450 & 0 & 2550 & 340000 & 240,0000
\end{array}\right]
$$

Load vector

The load vector in global coordinates for each member can be written as follows :

Member 2

$$P_5 = +40\text{ kN},\quad P_6 = +\frac{wL^2}{12} = +\frac{20\times 400^2}{12} = +26,6667\text{ kNcm}$$

$$P_8 = +40\text{ kN}\quad P_9 = -26,6667\text{ kN cm}$$

Member 3

$$P_8 = +40\text{ kN}, P_9 = +26,6667\text{ kNcm},\ P_{11} = +40\text{ kN},\ P_{12} = -26,6667\text{ kNcm}$$

$$\mathbf{P} = \mathbf{P}_0 - \mathbf{P}_e$$

The reduced system load vector is given by

$$
\mathbf{P}_r = \begin{Bmatrix} 15 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 26,6667 \\ 40+40 \\ 0 \\ -26,6667 \end{Bmatrix} = \begin{Bmatrix} 15 \\ -26,6667 \\ -80 \\ 0 \\ 26,6667 \end{Bmatrix}\begin{array}{c} 4 \\ 6 \\ 8 \\ 9 \\ 12 \end{array}
$$

Solution vector

The force-deformation relation can be written as : $\mathbf{P}_r = \mathbf{K}_r\,\Delta_r$

or, $\quad \Delta^T_r = \{0.0394 \quad -0.113 \quad -1.726 \quad 0.000057 \quad 0.113\}$

Member end forces

Typical member force vectors are as follows :

$$
\mathbf{P}_2 = \begin{Bmatrix} 743 \\ -226 \\ 1.94\times 10^5 \\ -743 \\ 306 \\ -3\times 10^5 \end{Bmatrix}\quad
\mathbf{P}_5 = \begin{Bmatrix} 298.14 \\ 298.14 \\ -298.14 \\ -298.1\ 4 \end{Bmatrix}\quad
\mathbf{P}_6 = \begin{Bmatrix} 312 \\ -312 \\ -312 \\ 312 \end{Bmatrix}
$$

Example 12.15

A truss is supported on a roller support inclined at 60° with the horizontal as shown in Fig. 12.38a. Analyze the truss and determine the movement of the roller support. Take $A = 10\text{ cm}^2$, $E = 15000\text{ kN/cm}^2$.

Solution

The roller support may be replaced by a very stiff spring which will not permit any movement perpendicular to the inclined surface. However, node 3 will be free to move parallel to the inclined surface. The d.o.f. are shown in Fig. 12.38b.

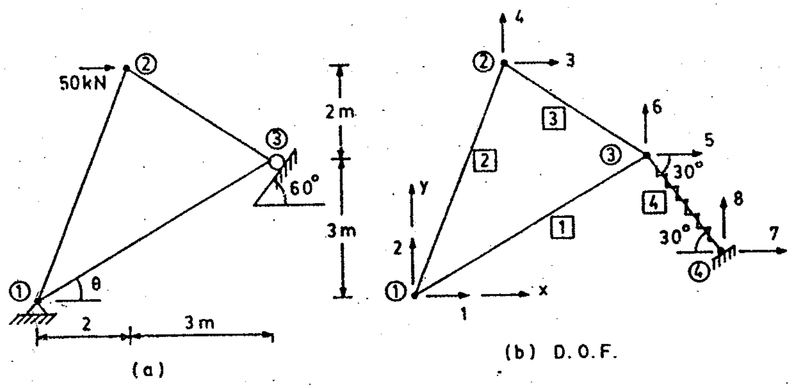


Fig. 12.38

Nodal data

Node	x (cm)	y (cm)
1	0	0
2	200	500
3	500	300
4	1019.60	0

Element data

Member	Connectivity		Geometric properties		Material properties	Direction cosine	
	I	J	L cm	A cm^2	E kN/cm^2	C	S
1	1	3	583.1	10	15000	0.857	0.515
2	1	2	538.5	10	15000	0.371	0.928
3	2	3	360.5	10	15000	0.833	−0.556
4	3	4	600	10^{10}	15000	0.866	−0.50

$$\mathbf{K}_1 = \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} & \\ \begin{bmatrix} 188.93 & & & \$ \\ 113.54 & 68.23 & & \\ -188.93 & -113.54 & 188.93 & \\ -113.54 & -68.23 & 113.54 & 68.23 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array} \end{array}$$

$$\mathbf{K}_2 = \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 38.34 & & & \$ \\ 95.90 & 239.88 & & \\ -38.34 & -95.90 & 38.34 & \\ -95.90 & -239.88 & 95.90 & 239.88 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

$$\mathbf{K}_3 = \begin{array}{cc} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} 288.72 & & & \$ \\ -192.7 & 128.63 & & \\ -288.72 & 192.7 & 288.72 & \\ 192.7 & -128.63 & -192.7 & 128.63 \end{bmatrix} & \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

$$\mathbf{K}_4 = 10^{10} \times \begin{array}{cc} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} & \\ \begin{bmatrix} 1.87 & & & \$ \\ -1.082 & 0.62 & & \\ -1.87 & 1.08 & 1.87 & \\ 1.082 & -0.62 & -1.082 & 0.62 \end{bmatrix} & \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array} \end{array}$$

Boundary conditions

The inclined roller at support 3 was replaced by a infinitely stiff axial spring. The boundary conditions are as follows :

$$u_1 = 0 = u_2 \quad \text{and} \quad u_7 = 0 = u_8$$

There will be no axial deformation in the spring, hence displacement normal to the inclined surface will automatically be zero. The reduced system stiffness matrix can be written as follows :

$$\mathbf{K}_r = \begin{array}{cc} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} (288.72+38.34 = 327.1) & & & \$ \\ (-192.7+95.9 = -96.8) & (128.63+239.88 = 368.5) & & \\ -288.72 & 192.7 & (288.72+188.93+1.87\times10^{10}) & \\ 192.7 & -128.63 & (-192.7+113.54-1.082\times10^{10}) & (128.63+68.23+0.62\times10^{10}) \end{bmatrix} & \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Load vector

The reduced system load vector can be written as : $\mathbf{P'}^T = \{50 \quad 0 \quad 0 \quad 0\}$

Solution vector

The solution vector Δ_r is given by $\mathbf{P}_r = \mathbf{K}_r\, \Delta_r$

Since, there are few very large numbers in [K], a suitable scaling scheme should be used to obtain the correct solution of the four linear simultaneous equations.

or, $\Delta^T = \{0.1658 \quad 0.0435 \quad -0.0149 \quad -0.026\}$ cm

Member end forces

The member end forces can be determined using the member force-deformation relations as follows :

$$\mathbf{P}_1 = \begin{Bmatrix} 5.78 \\ 3.47 \\ -5.78 \\ -3.47 \end{Bmatrix}, \quad \mathbf{P}_2 = \begin{Bmatrix} -10.53 \\ -26.33 \\ 10.53 \\ 26.33 \end{Bmatrix}, \quad \mathbf{P}_3 = \begin{Bmatrix} +38.8 \\ -25.9 \\ -38.8 \\ +25.9 \end{Bmatrix}$$

The net forces in the three truss members are – 6.74 kN, 28.35 kN, and – 46.65 kN, respectively. The axial force in the spring will be zero. The movement of the roller support is 0.03 cm down the slope.

12.12 COMPARISON OF FLEXIBILITY AND STIFFNESS METHODS

The various flexibility methods have been discussed in Chapters 3, 4, 5, 6 and 7, while the stiffness methods have been discussed in Chapters 10, 11 and 12. Let us first summarize the main steps of these two methods :

Step	Flexibility Methods	Stiffness Method
1.	Determine the degree of static indeterminacy, identify redundants and obtain released structures (statically determinate).	Determine the degree of kinematic indeterminacy, identify the degrees of freedom and obtain a restrained structure (kinematically determinate).
2.	Compute displacements at the points and directions of the releases due to the applied loads.	Compute fixed end forces at the ends of each member along the selected degrees of freedom due to the applied loads.
3.	Compute displacements at the points and directions of the releases due to the unit redundants. In other words, develop the flexibility matrix.	Compute forces at the ends of each member due to unit displacement along the selected degrees of freedom. In other words, develop the stiffness matrix.
4.	Impose compatibility conditions and obtain the redundants.	Impose equilibrium conditions and obtain the displacements.
5.	Compute member forces by using the equations of equilibrium.	Compute member forces by using the force-deformation equations.

The choice of a method for a given problem depends upon the following factors :

1. Ease in computations 2. Accuracy

Based on the experience gained while working through the various methods in each category, it can be concluded, in general, that :

(1) The choice of redundants is very crucial in any flexibility method because the computational efforts depend upon it. On the other hand, in any stiffness method, there is only one restrained structure.

(2) The computational effort required to compute deflections is much more than that required to compute forces at the ends of members.

(3) The chances of making mistakes are much more in computing deflections in, say, the unit load method or strain energy method. The chances of making mistakes, say, in the slope-deflection method or moment distribution method are relatively less.

(4) Most elements of the influence coefficient or flexibility matrix are non-zero, while, stiffness matrix is usually banded. Thus, the number of arithmatic operations required for the solution of linear simultaneous equations is far less in the stiffness method than that in the flexibility method.

(5) Both the flexibility and stiffness matrices are symmetric about the diagonal.

(6) In the flexibility method, once the member forces are known, additional computations become necessary to determine the slopes and deflections. On the contrary, in the stiffness method, deformations precede the computations of members forces , with the exception of moment distribution method which is iterative.

(7) Physical concepts become quite clear while working with any of the flexibility methods. On the contrary, stiffness methods are mechanical in nature.

(8) It is difficult to write a general purpose computer program for flexibility methods, whereas, stiffness methods are very computer friendly. The direct stiffness matrix method based computer programs are now used extensively even in small design offices.

PROBLEMS

In the following problems take $E = 200$ GPa and $I = 150 \times 10^{-6}$ m^4 unless specified otherwise. The stiffness matrices and load vectors may be generated using LOTUS 1-2-3 spread sheet. Alternatively, the solution of linear simultaneous equations may be obtained using the programs given in Appendix A. Verify the results using program STAP discussed in Chapter 14 and available on a floppy.

12.1 For the beams shown in Figs P 3.3, P3.4 and P3.6

(a) assemble the stiffness matrix, with and without axial deformations and mark the various degrees of freedom in the structure,
(b) assemble the load vector,
(c) evaluate reduced structural stiffness matrix and load vector after introducing the boundary conditions,
(d) solve for structural displacements and determine member forces.

12.2 Determine the support reactions and member forces due to a vertical settlement of 7.5 mm at support B of the beam in Fig. P3.4.

12.3 Analyze the frame shown in Fig. P3.8 c and determine the support reactions. Draw shear force and bending moment diagrams.

12.4 For the trusses shown in Fig. P5.7. and P5.8

(a) number the nodes so as to get the minimum band width,
(b) simulate the supports using boundary elements,
(c) assemble the global stiffness matrix,
(d) assemble the load vector,
(e) evaluate reduced structural stiffness matrix and load vector, and
(f) solve for structural displacements and determine member forces.

12.5 Analyze the closed frames shown in Fig. P5.11 and P 5.12 making use of symmetry and taking appropriate boundary conditions.

12.6 Analyze the circular arch shown in Fig. P7.3 by dividing the arch axis in ten beam elements. Taken $E = 26000$ MPa, $A = 0.20$ m^2, and $I = 0.67 \times 10^{-3}$ m^4.

12.7 For the frames shown in Fig. P10.1 and P10.2

(a) Mark the various degrees of freedom on the structure ignoring the axial deformations and assuming support D is hinged,
(b) introduce boundary elements,
(c) assemble the global stiffness matrix,
(d) assemble the load vector, and
(e) solve for structural displacements and evaluate support reactions.

CHAPTER
thirteen

DIRECT STIFFNESS METHOD- 3D ELEMENTS

13.1 STIFFNESS MATRIX- TRUSS ELEMENT

A truss element is assumed to resist only axial force. The deformation at the ends are translations in the x, y and z directions. With three possible degrees of freedom at each end, Fig. 13.1 the member stiffness matrix is of the order 6×6. It may be obtained exactly in the same manner as in the case of a plane truss element discussed in section 12.1, and looks as follows :

$$K_{6x6} = \left[\begin{array}{ccc|ccc} \frac{AE}{L} & & & & \text{Symmetric} & \\ 0 & 0 & & & & \\ 0 & 0 & 0 & & & \\ \hline -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

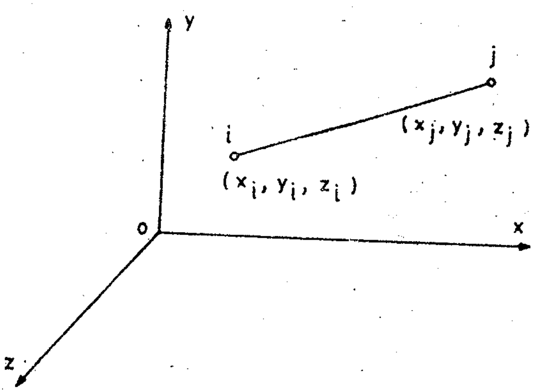


Fig. 13.1 3-D truss element

13.2 STIFFNESS MATRIX- BEAM ELEMENT

A 3-D beam element has six degrees of freedom per joint as shown in Fig. 13.2. There are total twelve degrees of freedom per element. In order to derive its stiffness matrix, the local axes are chosen to coincide with the principal axes of the cross section. Hence bending and shear in the x-y plane are independent of those in the x-z plane. The centroidal axis of the beam coincides with the o-x axis of the coordinate system. For an arbitrary choice of bending planes, the above simplification is invalid. Now, the forces can be arranged in six groups which can be considered independent of each other as follows :

(i)	axial forces	P_1 and P_7
(ii)	shear forces	P_2 and P_8
(iii)	shear forces	P_3 and P_9
(iv)	bending moments	P_5 and P_{11}
(v)	bending moments	P_6 and P_{12}
(vi)	bending moments	P_4 and P_{10}

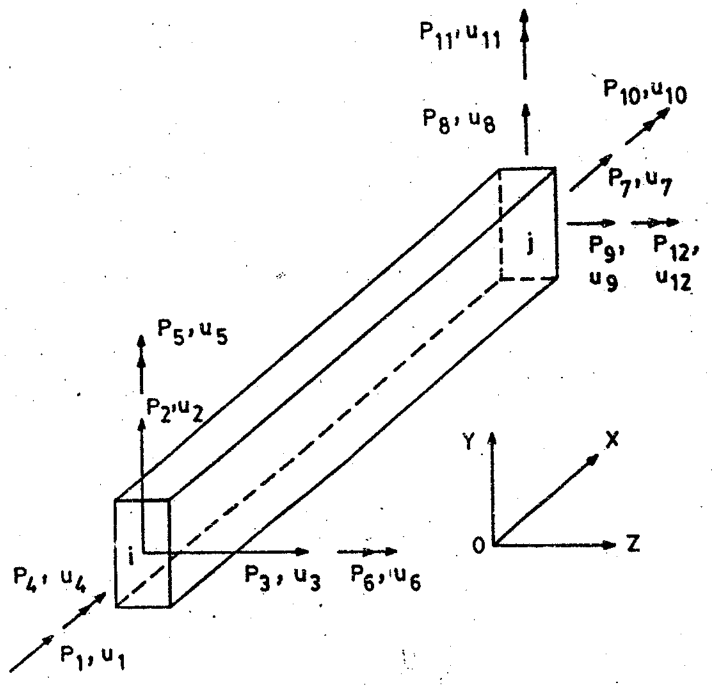


Fig. 13.2 3-D beam element

The corresponding displacements are u_1 to u_{12}. Each force can be imposed on the element and the corresponding reactions at the other end and displacements can be determined in the same manner as discussed in Chapter 12. Thus, a 12 × 12 stiffness matrix of the beam element can be generated. Such a formulation does not account for shear deformations which may be important for deep flexural members.

Przemieniecki [1968] used the differential equations relating the force and the corresponding deformations in each of the above six sets and derived a general 12 × 12 stiffness matrix for a beam element. This accounts for the shear deformations about the two axes of bending. The derivation of the stiffness matrix is not presented here but only the final expression is given in Eq. 13.2 a to d. The Structural Analysis Program STAP-3D presented in the next chapter makes use of the same stiffness matrix.

$$\mathbf{K}_{12\times12} = \begin{bmatrix} \mathbf{K}_{11} & : & \mathbf{K}_{12} \\ \ldots\ldots & : & \ldots\ldots \\ \mathbf{K}_{21} & : & \mathbf{K}_{22} \end{bmatrix}_{12\times12} \qquad (13.2a)$$

Each sub-matrix is a 6 × 6 matrix and is written as follows :

$$\mathbf{K}_{11} = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} \frac{EA}{L} & & & & \text{SYMMETRIC} & \\ 0 & \frac{12EI_z}{L^3(1+\alpha_y)} & & & & \\ 0 & 0 & \frac{12EI_y}{L^3(1+\alpha_y)} & & & \\ 0 & 0 & 0 & \frac{GJ}{L} & & \\ 0 & 0 & -\frac{6EI_y}{L^2(1+\alpha_z)} & 0 & \frac{(4+\alpha_z)EI_y}{(1+\alpha_z)L} & \\ 0 & \frac{6EI_z}{L^2(1+\alpha_y)} & 0 & 0 & 0 & \frac{(4+\alpha_y)EI_z}{(1+\alpha_y)L} \end{bmatrix} \end{array} \qquad (13.2b)$$

$$K_{21} = \begin{array}{c|c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} & \begin{bmatrix} \frac{-EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-12EI_z}{L^3(1+\alpha_y)} & 0 & 0 & 0 & \frac{-6EI_z}{L^2(1+\alpha_y)} \\ 0 & 0 & \frac{-12EI_y}{L^3(1+\alpha_z)} & 0 & \frac{6EI_y}{L^2(1+\alpha_z)} & 0 \\ 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\ 0 & 0 & \frac{-6EI_y}{L^2(1+\alpha_z)} & 0 & \frac{(2-\alpha_z)EI_y}{(1+\alpha_z)L} & 0 \\ 0 & \frac{6EI_z}{L^2(1+\alpha_y)} & 0 & 0 & 0 & \frac{(2-\alpha_y)EI_z}{(1+\alpha_y)L} \end{bmatrix} \end{array} = K_{12} \qquad (13.2d)$$

$$K_{22} = \begin{array}{c|c} & \begin{array}{cccccc} 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{array}{c} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} & \begin{bmatrix} \frac{EA}{L} & & & & & \\ 0 & \frac{12EI_z}{L^3(1+\alpha_y)} & & \text{SYMMETRIC} & & \\ 0 & 0 & \frac{12EI_y}{L^3(1+\alpha_z)} & & & \\ 0 & 0 & 0 & \frac{GJ}{L} & & \\ 0 & 0 & \frac{6EI_y}{L^2(1+\alpha_z)} & 0 & \frac{(4+\alpha_z)EI_y}{(1+\alpha_z)L} & \\ 0 & \frac{-6EI_z}{L^2(1+\alpha_y)} & 0 & 0 & 0 & \frac{(4+\alpha_y)EI_z}{(1+\alpha_y)L} \end{bmatrix} \end{array} \qquad (13.2d)$$

In these equations, α_y and α_z are shape factors along the y and z axes and are given as follows :

$$\alpha_y = \frac{12EI_z}{GA_{sy}L^2} = 24(1+\nu)\frac{A}{A_{sy}}\left(\frac{r_z}{L}\right)^2 \qquad (13.2e)$$

and

$$\alpha_z = \frac{12EI_y}{GA_{sz}L^2} = 24(1+\nu)\frac{A}{A_{sz}}\left(\frac{r_y}{L}\right)^2 \qquad (13.2f)$$

where, $\quad r_y, r_z$ = radii of gyration

If $r_y / L \ll 1$ and $r_z / L \ll 1$ than α_y and α_z may be taken as zero, that is, shear deformations can be neglected as is the case with slender beams.

13.3 STIFFNESS MATRIX- GRID ELEMENT

A structural grid is defined as a frame structure with rigid joints whose members and joints all lie in a common plane, say X-Y, with all applied loads being out of the plane and normal to the plane of the structure, that is, along z-axis. The moment vectors are, therefore, in the plane of the structure. In the case of a 2-D beam element and a 3-D beam element, the member end deformations were designated in the order of translation in the x, y and z directions, and then in the order of rotations about the x, y and z axes at each end. However, in the case of a grid member, a slightly different scheme of designations is more desirable as shown in Fig. 13.3. u_1 and u_2 represent rotation about x and y axes, and u_3 represent translation along the z-axis at end i. Similarly, u_4 and u_5 represent rotation about x and y axes and u_6 represent translation along the z axis. The stiffness matrix in local coordinate system can be generated using the basic principles enumerated earlier. The stiffness matrix of a grid element is given by Eq. 13.3

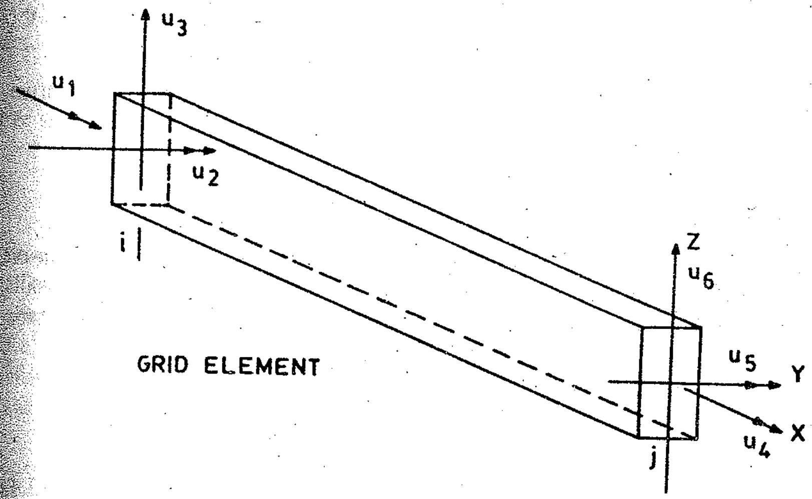


Fig. 13.3 Grid element

$$\mathbf{K} = \begin{bmatrix} \frac{GI_x}{L} & & & & & \\ 0 & \frac{4EI_y}{L} & & \$ & & \\ 0 & \frac{-6EI_y}{L^2} & \frac{12EI_y}{L^3} & & & \\ \frac{-GI_x}{L} & 0 & 0 & \frac{GI_x}{L} & & \\ 0 & \frac{2EI_y}{L} & \frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & \\ 0 & \frac{6EI_y}{L^2} & \frac{-12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} \end{bmatrix} \qquad (13.3)$$

The change in the sequence of numbering of the degrees of freedom has a disti advantage. The grid member stiffness is identical to that of a plane beam element terms of form and arrangement of stiffness elements. Therefore, the transformati matrix follows the same pattern for both elements as both require a rotation about the axis. Thus the transformation matrix for a grid element is given by Eqs. 12.13.

It may be noted that the stiffness matrix for 3-D beam element derived earlier equally applicable to a grid element or even a 2-D beam element. Care needs to taken to restrain *undesirable* degrees of freedom.

13.4 STIFFNESS MATRIX- SHEAR WALL ELEMENT

A shear wall may be treated as a vertical deep wall transmitting vertical and in-pl lateral loads to the foundations. The effect of shear deformations in these wall enormous as compared to that in conventional beams where the span-to-depth ratio much larger. Consider a shear wall between two adjacent floors shown in Fig. 13 which may be treated as a cantilever beam shown in Fig. 13.4b. The flexibility matrix a cantilever corresponding to the two coordinates indicated at end A can be written as

$$\mathbf{F} = \frac{h}{6EI} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 2h^2 & 3h \\ 3h & 6 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \qquad (13.4$$

If shear deformations are included,

$$\mathbf{F} = \begin{bmatrix} \left(\frac{h^3}{3EI} + \frac{h}{GA_e}\right) & \frac{h^2}{2EI} \\ \frac{h^2}{2EI} & \frac{h}{EI} \end{bmatrix} \qquad (13.4$$

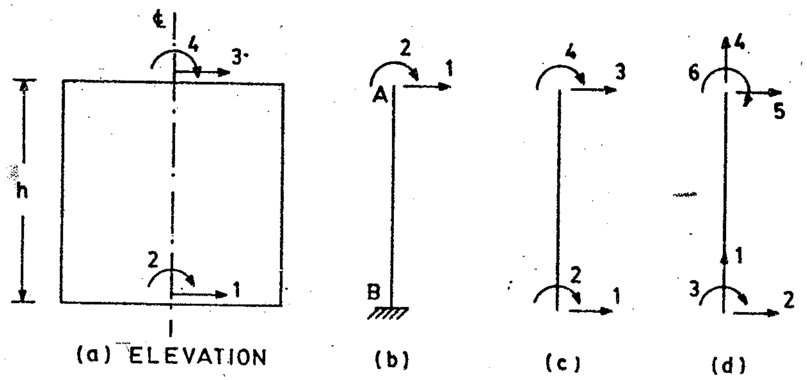


Fig. 13.4 Shear wall element

The stiffness matrix corresponding to the coordinates in Fig. 13.4b is obtained by inverting the flexibility matrix, that is,

$$\mathbf{K} = \frac{1}{(1+\alpha)} \begin{bmatrix} \frac{12EI}{h^3} & \frac{-6EI}{h^2} \\ \frac{-6EI}{h^2} & \frac{(4+\alpha)EI}{h} \end{bmatrix} \qquad (13.5a)$$

where $$\alpha = \frac{12EI}{h^2 GA_e} \qquad (13.5b)$$

From the elements of the stiffness matrix of Eq. 13.5 and by considering equilibrium, the stiffness matrix corresponding to the coordinates in Fig. 13.4c can be developed. Thus, the stiffness matrix of a shear wall element as shown in Fig. 13.4d can be written as :

$$K = \frac{1}{(1+\alpha)} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{12EI}{h^3} & & & \$ \\ \frac{6EI}{h^2} & (4+\alpha)\frac{EI}{h} & & \\ \frac{-12EI}{h^3} & \frac{-6EI}{h^2} & \frac{12EI}{h^3} & \\ \frac{6EI}{h^2} & (2-\alpha)\frac{EI}{h} & \frac{-6EI}{h^2} & (4+\alpha)\frac{EI}{h} \end{bmatrix}_{4\times 4} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad (13.6)$$

In case, the axial deformations are included :

$$
K_{6\times6}=\left[\begin{array}{ccc|ccc}
\frac{AE}{h} & & & & \$ & \\
0 & \frac{12EI}{(1+\alpha)h^3} & & & & \\
0 & \frac{6EI}{(1+\alpha)h^2} & \frac{(4+\alpha)}{(1+\alpha)}\frac{EI}{h} & & & \\
\hline
\frac{-AE}{h} & 0 & 0 & \frac{AE}{h} & & \\
0 & \frac{-12EI}{(1+\alpha)h^3} & \frac{-6EI}{(1+\alpha)h^2} & 0 & \frac{12EI}{(1+\alpha)h^3} & \\
0 & \frac{6EI}{(1+\alpha)h^2} & \frac{(2-\alpha)}{(1+\alpha)}\frac{EI}{h} & 0 & \frac{-6EI}{(1+\alpha)h^2} & \frac{(4+\alpha)}{(1+\alpha)}\frac{EI}{h}
\end{array}\right] \tag{13.7}
$$

In case the shear deformations are ignored, that is $\alpha = 0$ the stiffness matrix becomes identical with the matrix in Eq. 12.4 for a beam element.

13.5 STIFFNESS MATRIX- BEAM WITH RIGID ENDS

Shear walls are usually connected by beams, and the stiffness of such elements is required for the analysis of the structure. The stiffness matrix of an element is written based on its center line dimensions. In the case of a beam-column joint, it is usual to ignore the width of the joint. However, the width of a shear wall is usually quite large, and it cannot be ignored. Thus, there is a need to modify the stiffness of a beam connected to a shear wall. Consider the beam AB connected to the shear wall as shown in Fig. 13.5a. The beam has two rigid ends AA' and BB' and the corresponding degrees of freedom are shown in Fig. 13.5c. The displacements Δ^* at A and B are related to the displacements Δ at A' and B' by geometry as follows :

$$\{\Delta\} = [R]\ \{\Delta^*\} \quad \text{or,} \quad \Delta = \mathbf{R}\ \Delta^* \tag{13.8a}$$

where,

$$
\mathbf{R} = \begin{bmatrix} 1 & t_a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -t_b \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13.8b}
$$

Stiffness matrix of a prismatic beam corresponding to the A'B' coordinates is the same as given by Eq. 13.6 or 13.7. The stiffness matrix of a beam with rigid ends corresponding to the AB coordinates is given by Eq. 13.9 [Macleod 1967]. It is obtained by including the effect of finite rigid joints through Eq. 13.8 in the stiffness matrix of a 2-D beam element given by Eq. 12.4. Shear walls in a frame can also be idealized as wide columns through their center lines. Shear deformations must be included. Such a modelling gives reasonably accurate results.

$$
K=\begin{bmatrix}
\frac{EA}{L} & & & & \$ & \\
0 & \frac{2EI}{SL^3} & & & & \\
0 & \left(\frac{2EIt_a}{SL^3}+\frac{EI}{SL^2}\right) & \left(\frac{2EIt_a^2}{SL^3}+\frac{2EIt_a}{SL^2}+k_1\right) & & & \\
\frac{-EA}{L} & 0 & 0 & \frac{EA}{L} & & \\
0 & \frac{-2EI}{SL^3} & -\left(\frac{2EIt_a}{SL^3}+\frac{EI}{SL^2}\right) & 0 & \frac{2EI}{SL^3} & \\
0 & \left(\frac{2EIt_b}{SL^3}+\frac{EI}{SL^2}\right) & \left(\frac{2EIt_at_b}{SL^3}+\frac{EI}{SL^2}(t_a+t_b)+k_2\right) & 0 & \left(\frac{-2EIt_b}{SL^3}+\frac{EI}{SL^2}\right) & \left(\frac{2EIt_b^2}{SL^3}+\frac{2EI}{SL^2}t_b+k_1\right)
\end{bmatrix} \tag{139}
$$

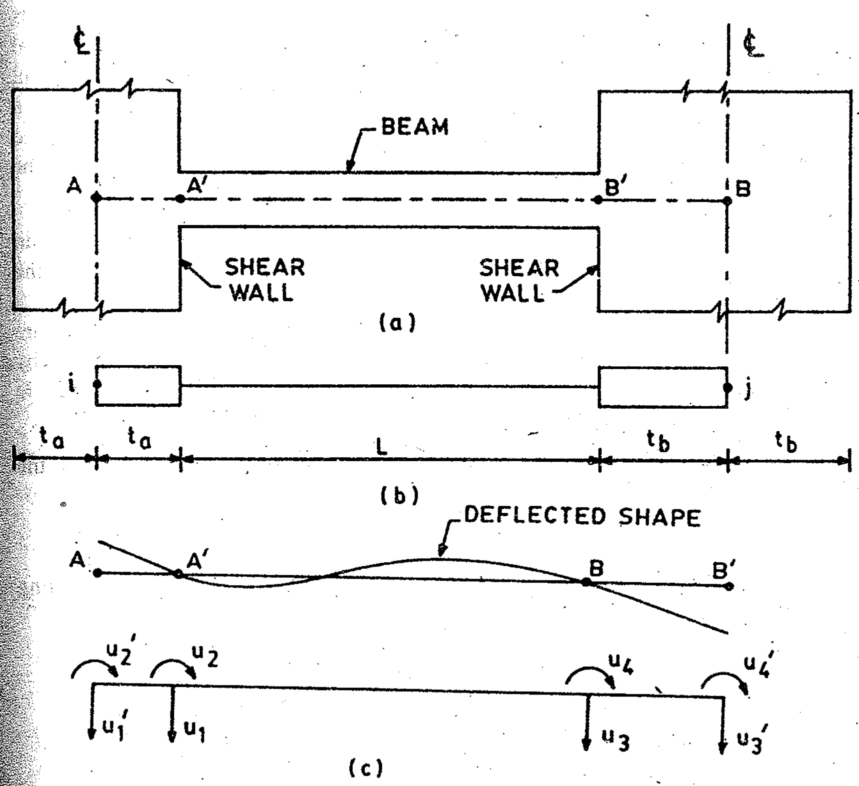


Fig. 13.5 Beam with rigid ends

where, $k_1 = \dfrac{2EI\left(\frac{1}{3}+b\right)}{SL}$ $\quad t_a$ = half width of wall/column at one end

$k_2 = \dfrac{EI\left(\frac{1}{3}-2b\right)}{SL}$ $\quad t_b$ = half width of wall/column at other end

$S = \dfrac{1}{6}+2b$ $\quad$ L = clear span of beam

$b = \dfrac{2I(1+\nu)}{A_e L^2}$

A_e = equivalent shear area, ν = poisson's ratio

13.6 STEPPED MEMBERS

Quite often the required strength of a member is not the same from one end to its other end. At certain sections, the member is expected to be stronger since the internal forces developed in the member vary from one end to the other. Provision of a constant cross-section would mean a waste of material. Therefore, either a tapered section or a stepped section is more desirable. The former is costly to build, therefore, a stepped section is more commonly used. The stiffness matrix of a stepped member can be obtained in two manners :

1. The conjugate beam method or the column analogy method is used to obtain the flexibility matrix. The stiffness matrix is then obtained by inversion of the flexibility matrix.
2. The second method treats the stepped member as a one-dimensional structure which consists of prismatic members connected in series and analyzes it by using the stiffness of each segment. Thus, the knowledge already acquired in the previous chapter is sufficient.

Let us derive the stiffness matrix of a stepped beam shown in Fig. 13.6 using the second approach. There are a few simplifications :

1. The entire structure lies on a straight line.
2. The global coordinate axes coincides with the local axes at end i, eliminating the coordinate transformations.

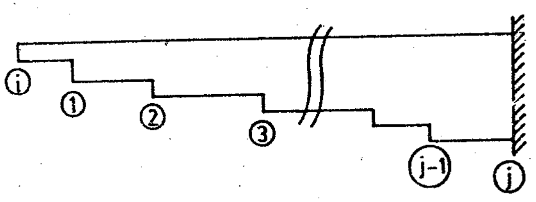


Fig. 13.6 Stepped beam

3. The analysis of the system due to axial displacements at the ends can be separated from the rest of the analysis because forces developed at the ends due to axial deformations are :

$$u = P\sum_{i=1}^{n}\left(\frac{L}{AE}\right)_i \qquad (13.10a)$$

or,

$$P = \frac{u}{\sum_{i=1}^{n}\left(\frac{L}{AE}\right)_i} \qquad (13.10b)$$

or,

$$k_{11} = \frac{1}{\sum_{i=1}^{n}\left(\frac{L}{AE}\right)_i} \qquad (13.11a)$$

$$k_{12} = -k_{11} = k_{21} \qquad (13.11b)$$

where, $\left(\dfrac{AE}{L}\right)_i$ = stiffness of i th segment

n = number of total segments

This simplifies the analysis due to reduction in size of the stiffness matrix. The stiffness matrix of any segment r between joints l and $l+1$ is

$$k^r = \left[\begin{array}{cc:cc} \frac{12EI}{L^3} & & & \\ \frac{6EI}{L^2} & \frac{4EI}{L} & & \\ \hdashline -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array}\right] \qquad (13.12a)$$

$$k^r = \begin{bmatrix} K_{l,l} & : & K_{l,l+1} \\ \ldots & : & \ldots \\ K_{l+1,l} & : & K_{l+1,l+1} \end{bmatrix} \qquad (13.12b)$$

where, $[K_{l,\,l+1}] = [K_{l+1,\,l}]^T$

We can now assemble the stiffness matrix of, say, first two segments :

$$K = \left[\begin{array}{c:c:c} K_{11} & K_{12} & 0 \\ \hdashline K_{21} & K_{22}+K_{22} & K_{23} \\ \hdashline 0 & K_{32} & K_{33} \end{array}\right]_{6\times 6}$$

Each sub matrix K_{ij} is of 2×2 size.

In this manner the stiffness matrix of the entire structure can be assembled. The segment numbers are shown in Fig. 13.6. It is a tri-diagonal matrix as shown below :

$$\begin{Bmatrix} P_{ij} \\ 0 \\ 0 \\ 0 \\ \\ 0 \\ P_{ji} \end{Bmatrix} = \begin{bmatrix} K_{ii} & K_{i1} & & & & 0 & \\ K_{1i} & K_{11} & K_{12} & & & & \\ & K_{21} & K_{22} & K_{23} & & & \\ & & & & & & \\ & 0 & \cdot & & K_{j-1,j-2} & K_{j-1,j-1} & K_{j-1,j} \\ & & & & & K_{j-1,j} & K_{j,j} \end{bmatrix} \times \begin{Bmatrix} u_{ij} \\ u_1 \\ u_2 \\ \\ \\ u_{j-1} \\ u_{ji} \end{Bmatrix} \qquad (13.13)$$

If the element is fixed at j, the boundary conditions are :

$$u_{ji} = 0$$

Thus, eliminating the last row and last column, the final matrix becomes ,

$$\begin{Bmatrix} P_{ij} \\ \cdots \\ 0 \\ 0 \\ : \\ : \\ 0 \end{Bmatrix} = \begin{bmatrix} K_{ii} & : & K_{i1} & & & & \\ \cdots & : & \cdots & \cdots & \cdots & \cdots\cdots & \cdots\cdots \\ K_{1i} & : & K_{11} & & & 0 & \\ & : & K_{21} & K_{22} & K_{23} & & \\ & : & & & & & \\ & : & & & & & \\ & : & & 0 & & K_{j-1,j-2} & K_{j-1,j-1} \end{bmatrix} \times \begin{Bmatrix} u_{ij} \\ \cdots \\ u_1 \\ u_2 \\ : \\ : \\ u_{j-1} \end{Bmatrix} \qquad (13.14a)$$

It can be rewritten in the partitioned form as :

$$\begin{Bmatrix} P_I \\ \hdashline 0 \end{Bmatrix} = \left[\begin{array}{c:c} K_{I,I} & K_{I,II} \\ \hdashline K_{II,I} & K_{II,II} \end{array}\right] \begin{Bmatrix} \Delta_I \\ \hdashline \Delta_{II} \end{Bmatrix}$$

where, $K_{I,I}$ is a 2×2 sub-matrix
$K_{II,II}$ is a $(2j-4) \times (2j-4)$ sub-matrix

The solution of this equation is

$$P_I = [K_{I,I} - K_{I,II} K^{-1}_{II,II} K_{II,I}] \Delta_I \qquad (13.15)$$

Thus $\quad k_{ii} = [K_{I,I} - K_{I,II} K^{-1}_{II,II} K_{II,I}]_{2 \times 2} \qquad (13.16)$

The cross-stiffness matrix k_{ij} and the direct stiffness matrix k_{jj} at the other end can be obtained by the equilibrium of the member, that is,

$$\begin{Bmatrix} P_2 \\ P_3 \end{Bmatrix}_j = -\begin{bmatrix} +1 & 0 \\ -L & 1 \end{bmatrix} \begin{Bmatrix} P_2 \\ P_3 \end{Bmatrix}_i$$

or, $\quad p_{ji} = -h_{ij} \, p_{ij} \qquad (13.17)$

where, h_{ij} = stress matrix relating the force vector from end i to end j in local coordinate axes at j.

The equilibrium equation of the member can be written as :

$$p_{ji} = -h_{ij} k_{ii} u_{ij} - h_{ij} k_{ij} u_{ji}$$

It is similar to, (for any member)

$$p_{ji} = -k_{ji} u_i + k_{jj} u_j$$

therefore, $\quad k_{ji} = -h_{ij} k_{ii} \qquad (13.18a)$

and $\quad k_{jj} = -h_{ij} k_{ij} \qquad (13.18b)$

Thus, Eqs. 13.16, 13.17 and 13.18 completely define the stiffness matrix of a stepped member in local coordinates. The calculation of fixed end forces follows the same procedure.

Example 13.1

Determine the stiffness matrix of the stepped beam shown in Fig. 13.7a. If A = 900 cm², and I = 64000 cm⁴.

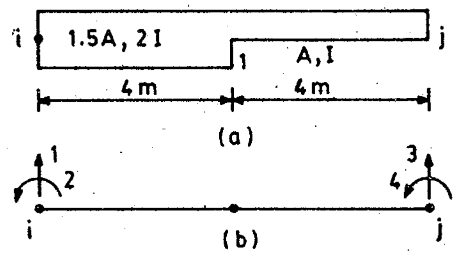


Fig. 13.7

Solution

The force-deformation relation is given by Eq.13.13,

$$\begin{Bmatrix} P_i \\ \hdashline P_1 \\ P_j \end{Bmatrix} = \left[\begin{array}{c:cc} K_{ii} & K_{i1} & 0 \\ \hdashline K_{1i} & K_{11} & K_{1j} \\ 0 & K_{j1} & K_{jj} \end{array}\right] \begin{Bmatrix} u_{ij} \\ \hdashline u_1 \\ u_{ji} \end{Bmatrix}$$

$$K_{ii} = \begin{bmatrix} \frac{12EI_1}{L^3_1} & \frac{6EI_1}{L^2_1} \\ \frac{6EI_1}{L^2_1} & \frac{4EI_1}{L_1} \end{bmatrix}, \qquad K_{i1} = \begin{bmatrix} \frac{-12EI_1}{L^3_1} & \frac{6EI_1}{L^2_1} \\ \frac{-6EI_1}{L^2_1} & \frac{2EI_1}{L_1} \end{bmatrix} = K^T_{1i}$$

$$\therefore \quad K_{11} = K^{i}_{11} + K^{j}_{11} = \begin{bmatrix} \dfrac{12EI_1}{L^3_1}+\dfrac{12EI_2}{L^3_2} & \dfrac{-6EI_1}{L^2_1}+\dfrac{6EI_2}{L^2_2} \\ \dfrac{-6EI_1}{L^2_1}+\dfrac{6EI_2}{L^2_2} & \dfrac{4EI_1}{L_1}+\dfrac{4EI_2}{L_2} \end{bmatrix}$$

$$\frac{EI_1}{L^3_1} = \frac{2\times 64000}{400^3} = 0.002, \quad \frac{EI_1}{L^2_1} = \frac{2\times 64000}{400^2} = 0.8; \quad \frac{EI_1}{L_1} = \frac{2\times 64000}{400} = 320$$

$$\frac{EI_2}{L^3_2} = \frac{64000}{400^3} = 0.001, \quad \frac{EI_2}{L^2_2} = 0.4, \quad \frac{EI_2}{L_2} = 160$$

$$K_{ii} = \begin{bmatrix} 0.024 & 4.8 \\ 4.8 & 1280 \end{bmatrix}, \qquad K_{i1} = \begin{bmatrix} -0.024 & 4.8 \\ -4.8 & 640 \end{bmatrix}$$

$$K_{11} = \begin{bmatrix} 0.024+0.012 & -4.8+2.4 \\ -4.8+2.4 & 1280+640 \end{bmatrix} = \begin{bmatrix} 0.036 & -2.4 \\ -2.4 & 1920 \end{bmatrix}$$

$$K^{-1}_{11} = \begin{bmatrix} 30.303 & 0.03787 \\ 0.03787 & 0.000568 \end{bmatrix}$$

If end j is fully restrained, stiffness at end i is given by Eq.13.16,

$$\therefore \quad k_{ii} = K_{ii} - K_{i1}\, K_{11}^{-1}\, K_{1i}$$

where

$$K_{i1}\, K_{11}^{-1}\, K_{1i} = \begin{bmatrix} -0.024 & 4.8 \\ -4.8 & 640 \end{bmatrix} \begin{bmatrix} 30.303 & 0.03787 \\ 0.03787 & 0.000568 \end{bmatrix} \begin{bmatrix} -0.024 & -4.8 \\ 4.8 & 640 \end{bmatrix}$$

$$= \begin{bmatrix} 0.02182 & 3.78 \\ 3.78 & 698.13 \end{bmatrix}$$

Direct stiffness matrix at end i without the axial component is

$$\therefore \quad k_{ii} = \begin{bmatrix} 0.024 & 4.8 \\ 4.8 & 1280 \end{bmatrix} - \begin{bmatrix} 0.02182 & 3.78 \\ 3.78 & 698.13 \end{bmatrix} = \begin{bmatrix} 0.00218 & 1.02 \\ 1.02 & 581.87 \end{bmatrix}$$

The axial stiffness is given by Eq. 13.11,

$$k_{11} = \frac{1}{\Sigma \dfrac{L}{AE}} = \frac{1}{\dfrac{400}{1.5\times 900} + \dfrac{400}{900}} = 1.35$$

The direct stiffness matrix at end i including the axial component is

$$k_{ii} = \begin{bmatrix} 1.35 & 0 & 0 \\ 0 & 0.00218 & 1.02 \\ 0 & 1.02 & 581.87 \end{bmatrix}$$

$$k_{ji} = -\, h_{ij}\, k_{ii} = -\begin{bmatrix} +1 & 0 \\ -800 & 1 \end{bmatrix} \begin{bmatrix} 0.00218 & 1.02 \\ 1.02 & 581.87 \end{bmatrix} = \begin{bmatrix} -0.00218 & -1.02 \\ 0.726 & +232.53 \end{bmatrix}$$

and direct stiffness component at end j is given by

$$k_{jj} = -\, k_{ji}\, h_{ij}^{T} = -\begin{bmatrix} -0.00218 & -1.02 \\ 0.726 & 232.53 \end{bmatrix} \begin{bmatrix} +1 & -800 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.00218 & -0.726 \\ -0.726 & 348.27 \end{bmatrix}$$

The final stiffness matrix is given by

$$K = \begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{bmatrix}$$

Example 13.2

Determine the stiffness matrix of the stepped beam of example 13.1 using the influence coefficient method. Take $L_1 = L_2 = 400$ cm, $I_2 = 64000$ cm^4, and $I_1 = 2I_2$.

Solution

Let us first determine the flexibility matrix of the stepped beam whose inverse will give the desired stiffness matrix.

(i) Let us restrain end j of the beam, and apply a unit force at d.o.f. 1, and compute deflections at, d.o.f. 1 and 2 using the moment-area theorem, (Fig. 13.8 a,b)

$$\delta_{11} = \frac{1}{2}\times 400 \times \frac{400}{EI_1} \times \frac{2}{3}\times 400 + \frac{600}{EI_2}\times 400 \times \left[400 + \frac{400}{3}\left(\frac{400+1600}{400+800}\right)\right]$$

$$= \frac{64\times 10^6}{3\times 2\times 64\times 10^3} + \frac{96\times 10^6}{64\times 10^3}\left[1 + \frac{1}{3}\times 1.667\right]$$

$$\delta_{11} = 2500.17$$

$$\delta_{21} = \frac{1}{2}\times 400 \times \frac{400}{EI_1} + \frac{600}{EI_2}\times 400 = \frac{80000}{2\times 64\times 10^3} + \frac{24\times 10^4}{64\times 10^3}$$

$$= (-)\ 4.375$$

Now apply a unit force at d.o.f. 2 and compute deflection at d.o.f. 1 and 2,

$$\delta_{12} = 200\times\frac{400}{EI_1} + 600\times\frac{400}{EI_2} = \frac{8\times 10^4}{2\times 64\times 10^3} + \frac{24\times 10^4}{64\times 10^3} = (-)\ 4.375$$

$$\delta_{22} = \frac{400}{EI_1} + \frac{400}{EI_2} = \frac{400}{2\times 64\times 10^3} + \frac{400}{64\times 10^3} = 0.00937$$

Thus,

$$F = \begin{bmatrix} 2500.17 & -4.375 \\ -4.375 & -0.00937 \end{bmatrix}$$

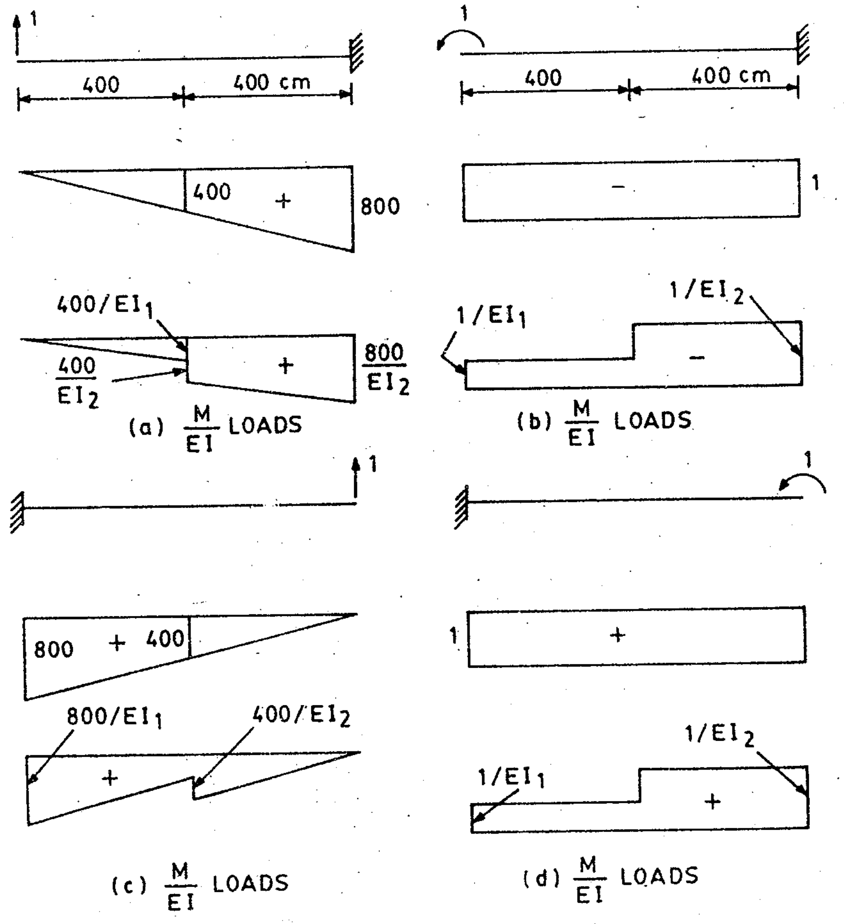


Fig. 13.8

or,
$$F^{-1} = K = \begin{bmatrix} 0.00218 & +1.018 \\ +1.018 & 581.78 \end{bmatrix} \qquad \text{O.K.}$$

(ii) Now let us restrain end i of the beam, and apply a unit force at d.o.f. 3 and compute the deflections at d.o.f. 3 and 4. (Fig. 13.8c, d)

$$\delta_{33} = \frac{1}{2} \times 400 \times \frac{400}{EI_2} \times \frac{2}{3} \times 400 + \frac{600}{EI_1} \times 400 \times \left[400 + \frac{400}{3}\left(\frac{400+1600}{400+800}\right)\right]$$

$$= \frac{64 \times 10^6}{3 \times 64000} + \frac{96 \times 10^6}{2 \times 64000} \times 1.555 = 1500$$

$$\delta_{43} = \frac{1}{2} \times 400 \times \frac{400}{EI_2} + \frac{600}{EI_1} \times 400 = \frac{80000}{64000} + \frac{240000}{2 \times 64000} = 3.125$$

$$\delta_{44} = \frac{400 \times 1}{2 \times 64000} + \frac{400 \times 1}{64000} = 0.00937$$

or,
$$F_j = \begin{bmatrix} 1500 & 3.125 \\ 3.125 & 0.00937 \end{bmatrix}$$

$$F_j^{-1} = K_j = \begin{bmatrix} 0.00218 & -0.728 \\ -0.728 & 349.70 \end{bmatrix}$$

The cross-stiffness terms k_{ij} can be computed by considering the equilibrium of the member as given by Eq. 13.17 and 13.18. The results are the same as in Example 13.1.

O.K.

13.7 TRANSFORMATION MATRIX - 3-D TRUSS ELEMENT

The system stiffness matrix of a space truss is assembled from the various elements of the member stiffness matrices. The member stiffness matrix must be expressed in terms of the structure or global system of coordinates before the stiffness can be assembled. The two reference systems of coordinates are shown in Fig. 13.9. Also shown in this figure is a general, vector A with its components along the global axes x', y' and z'.

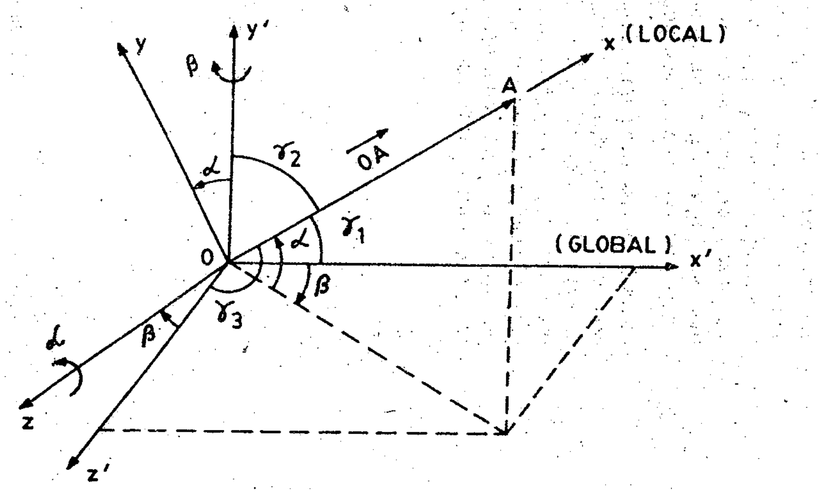


Fig. 13.9 Local and global axes in 3-D

To obtain the components of vector A along one of the local axes x , y , or z , it is necessary to add the projections of x', y' and z' components along that axis. Thus, the component x of vector A along the x-axis is given by :

$$x = x' \cos x x' + y' \cos x y' + z' \cos x z'$$

where, $\cos x x' = \cos \gamma_1$ = cosine of angle between axes x and x'

$\cos x y' = \cos \gamma_2$ = cosine of angle between axes x and y'

and $\cos x z' = \cos \gamma_3$ = cosine of angle between axes x and z'

Similarly, the y and z components of vector A can be written as :

$$y = x' \cos y x' + y' \cos y y' + z' \cos y z'$$

and

$$z = x' \cos z x' + y' \cos z y' + z' \cos z z'$$

These equations can be written in the matrix form as :

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \cos xx' & \cos xy' & \cos xz' \\ \cos yx' & \cos yy' & \cos yz' \\ \cos zx' & \cos zy' & \cos zz' \end{bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}$$

or,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} \tag{13.19a}$$

or,

$$A = \lambda A' \tag{13.19b}$$

where, l_1, m_1, and n_1 are the direction cosines for the x-axis.
l_2, m_2, and n_2 are the direction cosines for the y-axis.
l_3, m_3, and n_3 are the direction cosines for the z-axis.

These direction cosines may be determined by geometry from the physical orientation of the member. The complete transformation matrix R for a space truss is of the form :

$$R = \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & \lambda \end{array}\right] \tag{13.20}$$

where each element λ is given by Eq. 13.19.

The relation between global and local stiffness matrix can be written as before, that is,

$$K'_{6\times6} = R^T_{6\times6}\, K_{6\times6}\, R_{6\times6} \tag{13.21}$$

where, K is given by Eq. 13.1.

Substituting Eqs. 13.1 and 13.20, into Eq. 13.21 yields an expression for the member stiffness matrix in global system for a space truss.

A systematic procedure for the evaluation of the direction cosines in Eq. 13.19 involves rotation about three axes. The final rotation matrix λ is obtained by multiplying the three separately formed rotation matrices in a given order. Consider a global axes system x'-y'-z' and a local axes system 1'-2'-3' which coincides with each other. Let the first rotation consists of a rotation β about the 2' axis as shown in Figs. 13.9 and 13.10a. The direction cosine terms in Eq. 13.19 a can be readily computed as :

$$\lambda_\beta = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{13.22a}$$

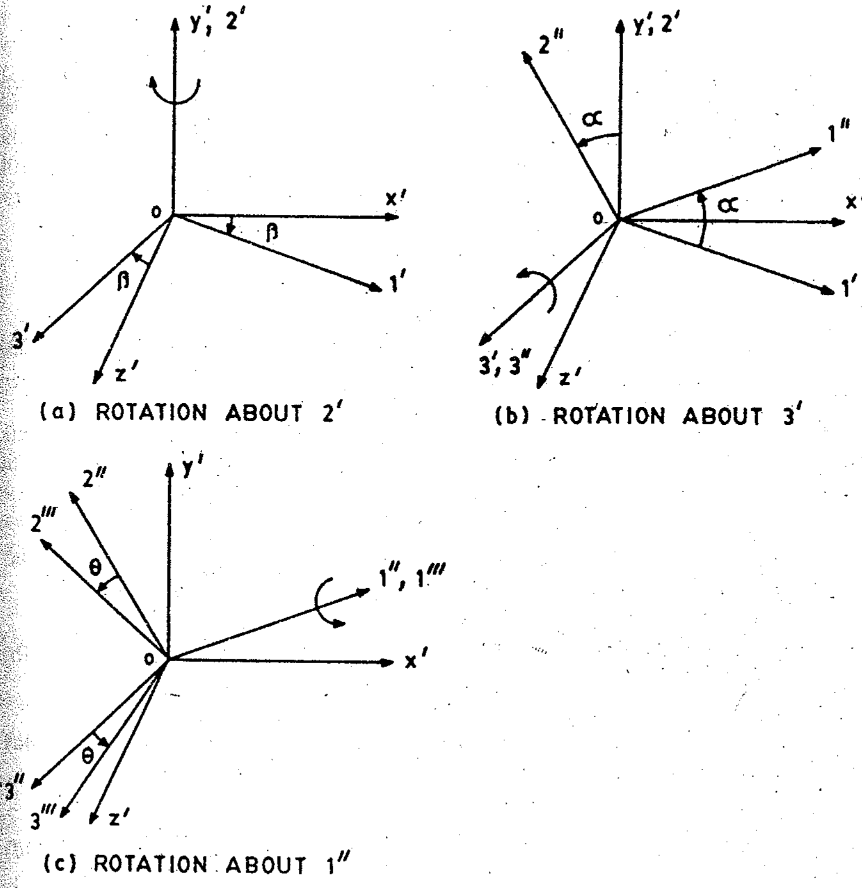


Fig. 13.10 Transformation of axes

Now let a rotation α takes place about the 3' axis as shown in Fig. 13.9 and 13.10b. The rotated coordinate system is given by double primes. The angle between the 1" axis and the plane 1' - 3' is α. The direction cosines are given as :

$$\lambda_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13.22b}$$

Now let a third rotation θ take place about the axis 1" as shown in Fig. 13.10c. The rotated coordinate system is given by triple primes which also represents the local axes of a element in space. The direction cosines are given as :

$$\lambda_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{13.22c}$$

The final rotation matrix is given by the triple product :

$$\lambda = \lambda_\theta\, \lambda_\alpha\, \lambda_\beta \tag{13.22d}$$

For a truss member, the orientation of the principal planes is immaterial. Hence, the third rotation may be ignored, that is θ = 0. 1" – 2" – 3" axes as shown in Fig. 13.10b represents the local member axes x-y-z as shown in Fig. 13.9. The final rotation matrix reduces to

$$\lambda = \lambda_\alpha\, \lambda_\beta$$

$$\lambda = \begin{bmatrix} \cos\beta\cos\alpha & \sin\alpha & \sin\beta\cos\alpha \\ -\cos\beta\sin\alpha & \cos\alpha & -\sin\beta\sin\alpha \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{13.23}$$

All elements of this matrix happen to be in terms of the direction cosines of the x-axis, which are also the direction cosines of the member itself. These direction cosines may be determined from the coordinates of the member as (Fig. 13.1 and Fig. 13.9) follows:

$$l_1 = \cos\gamma_1 = \frac{x_j - x_i}{L}, \quad m_1 = \cos\gamma_2 = \frac{y_j - y_i}{L}, \quad n_1 = \cos\gamma_3 = \frac{z_j - z_i}{L} \tag{13.24a}$$

where,

$$L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2} \tag{13.24b}$$

$$\sin\beta = \frac{n_1}{\sqrt{l_1^2 + n_1^2}} \qquad \cos\beta = \frac{l_1}{\sqrt{l_1^2 + n_1^2}} \tag{13.24c}$$

and

$$\sin\alpha = m_1 \qquad \cos\alpha = \sqrt{l_1^2 + n_1^2} \tag{13.24d}$$

Therefore

$$\lambda = \begin{bmatrix} l_1 & m_1 & n_1 \\ \dfrac{-l_1 m_1}{\sqrt{l_1^2 + n_1^2}} & \sqrt{l_1^2 + n_1^2} & \dfrac{-m_1 n_1}{\sqrt{l_1^2 + n_1^2}} \\ \dfrac{-n_1}{\sqrt{l_1^2 + n_1^2}} & 0 & \dfrac{l_1}{\sqrt{l_1^2 + n_1^2}} \end{bmatrix} \tag{13.19c}$$

13.8 TRANSFORMATION MATRIX - 3-D BEAM ELEMENT

The transformation matrix of a 3-D beam element is identical to the general transformation matrix used in 3-D truss element. In a truss element, there are three degrees of freedom at a node, whereas, in a beam element there are six degrees of freedom per node. The resulting stiffness matrix and the transformation matrix are of the order 12 × 12. For a beam element, the transformation of the nodal displacement vectors involve the transformation of linear and angular displacement vectors at each node. Therefore, it requires the transformation of total of four displacement vectors, two at each of the two nodes. The transformation or rotation matrix can be written as :

$$\mathbf{R}_{12\times 12} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}_{12\times 12} \tag{13.25}$$

where λ is given by Eq. 13.22. The individual elements of this rotation matrix are quite complicated. Therefore, an alternate approach is used to determine the direction cosines.

The nine direction cosine elements for the matrix $\lambda_{3\times 3}$, Eq. 13.19 are usually calculated from the global coordinates of three points. Two points are the two ends of the beam element along the local x-axis, and third point is taken any where in the x-y local plane, where y is one of the principal axes of the cross-sectional area of the member.

Let us consider the three points I, J at the two ends of a beam element and L, a point on the local x-y plane as shown in Fig. 13.11. Their coordinates are : (x_i, y_i, z_i), (x_j, y_j, z_j) and (x_l, y_l, z_l). The direction cosines of local x-axis along the beam element are given by

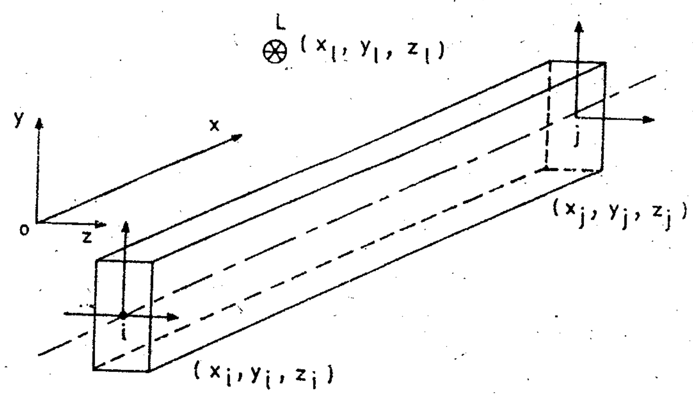


Fig. 13.11 L - node of a 3-D beam element

$$\cos xx' = l_1 = \frac{x_j - x_i}{L}, \quad m_1 = \frac{y_j - y_i}{L}, \quad n_1 = \frac{z_j - z_i}{L} \tag{13.26}$$

The direction cosines of the z-axis can be calculated from the condition that any vector $\vec{Z}$ along the z-axis must be perpendicular to the plane formed by any two vectors in the local x-y plane. These two vectors could be $\vec{X}$ from I to J, along the x-axis, and $\vec{L}$ from point I to point L. Thus, the cross-product of these three vectors can be written as:

$$\vec{Z} = \vec{X} \times \vec{L} \tag{13.27a}$$

or,

$$\vec{Z} = z_x \hat{i} + z_y \hat{j} + z_z \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_j - x_i & y_j - y_i & z_j - z_i \\ x_l - x_i & y_l - y_i & z_l - z_i \end{vmatrix} \tag{13.27b}$$

where, $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along the global coordinate axes x', y' and z' respectively. Now the direction cosines of z axis are given by

$$\cos zx' = l_3 = \frac{z_x}{|Z|}, \quad m_3 = \frac{z_y}{|Z|}, \quad n_3 = \frac{z_z}{|Z|} \tag{13.27c}$$

where,

$$z_x = (y_j - y_i)(z_l - z_i) - (z_j - z_i)(y_l - y_i)$$
$$z_y = (z_j - z_i)(x_l - x_i) - (x_j - x_i)(z_l - z_i)$$
$$z_z = (x_j - x_i)(y_l - y_i) - (y_j - y_i)(x_l - x_i)$$

and,

$$|Z| = \sqrt{z_x^2 + z_y^2 + z_z^2} \tag{13.27d}$$

Once the z-axis is established, the direction cosines of the y-axis can be calculated using the similar condition of orthogonality.

Thus,

$$\vec{Y} = \vec{X}_1 \times \vec{Z}_1 \tag{13.28a}$$

where, $\vec{X}_1$ and $\vec{Z}_1$ are unit vectors along the x and z-axes, respectively.

$$\vec{Y} = y_x \hat{i} + y_y \hat{j} + y_z \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ l_1 & m_1 & n_1 \\ l_3 & m_3 & n_3 \end{vmatrix} \tag{13.28b}$$

Therefore,

$$l_2 = \frac{y_x}{|Y|}, \quad m_2 = \frac{y_y}{|Y|}, \quad n_2 = \frac{y_z}{|Y|} \tag{13.28c}$$

where,

$$y_x = m_1 n_3 - n_1 m_3$$
$$y_y = n_1 l_3 - l_1 n_3$$
$$y_z = l_1 m_3 - m_1 l_3$$

and,

$$|Y| = \sqrt{y_x^2 + y_y^2 + y_z^2} \tag{13.28d}$$

Alternatively, once the z-axis is established, the direction cosines of the y-axis can be calculated by using the following expressions and solving simultaneously :

$$\begin{aligned} l_2^2 + m_2^2 + n_2^2 &= 1 \\ l_2 l_1 + m_2 m_1 + n_2 n_1 &= 0 \\ l_2 l_3 + m_2 m_3 + n_2 n_3 &= 0 \end{aligned} \tag{13.29}$$

These are again the conditions of orthogonality.

Quite often the point L is selected as a known point in the structure which is placed on the local y-axis, although it could be any point in the local x-y plane. The only condition is that this point L should not lie on the local x-axis. If this point does not lie on the structure itself, its boundary condition is given as restrained in all the six-degrees - of - freedom. Thus, as many L-points may be selected as are required to define the orientation of the beam elements used in the structure, without affecting the storage requirements or the band width.

REFERENCES :

Macleod, I. A. (1967) *Shear Wall Interaction*, Portland Cement Association, Chicago.

Przemieniecki, J. S. (1968). *Theory of Matrix Structural Analysis*, McGraw Hill Book Co., New York.

CHAPTER

fourteen

STAP-3D COMPUTER PROGRAM

14.1 INTRODUCTION

STAP-3D (Structural Analysis Program) is a general purpose program for the analysis of any structure consisting of truss or beam or both these elements. The program is organized in such a manner that any new element can be easily included in its element library. The program is based on the stiffness matrix method as developed in chapters 12 and 13. The user instructions along with typical test problems with input and output are provided in this chapter. The FORTRAN listing of the STAP program is available on a floppy.

Originally written by Prof. E.L.Wilson, University of California, Berkeley for main frame computers, it has bean modified for execution on a Personal Computer. The program can be compiled using a Microsoft FORTRAN Compiler. The program is interactive although the data is read from a user- specified data file for convenience. It has a data check option where the program reads the input data, generates missing data, if any and prints the data. The user can examine if the input data is correct and re-run the program with the execution option.

14.2 SALIENT FEATURES

The program is divided in a number of segments as follows :

1.	CORE segment	- Main routine	
		- subroutines	STAP, NODES, ELIB, LOAD, GSTIFF, SOLVE, PRINTD, and FORCE
2.	ELEMENT segment	- subroutines	TRUSS and TRUSS1
		- subroutines	BEAM, BEAM1 and STIFF
		- subroutines	BOUND, and BOUND1

Main Routine

The program makes use of the dynamic dimensioning concept, which allocates dimensions to various variables in a single COMMON array A depending upon the requirement of the given problem. The program checks the memory requirement with the available size of the COMMON array A. In case, the memory exceeds, a message is displayed and the execution stops. It can be restarted by increasing the size of array A and the variable MTOT.

This program employs a total of eight files on the hard disc during the various stages of execution. These files are opened and closed in the main routine.

Subroutine STAP

It is, in fact, the controlling routine for the entire program. It reads the control data, initializes the named COMMON statements, and calls the core subroutines. COMMON block A consists of control parameters, whereas, COMMON blocks B and C are meant for working arrays which may vary in different subroutines. The important variables are as follows :

NPAR	=	number of parameters
NUMNP	=	number of nodal points
MBAND	=	band width
NELTYP	=	number of element types
MTOT	=	total dynamic memory,
NEQ	=	number of equations
LL	=	number of load cases
MODEX	=	execution mode
NBLOCK	=	number of blocks Due to limited dynamic memory size, the equations are stored in blocks in a file. They are called blockwise for further processing, as and when required.
DUMP	=	dummy or working variables
DUM	=	dummy or working variables
N1, N2, N3 etc.	=	indicator for starting location of different arrays in the dynamic storage array A.

Subroutine NODES

It reads the nodal data and generates missing nodal data, if necessary. The boundary conditions are also read herein. It generates one array ID (identity) which consists of equation numbers of all the active degrees of freedom. It is very helpful in locating any data error or instability in the structure.

Subroutine ELIB

This is the element library subroutine and calls the desired elements. At present there are three elements : Truss, Beam and Boundary. These will be discussed a little later.

Subroutine LOAD

The loads may be specified either at the element level or at the structure level or both. The element level loads are read in along with the corresponding element data. The structure level loads are specified through the nodal loads in the global system and are read in this subroutine.

There can be four cases of loads at the element level : case A, B, C, and D. These are specified through the fixed end forces due to dead load, live load, temperature, or any other cause. The user has the option to assign load cases A,B, C and D to dead load, live load or any imposed load. At the structural level, these load cases can be combined in a specified proportion with the concentrated loads to get the net loads for each structural load case 1, 2,....................., LL. The multipliers for each of the four element load cases corresponding to each of the structural load cases are read in. There are no multipliers for concentrated loads. If multipliers for element load cases A, B, C and D for structural load case 1 are q_{11}, q_{12}, q_{13} and q_{14}; for structural load case 2 are q_{21}, q_{22}, q_{23} and q_{24}; and so on, the load combinations may be written as follows :

$$
\begin{aligned}
\text{Structural load case } 1 &= q_{11}\, A_1 + q_{12}\, B_1 + q_{13}\, C_1 + q_{14}\, D_1 + P_1 \\
2 &= q_{21}\, A_2 + q_{22}\, B_2 + q_{23}\, C_2 + q_{24}\, D_2 + P_2 \\
3 &= q_{31}\, A_3 + q_{32}\, B_3 + q_{33}\, C_3 + q_{34}\, D_3 + P_2 \\
&\vdots \\
LL &= q_{LL,1}\, A_{LL} + q_{LL,2}\, B_{LL} + q_{LL,3}\, C_{LL} + q_{LL,4}\, D_{LL} + P_{LL}
\end{aligned}
$$

where, A_i, B_i, C_i, D_i = loads at element level in local coordinates, to be transformed in global coordinates within the program. (i = 1 to LL)

q_{ij} = element load multipliers (i = 1 to LL, and j = 1 to 4)

P_i = concentrated loads (i = 1 to LL)

LL = total number of structural load cases.

The loads due to self weight of the element along the global axes are computed and added separately.

Subroutine GSTIFF

The element stiffness matrix, element load vector and the corresponding transformation matrices are generated in the respective element routines. These are stored in a file. This subroutine reads back this information and assembles global stiffness matrix and global load vector taking due care of the specified boundary conditions. It generates a banded symmetric stiffness matrix

Subroutine SOLVE

It is based on the standard Gauss elimination method and solves for the nodal displacements.

Subroutine PRINTD

It extracts the nodal displacements and prints them for each load case.

Subroutine FORCE

This subroutine extracts and prints the member forces in the local axis for each element. It calls the element library for transformation of element forces from global to local axes.

Element Segment

Subroutine TRUSS/TRUSS1

This is a main subroutine which checks the memory requirements and calls TRUSS1. This latter routine reads element data, generates missing data, if any, and generates element stiffness matrix and load vector.

The subroutine FORCE again calls TRUSS to extract the element forces in local coordinates.

Subroutine BEAM

This is a main routine which checks the memory requirements and calls BEAM1. When called by FORCE, it helps extract the element forces in local coordinates.

Subroutine BEAM1/STIFF

It reads element data, generates missing data, if any, and generates transformation matrix for each element. The stiffness matrix and load vectors are generated in the routine STIFF.

Subroutine BOUND/BOUND1

This is a spring element for determining support reactions and imposing known displacements on the structure. It is also used to model an inclined roller support. The element data is read in BOUND1 which also generates the stiffness matrix. A boundary element is defined by a single directed axis through a specified nodal point by a linear extensional stiffness along the axis or by a linear rotational stiffness about the axis. It should be aligned with the global directions to avoid ill-conditioning of the stiffness matrix.

The important variables used in the program are as follows :

LM	=	location matrix array for an element which depends on its connectivity
NS	=	maximum number of stress or force components of the current element
ND	=	degree of freedom of the current element
NDM	=	max. degree of freedom of any element used in the library
SA, ST, ASA	=	member stiffness matrices
X, Y, Z, T	=	x-, y-, z- coordinates and T-temperature arrays
SIG	=	member force array
MATTYP	=	material type
NUME	=	number of elements in a given group

NUMMAT	=	number of materials in a given group
SF	=	fixed end force in local coordinates for the four load cases A,B,C,D.
RF	=	fixed end force in global coordinates
EMUL	=	element load multipliers for self weight of an element along x-, y-, z-directions for each of the four load cases A, B, C and D.

14.3 ADDING NEW ELEMENTS TO THE LIBRARY

A new element (one or more) can be easily added to the current library. It will require some changes in subroutine ELIB so that a new element is called. If the maximum d.o.f. of the new element is 12 or less, the control COMMON blocks A and B as shown in subroutine TRUSS1 remain unchanged. Otherwise, a suitable change is necessary in COMMON block B. There will be at least two subroutines per element : a main routine and a working routine. The element geometry, material property, connectivity, load etc. will be read in, stiffness matrix and load vector generated and saved on a file in the working routine. The band width is computed by calling subroutine BAND residing in the core segment of the program. The location indicators N1, N2, N3 etc. are to be appropriately defined in the main element routine as well as in the core segment routine STAP. The nodal coordinates, ID array and material properties are stored in array A which has to be carefully transferred to the working element routine through the calling arguments.

There are sufficient comment cards in the current element library and should be helpful in developing new elements.

14.4 USER'S INSTRUCTIONS

The following questions must be answered while executing the program on a Personal Computer :

1. The names of the input and output files.
2. Do you want to print generated nodal data : (Y/N)
   Enter Y or y for yes; N or n for no.
3. Do you want to print ID matrix : (Y/N)
   The Identification matrix is helpful in checking the generated nodal load vectors for each load case and in debugging.
4. Do you want to print final nodal loads : (Y/N)
5. Do you want to print nodal displacements : (Y/N)

DATA PREPARATION

I. Number of Problems (I5)

Columns 1 - 5 Total number of problems

II. Heading Line (20A4)

Columns 1 - 80 Problem Title

NOTE :

1. The user may also indicate his/her name, units employed and date.

2. The program accepts data in any set of consistent units.

III. Control Data (4I5)

Columns	1 - 5	Total number of nodes (NUMNP)
	6 - 10	Number of different element types (NELTYP)
	11 - 15	Number of different load cases (LL)
	20	Execution mode (MODEX) 0 : Solution 1 : Data check only

IV. Load Case Headings (15A4)

Give as many lines as is the number of load cases.

Columns 1 - 60 Title for load case 1

V. Nodal Data (7I5,3F10.0,I5,F10.0)

Columns	1 - 5	Node number
	6 - 10	Boundary condition for displacement in x-direction 0 : free 1 : restrained
	11 - 15	......displacement in y-direction
	16 - 20	......displacement in z-direction
	21 - 25	Boundary condition for rotation about x-axis
	26 - 30	......rotation about y-axis
	31 - 35	......rotation about z-axis
	36 - 45	x-coordinate
	46 - 55	y-coordinate
	56 - 65	z-coordinate
	66 - 70	Node generation parameter KN
	71 - 80	Nodal temperature

NOTES :

1. The coordinate axes correspond to the right hand thumb rule.
2. Nodal data lines need not be in ascending order. If lines are omitted the nodal data for the missing lines will be automatically genarated using KN.
3. KN is the increment to be added to the previous node number.
4. KN should be given on the last line of a generation sequence.
5. The lines so generated have the same boundary conditions as the first line in the generation sequence.
6. KN = 0 means no generation.
7. The data of the last node must be given at the end .

VI. Element Data

1. Truss element : Give its data only if required.
2. Beam element : Give its data only if required.
3. Boundary element : Give its data only if required.

VII. Nodal Load Data with respect to global axis (2I5,6F10.0)

Columns 1 - 5 Node number

6 - 10 Load case number
11 - 20 Load in x-direction
21 - 30 Load in y-direction
31 - 40 Load in z-direction
41 - 50 Moment about x-axis
51 - 60 Moment about y-axis
61 - 70 Moment about z-axis

NOTES :

1. The lines must be in node number and load case sequence .
2. The data for nodes having non-zero concentrated loads only need be given.
3. This set of lines, if any, must be terminated with a line containing only zero values.

VIII. Element Load Multipliers for Structural Load Cases (4F10.0)

Give as many lines as the number of load cases defined in line type III.

Columns 1 - 10 Multiplier for element load case A
11 - 20 Multiplier for element load case B
21 - 30 Multiplier for element load case C
31 - 40 Multiplier for element load case D

IX. For next problem, if any, repeat lines type II to VIII.

DATA Set VI.1 3-D TRUSS ELEMENT

A. Control Data (3I5)

Columns 1 - 5 Type 1
6 - 10 Number of truss elements (NUME)
11 - 15 Number of different geometric/material properties (NUMMAT)

B. Geometric Property Data (I5, 4F10.0)

One line is required for each different geometric or material property.

Columns 1 - 5 Material number
6 - 15 Modulus of elasticity
16 - 25 Cross-sectional area
26 - 35 Weight per unit length
36 - 45 Coeff. of thermal expansion

C. Element Load Multipliers (4F10.0)

Line 1 : Multiplier of gravity loads in the +ve x-direction.

Columns 1 - 10 Element load case A
11 - 20 Element load case B
21 - 30 Element load case C
31 - 40 Element load case D

Line 2 : As above for gravity load in the +ve y-direction.
Line 3 : As above for gravity load in the +ve z-direction.
Line 4 : Multiplier of thermal loads to be considered for each element load case.

D. Member Data (4I5, F10.0,I5)

Columns 1 - 5 Member number

6 - 10 Node I
11 - 15 Node J
16 - 20 Geometry identification number
21 - 30 Reference temperature for zero stress
31 - 35 Member generation code KN

NOTES :

1. KN is equal to node difference which must be constant for a given generation sequence.
2. KN must be specified on the first line of the generation sequence.
3. Member number to be generated must be one greater than the previous member number. The node numbering is done as follows :

$$I_{i+1} = I_i + KN$$
$$J_{i+1} = J_i + KN$$

4. The data for first element and last element in the series must be provided.
5. Automatic generation should be used only if all the elements to be generated in a group have the same material and geometric properties.
6. The change in temperature in a member used to calculate thermal loads is given by:

$$dT = \frac{T_i + T_j}{2} - T_{ref}$$

where T_i and T_j are the temperatures specified on the nodal data lines for nodes I and J.

7. The initial lack of fit or initial prestress can be specified using an equivalent change in temperature.

(a) If member is too long by r, then

$$dT = r / \alpha L$$

(b) If initial prestress = P (tensile) which is released after the member is connected to the rest of the structure, then

$$dT = -P/(\alpha AE)$$

where α = coeff of thermal expansion, A = area of cross-section and E = Youngs modulus of elasticity.

Data Set VI.2 3-D BEAM ELEMENT

A. Control Data (5I5)

Columns 1 - 5 Type 2
6 - 10 Number of beam elements (NBEAM)
11 - 15 Number of different materials (NUMMAT)
16 - 20 Number of geometric properties (NUMETP)
21 - 25 Number of fixed end forces (NUMFIX)

B. Material Property Data (I5,3F10.0)

Give as many lines as the number of different materials.

Columns 1 - 5 Material number
6 - 15 Modulus of elasticity
16 - 25 Poisson's ratio

26 - 35 Weight per unit volume

C. Geometric Property Data (I5,6F10.0)

Give as many lines as the number of different geometric properties about local axis of a member.

Columns	1 - 5	Geometric property number
	6 - 15	Axial area A1
	16 - 25	Shear area A2 associated with shear force in local 2-axis
	26 - 35	Shear area A3 associated with shear force in local 3- axis
	36 - 45	Torsional inertia about local 1-axis
	46 - 55	Moment of inertia about local 2-axis
	56 - 65	Moment of inertia about local 3-axis

NOTES :

1. Any value except shear areas must not be zero.
2. Shear area must be included (non-zero) only if shear deformations need be considered in the analysis.
3. Shear area can be calculated using the following expressions for shape factors:

for rectangular sections $= \dfrac{10(1+\nu)}{(11\nu+12)}$

circular sections $= \dfrac{6(1+\nu)}{(6\nu+7)}$

where ν = Poisson's ratio

4. Local 1, 2 and 3 axis indicate principal directions of a member.

D. Element Load Multipliers for Gravity Loads (4F10.0)

Line 1 : Multiplier for gravity load in +ve global x- direction.

Columns	1 - 10	Element load case A
	11 - 20	Element load case B
	21 - 30	Element load case C
	31 - 40	Element load case D

Line 2 : As above for gravity load in +ve y-direction.

Line 3 : As above for gravity load in +ve z-direction.

NOTE :

1. This program does not compute fixed end forces due to gravity loads but only reactions.

E. Fixed End Forces (I5,6F10.0)

Two lines are required for each unique set of fixed end forces in local axis. Skip if there are no fixed end forces.

Line 1

Columns	1 - 5	Fixed end force number
	6 - 15	F.E. Force in local 1-axis at node I
	16 - 25	F.E. Force in local 2-axis at node I
	26 - 35	F.E. Force in local 3-axis at node I
	36 - 45	F.E. Moment about local 1-axis at node I
	46 - 55	F.E. Moment about local 2-axis at node I
	56 - 66	F.E. Moment about local 3-axis at node I

Line 2

Columns	1 - 5	Fixed end force number
	6 - 15	F.E. Force in local 1-axis at node J
	16 - 25	F.E. Force in local 2-axis at node J
	26 - 35	F.E. Force in local 3-axis at node J
	36 - 45	F.E. Moment about local 1-axis at node J
	46 - 55	F.E. Moment about local 2-axis at node J
	56 - 66	F.E. Moment about local 3-axis at node J

NOTE :

1. Thermal forces may be input through fixed end forces.

F. Member Data (11I5)

Columns	1 - 5	Member number
	6 - 10	Node I
	11 - 15	Node J
	16 - 20	Node L
	21 - 25	Material number
	26 - 30	Geometry number
	31 - 35	F.E. Force set number for load case A
	36 - 40	F.E. Force set number for load case B
	41 - 45	F.E. Force set number for load case C
	46 - 50	F.E. Force set number for load case D
	51 - 55	Member generation code KN

NOTES :

1. KN is equal to node difference which must be constant for a given generation sequence.
2. KN must be specified on the first line of the generation sequence.
3. Member number to be generated must be one greater than the previous member number.
4. The data for first element and last element in the series must be provided.
5. Automatic generation should be used only if all the elements to be generated in a group have the same material and geometric properties.
6. The details for specifying the L-node were discussed in section 13.5 and Fig. 13.4.

Data Set VI.3 BOUNDARY ELEMENT

A. Control Data (2I5)

Columns	1 - 5	Type 3
	6 - 10	Number of boundary elements

B. Member Data (5I5,3F10.0)

Columns	1 - 5	Node N at which the boundary element is placed
	6 - 10	Node I , far end of the spring

11 - 15	Code for displacement	
	0 :	displacement not specified
	1 :	displacement is specified
16 - 20	Code for rotation	
	0 :	rotation not specified
	1 :	rotation is specified
21 - 25	Member generation code KN	
26 - 35	Specified displacement along member axis	
36 - 45	Specified rotation about member axis	
46 - 55	Spring stiffness	

NOTE :

1. The positive direction of the element is from node I to N.
2. If spring stiffness is input as zero, it is taken as 10^{10}.

14.5 ILLUSTRATIVE EXAMPLES

The following examples will help develop an understanding of the various capabilities of the program STAP-3D.

Example 14.1

Analyze the Warren truss shown in Fig. 14.1 using the truss element in STAP-3D.

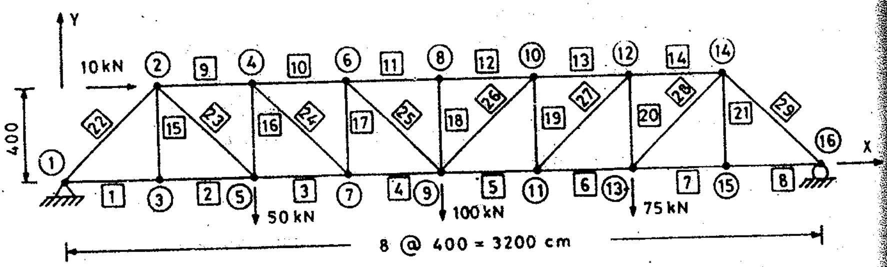


Fig. 14.1 Warren truss

Solution

The node numbers are shown in circles and element numbers are shown in square boxes in Fig. 14.1. The member properties are given in Table 14.1

Table 14.1 Member geometric properties

Member no.	Area, cm^2	Remark
1 to 14	30.78	2ISA 100 × 100 × 8
15 - 21	5.68	ISA 50 × 50 × 6
22 - 29	11.36	2ISA 50 × 50 × 6

Modulus of elasticity E = 20000 kN/cm^2

The input data prepared in accordance with the instructions given earlier and the complete input and output are shown in EX141.FIL and EX141.DOC.

EX141.FIL

```
    1
 Analysis of a Warren Truss   units are kN-cm  Example 14.1
   16     1     2     0
loads at only four points
loads at all panel points of the bottom chord
    1     1     1     1     1     1     1 0.        0.
    2     0     0     1     1     1     1 400.      400.
   14                 1     1     1     1 2800.     400.                   2
    3                 1     1     1     1 400.
   15                 1     1     1     1 2800.     0.                     2
   16           1     1     1     1     1 3200.
    1    29     3
    1 20000.    30.78      0.       0.
    2 20000.    5.68
    3 20000.    11.36
0.        0.        0.        0.
0.        0.        0.        0.
0.        0.        0.        0.
0.        0.        0.        0.
    1     1     3     1 0.          2
    7    13    15     1 0.
    8    15    16     1
    9     2     4     1 0.          2
   14    12    14     1 0.
   15     2     3     2 0.          2
   21    14    15     2 0.
   22     1     2     3
   23     2     5     3 0.          2
   25     6     9     3
   26     9    10     3 0.          2
   28    13    14     3 0.
   29    14    16     3
    2     1 10.       0.
    3     2          -25.
    5     1          -50.
    5     2          -25.
    7     2          -25.
    9     1          -100.
    9     2          -25.
   11     2          -25.
   13     1          -75.
   13     2          -25.
   15     2          -25.
    0     0
0.        0.        0.        0.
0.        0.        0.        0.
```

EX141.DOC

****

STRUCTURAL ANALYSIS PROGRAM STAP-3D
Analysis of a Warren Truss units are kN-cm Example 14.1

****

NUMBER OF NODAL POINTS = 16
NUMBER OF ELEMENT TYPES = 1
NUMBER OF LOAD CASES = 2
SOLUTION MODE (MODEX) = 0
EQ.0 : EXECUTION
EQ.1 : DATA CHECK

LOAD CASE NO 1 IS loads at only four points
LOAD CASE NO 2 IS loads at all panel points of the bottom chord

NODAL POINT INPUT DATA

NODE	BOUNDARY CONDITION CODES						NODAL POINT COORDINATES				
NUMBER	X	Y	Z	XX	YY	ZZ	X	Y	Z	KN	TEMP
1	1	1	1	1	1	1	.000	.000	.000	0	.000
2	0	0	1	1	1	1	400.000	400.000	.000	0	.000
14	0	0	1	1	1	1	2800.000	400.000	.000	2	.000
3	0	0	1	1	1	1	400.000	.000	.000	0	.000
15	0	0	1	1	1	1	2800.000	.000	.000	2	.000
16	0	1	1	1	1	1	3200.000	.000	.000	0	.000

GENERATED NODAL DATA

NODE	BOUNDARY CONDITION CODES						NODAL POINT COORDINATES			TEMP
NUMBER	X	Y	Z	XX	YY	ZZ	X	Y	Z	
1	1	1	1	1	1	1	.000	.000	.000	.000
2	0	0	1	1	1	1	400.000	400.000	.000	.000
3	0	0	1	1	1	1	400.000	.000	.000	.000
4	0	0	1	1	1	1	800.000	400.000	.000	.000
5	0	0	1	1	1	1	800.000	.000	.000	.000
6	0	0	1	1	1	1	1200.000	400.000	.000	.000
7	0	0	1	1	1	1	1200.000	.000	.000	.000
8	0	0	1	1	1	1	1600.000	400.000	.000	.000
9	0	0	1	1	1	1	1600.000	.000	.000	.000
10	0	0	1	1	1	1	2000.000	400.000	.000	.000
11	0	0	1	1	1	1	2000.000	.000	.000	.000
12	0	0	1	1	1	1	2400.000	400.000	.000	.000
13	0	0	1	1	1	1	2400.000	.000	.000	.000
14	0	0	1	1	1	1	2800.000	400.000	.000	.000
15	0	0	1	1	1	1	2800.000	.000	.000	.000
16	0	1	1	1	1	1	3200.000	.000	.000	.000

EQUATION NUMBERS

N	X	Y	Z	XX	YY	ZZ
1	0	0	0	0	0	0
2	1	2	0	0	0	0
3	3	4	0	0	0	0
4	5	6	0	0	0	0
5	7	8	0	0	0	0
6	9	10	0	0	0	0
7	11	12	0	0	0	0
8	13	14	0	0	0	0
9	15	16	0	0	0	0
10	17	18	0	0	0	0
11	19	20	0	0	0	0
12	21	22	0	0	0	0
13	23	24	0	0	0	0
14	25	26	0	0	0	0
15	27	28	0	0	0	0
16	29	0	0	0	0	0

NUMBER OF TRUSS MEMBERS = 29
NUMBER OF DIFF. MEMBERS = 3

TYPE	E	AREA	Wt/LENGTH	TEMP. COEFF.
1	20000.000	30.780	.000	.000
2	20000.000	5.680	.000	.000
3	20000.000	11.360	.000	.000

ELEMENT LOAD MULTIPLIERS

	A	B	C	D
X-DIR	.000000E+00	.000000E+00	.000000E+00	.000000E+00
Y-DIR	.000000E+00	.000000E+00	.000000E+00	.000000E+00
Z-DIR	.000000E+00	.000000E+00	.000000E+00	.000000E+00
TEMP	.000000E+00	.000000E+00	.000000E+00	.000000E+00

N	I	J	TYPE	TEMP	BAND WIDTH	LENGTH
1	1	3	1	.00	2	400.000
2	3	5	1	.00	6	400.000
3	5	7	1	.00	6	400.000
4	7	9	1	.00	6	400.000
5	9	11	1	.00	6	400.000
6	11	13	1	.00	6	400.000
7	13	15	1	.00	6	400.000
8	15	16	1	.00	3	400.000
9	2	4	1	.00	6	400.000

10	4	6	1	.00	6	400.000
11	6	8	1	.00	6	400.000
12	8	10	1	.00	6	400.000
13	10	12	1	.00	6	400.000
14	12	14	1	.00	6	400.000
15	2	3	2	.00	4	400.000
16	4	5	2	.00	4	400.000
17	6	7	2	.00	4	400.000
18	8	9	2	.00	4	400.000
19	10	11	2	.00	4	400.000
20	12	13	2	.00	4	400.000
21	14	15	2	.00	4	400.000
22	1	2	3	.00	2	565.685
23	2	5	3	.00	8	565.685
24	4	7	3	.00	8	565.685
25	6	9	3	.00	8	565.685
26	9	10	3	.00	4	565.685
27	11	12	3	.00	4	565.685
28	13	14	3	.00	4	565.685
29	14	16	3	.00	5	565.685

TOTAL NUMBER OF EQUATIONS = 29
BANDWIDTH = 8
NUMBER OF EQUATIONS IN A BLOCK = 29
NUMBER OF BLOCKS = 1

NODAL POINT LOADS

NODE NO.	LOAD CASE	APPLIED LOADS RX	RY	RZ	MX	MY	MZ
2	1	10.000	.000	.000	.000	.000	.000
3	2	.000	−25.000	.000	.000	.000	.000
5	1	.000	−50.000	.000	.000	.000	.000
5	2	.000	−25.000	.000	.000	.000	.000
7	2	.000	−25.000	.000	.000	.000	.000
9	1	.000	−100.000	.000	.000	.000	.000
9	2	.000	−25.000	.000	.000	.000	.000
11	2	.000	−25.000	.000	.000	.000	.000
13	1	.000	−75.000	.000	.000	.000	.000
13	2	.000	−25.000	.000	.000	.000	.000
15	2	.000	−25.000	.000	.000	.000	.000

STRUCTURE LOAD CASE	ELEMENT LOAD MULTIPLIERS A	B	C	D
1	.000	.000	.000	.000

NODAL LOADS IN BLOCK NUMBER 1

EQUATION NO	LOAD CASE 1	2
1	10.00000	.00000
2	.00000	.00000
3	.00000	.00000
4	.00000	−25.00000
5	.00000	.00000
6	.00000	.00000
7	.00000	.00000
8	−50.00000	−25.00000
9	.00000	.00000
10	.00000	.00000
11	.00000	.00000
12	.00000	−25.00000
13	.00000	.00000
14	.00000	.00000
15	.00000	.00000
16	−100.00000	−25.00000
17	.00000	.00000
18	.00000	.00000
19	.00000	.00000
20	.00000	−25.00000
21	.00000	.00000
22	.00000	.00000
23	.00000	.00000
24	−75.00000	−25.00000
25	.00000	.00000
26	.00000	.00000
27	.00000	.00000
28	.00000	−25.00000
29	.00000	.00000

NODE DISPLACEMENTS AND ROTATIONS

NODE	LOAD CASE	X	Y	Z	XX	YY	ZZ
16	1	9.682E-01	.000E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	6.660E-01	.000E+00	.000E+00	.00E+00	.00E+00	.00E+00
15	1	8.902E-01	−1.641E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	6.092E-01	−1.206E+00	.000E+00	.00E+00	.00E+00	.00E+00
14	1	−7.486E-02	−1.641E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	−1.624E-02	−1.118E+00	.000E+00	.00E+00	.00E+00	.00E+00
13	1	8.122E-01	−3.125E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	5.523E-01	−1.998E+00	.000E+00	.00E+00	.00E+00	.00E+00
12	1	8.108E-02	−3.284E+00	.000E+00	.00E+00	.00E+00	.00E+00

	2	8.122E-02	-2.130E+00	.000E+00	.00E+00	.00E+00	.00E+00
11	1	6.563E-01	-4.083E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	4.548E-01	-2.690E+00	.000E+00	.00E+00	.00E+00	.00E+00
10	1	2.663E-01	-4.241E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	2.031E-01	-2.734E+00	.000E+00	.00E+00	.00E+00	.00E+00
9	1	4.711E-01	-4.670E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	3.330E-01	-2.926E+00	.000E+00	.00E+00	.00E+00	.00E+00
8	1	4.807E-01	-4.670E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	3.330E-01	-2.926E+00	.000E+00	.00E+00	.00E+00	.00E+00
7	1	2.924E-01	-3.979E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	2.112E-01	-2.690E+00	.000E+00	.00E+00	.00E+00	.00E+00
6	1	6.951E-01	-4.172E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	4.630E-01	-2.734E+00	.000E+00	.00E+00	.00E+00	.00E+00
5	1	1.494E-01	-2.930E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	1.137E-01	-1.998E+00	.000E+00	.00E+00	.00E+00	.00E+00
4	1	8.738E-01	-3.123E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	5.848E-01	-2.130E+00	.000E+00	.00E+00	.00E+00	.00E+00
3	1	7.472E-02	-1.540E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	5.686E-02	-1.206E+00	.000E+00	.00E+00	.00E+00	.00E+00
2	1	1.017E+00	-1.540E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	6.823E-01	-1.118E+00	.000E+00	.00E+00	.00E+00	.00E+00
1	1	.000E+00	.000E+00	.000E+00	.00E+00	.00E+00	.00E+00
	2	.000E+00	.000E+00	.000E+00	.00E+00	.00E+00	.00E+00

TRUSS MEMBER ACTIONS

MEMBER	LOAD CASE	STRESS	FORCE
1	1	3.73622	115.001
1	2	2.84277	87.500
2	1	3.73622	115.001
2	2	2.84277	87.500
3	1	7.14753	220.001
3	2	4.87332	150.001
4	1	8.93442	275.001
4	2	6.09165	187.501
5	1	9.25931	285.001
5	2	6.09165	187.501
6	1	7.79730	240.001
6	2	4.87331	150.001
7	1	3.89865	120.000
7	2	2.84276	87.500
8	1	3.89865	120.000
8	2	2.84276	87.500
9	1	-7.14752	-220.001
9	2	-4.87331	-150.000
10	1	-8.93441	-275.001

10	2	-6.09164	-187.501
11	1	-10.72129	-330.001
11	2	-6.49775	-200.001
12	1	-10.72129	-330.001
12	2	-6.49775	-200.001
13	1	-9.25929	-285.001
13	2	-6.09164	-187.501
14	1	-7.79729	-240.001
14	2	-4.87331	-150.000
15	1	.00000	.000
15	2	4.40141	25.000
16	1	-9.68316	-55.000
16	2	-6.60215	-37.500
17	1	-9.68316	-55.000
17	2	-2.20074	-12.500
18	1	.00000	.000
18	2	.00000	.000
19	1	-7.92260	-45.000
19	2	-2.20075	-12.500
20	1	-7.92259	-45.000
20	2	-6.60215	-37.500
21	1	.00000	.000
21	2	4.40141	25.000
22	1	-13.07157	-148.493
22	2	-10.89297	-123.744
23	1	13.07157	148.493
23	2	7.78070	88.389
24	1	6.84702	77.782
24	2	4.66843	53.033
25	1	6.84699	77.782
25	2	1.55614	17.678
26	1	5.60210	63.640
26	2	1.55616	17.678
27	1	5.60212	63.640
27	2	4.66843	53.033
28	1	14.93892	169.706
28	2	7.78069	88.389
29	1	-14.93892	-169.706
29	2	-10.89296	-123.744

Example 14.2

Analyze the warren truss shown in Fig. 14.2 using the truss and boundary elements in STAP-3D.

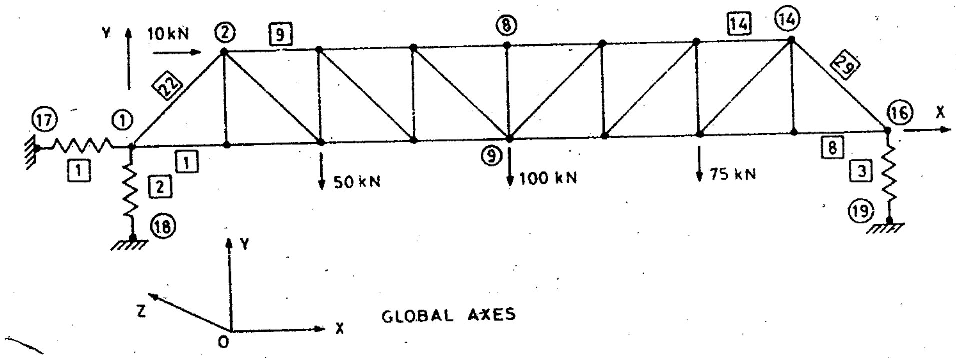


Fig. 14.2 Modelling of supports using boundary elements

Solution

The boundary elements are used to evaluate support reactions. The node and member numbering for the truss are shown in the same figure. The members in each group are numbered from 1 onward. Node numbers 1 and 16 are treated as free or unrestrained. The springs have infinite stiffness and the specified displacement in the direction of each spring is zero. The length of each spring is of the same order as other member lengths.

The input data and part output are shown in EX142.FIL and EX142.DOC. Three astricks *** indicate that only selected output is reproduced here.

EX142.FIL

```
    1
Warren Truss with boundary element, units are kN-cm  Example 14.2
  19    2    2    0
loads at only four points
loads at all panel points of the bottom chord
  1    0    0    1    1    1    1 0.        0.
  2    0    0    1    1    1    1 400.      400.
 14              1    1    1    1 2800.     400.                  2
  3              1    1    1    1 400.
 15              1    1    1    1 2800.     0.                    2
 16    0    0    1    1    1    1 3200.
 17    1    1    1    1    1    1 -400.
 18    1    1    1    1    1    1 0.        -400.
 19    1    1    1    1    1    1 3200.     -400.
  1   29    3
  1 20000.   30.78      0.        0.
  2 20000.   5.68
  3 20000.   11.36
```



```
 0.        0.        0.        0.
 0.        0.        0.        0.
 0.        0.        0.        0.
 0.        0.        0.        0.
   1    1    3    1 0.             2
   7   13   15    1 0.
   8   15   16    1
   9    2    4    1 0.             2
  14   12   14    1 0.
  15    2    3    2 0.             2
  21   14   15    2 0.
  22    1    2    3
  23    2    5    3 0.             2
  25    6    9    3
  26    9   10    3 0.             2
  28   13   14    3 0.
  29   14   16    3
   3    3
   1   17    1    0    0 0.
   1   18    1    0    0 0.
  16   19    1    0    0 0.
   2    1 10.       0.
   3    2         -25.
   5    1         -50.
   5    2         -25.
   7    2         -25.
   9    1         -100.
   9    2         -25.
  11    2         -25.
  13    1         -75.
  13    2         -25.
  15    2         -25.
   0    0
 0.        0.        0.        0.
 0.        0.        0.        0.
```

EX142.DOC

```
*******************************************************************************
STRUCTURAL ANALYSIS PROGRAM STAP-3D
  Warren Truss  with boundary element, units are kN-cm  Example 14.2
*******************************************************************************
NUMBER OF NODAL POINTS        =  19
NUMBER OF ELEMENT TYPES       =  2
NUMBER OF LOAD CASES          =  2
SOLUTION MODE (MODEX)         =  0
```

```
   EQ.0 : EXECUTION
   EQ.1 : DATA CHECK
LOAD CASE NO  1 IS   loads at only four points
LOAD CASE NO  2 IS   loads at all panel points of the bottom chord
***
    BOUNDARY ELEMENTS
    NUMBER OF ELEMENTS   =   3

 NODE   NODE   DEFINING      CODES        SPECF     SPECF.    STIFF     CONSTR.
        CONSTR.   DIR.                    DISPL     ROTAT
  N      NI           KD    KR    KN        D         R         S       NUMBER
  1      17           1     0     0     .00E+00   .00E+00   1.00E+10     1
  1      18           1     0     0     .00E+00   .00E+00   1.00E+10     2
 16      19           1     0     0     .00E+00   .00E+00   1.00E+10     3
***
 CONSTRAINT FORCES

 NUMBER   LOAD        FORCE        MOMENT
          CASE
   1       1      -.10000E+02   .00000E+00
   1       2      -.17108E-03   .00000E+00
   2       1       .10500E+03   .00000E+00
   2       2       .87500E+02   .00000E+00
   3       1       .12000E+03   .00000E+00
   3       2       .87500E+02   .00000E+00
```

Example 14.3

Analyze a three span continuous beam shown in Fig. 14.3a using the beam element of STAP-3D

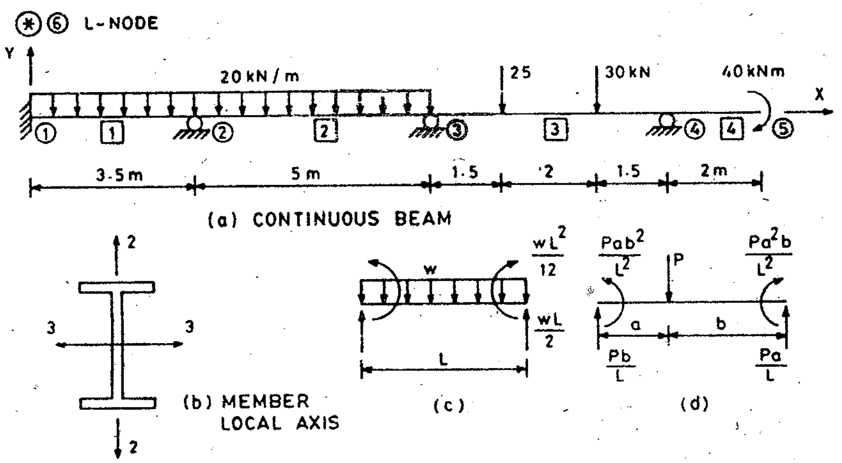


Fig. 14.3

Solution

The node numbers are shown in circles and the member numbers are shown in square boxes. The member properties are shown in Table 14.2

Table 14.2 Member geometric properties

Beam	A cm^2	$I_{1-1} = J$ cm^4	I_{2-2} cm^4	I_{3-3} cm^4	Remark
1,3,4	66.7	32.4	537.7	13630.3	ISMB 350
2	78.46	46.9	622.1	20458.4	ISMB 400

The member local axes are shown in Fig.14.3b. The equivalent nodal loads in the three members are shown in Table 14.3 with respect to Figs.14.3c and 14.3d.

Table 14.3 Fixed end forces

Member	Reaction kN	Moment, kN cm	Remark
1	35. 35.	2042.00 −2042.00	end i
2	50. 50.	4167.0(anticlockwise) − 4167.0(clockwise)	end j
3	26.5 28.5	2782.5 −2992.5	

The input data and complete output are shown in EX143.FIL and EX143.DOC.

EX143.FIL

```
 1
continuous beam kN and cm  Example 14.3
 6    1    1    0
dead load
 1    1    1    1    1    1    1 0.       0.
 2    0    1    1    1    1    0 350.     0.
 3         1    1    1    1    0 850.     0.
 4         1    1    1    1    0 1350.    0.
 5              1    1    1    0 1550.    0.
 6    1    1    1    1    1    1 0.       100.
 2    4    1    2    3
 1  21000.   0.      0.      0.
 1  66.7     0.      0.      32.4     537.7    13630.3
 2  78.46    0.      0.      46.9     622.1    20458.4
0.     0.      0.      0.
0.     0.      0.      0.
0.     0.      0.      0.
 1 0.        35.     0.      0.       0.       2042.0
             35.                               -2042.0
```

```
 2 0.      50.      0.       0.       0.        4167.0
           50.                                  -4167.0
 3 0.      26.5     0.       0.       0.        2782.5
           28.5                                 -2992.5
 1  1  2  6  1  1  1
 2  2  3  6  1  2  2
 3  3  4  6  1  1  3
 4  4  5  6  1  1
 5  1 0.     0.      0.      0.      0.        -4000.0
 0  0
1.    0.     0.      0.
```

EX143.DOC

```
*********************************************************************************
STRUCTURAL ANALYSIS PROGRAM STAP-3D
  continuous beam kN and cm  Example 14.3
*********************************************************************************
```

NUMBER OF NODAL POINTS = 6
NUMBER OF ELEMENT TYPES = 1
NUMBER OF LOAD CASES = 1
SOLUTION MODE (MODEX) = 0
 EQ.0 : EXECUTION
 EQ.1 : DATA CHECK
LOAD CASE NO 1 IS dead load

NODAL POINT INPUT DATA

NODE	BOUNDARY CONDITION CODES						NODAL POINT COORDINATES				
NUM.	X	Y	Z	XX	YY	ZZ	X	Y	Z	KN	TEMP
1	1	1	1	1	1	1	.000	.000	.000	0	.000
2	0	1	1	1	1	0	350.000	.000	.000	0	.000
3	0	1	1	1	1	0	850.000	.000	.000	0	.000
4	0	1	1	1	1	0	1350.000	.000	.000	0	.000
5	0	0	1	1	1	0	1550.000	.000	.000	0	.000
6	1	1	1	1	1	1	.000	100.000	.000	0	.000

THREE DIMENSIONAL BEAM ELEMENTS
NUMBER OF BEAMS = 4
NUMBER OF MATERIALS = 1
NUMBER OF GEOMETRIC PROPERTY SETS = 2
NUMBER OF FIXED END FORCE SETS = 3

MATERIAL	YOUNG S MODULUS	POISSON S RATIO	WEIGHT DENSITY
1	21000.	.00000	.00000

BEAM GEOMETRIC PROPERTIES

ELEMENT TYPE	AREA 1	SHEAR AREA 2	SHEAR AREA 3	INERTIA 1	INERTIA 2	INERTIA 3
1	66.700	.000	.000	32.400	537.700	13630.300
2	78.460	.000	.000	46.900	622.100	20458.400

ELEMENT LOAD MULTIPLIERS FOR GRAVITY LOADS

	A	B	C	D
X-DIR	.000000E+00	.000000E+00	.000000E+00	.000000E+00
Y-DIR	.000000E+00	.000000E+00	.000000E+00	.000000E+00
Z-DIR	.000000E+00	.000000E+00	.000000E+00	.000000E+00

FIXED END FORCES IN LOCAL COORDINATES

TYPE	NODE	FORCE 1	FORCE 2	FORCE3	MOMENT 1	MOMENT 2	MOMENT 3
1	I	.000	35.000	.000	.000	.000	2042.000
	J	.000	35.000	.000	.000	.000	-2042.000
2	I	.000	50.000	.000	.000	.000	4167.000
	J	.000	50.000	.000	.000	.000	-4167.000
3	I	.000	26.500	.000	.000	.000	2782.500
	J	.000	28.500	.000	.000	.000	-2992.500

BEAM	NODES			MATL	GEOM	ELEM		LOADS		LENGTH
NO	I	J	L	NO	NO	A	B	C	D	
1	1	2	6	1	1	1	0	0	0	350.000
2	2	3	6	1	2	2	0	0	0	500.000
3	3	4	6	1	1	3	0	0	0	500.000
4	4	5	6	1	1	0	0	0	0	200.000

TOTAL NUMBER OF EQUATIONS = 9
BANDWIDTH = 5
NUMBER OF EQUATIONS IN A BLOCK = 9
NUMBER OF BLOCKS = 1

NODAL POINT LOADS

NODE	LOAD	APPLIED LOADS					
NO.	CASE	RX	RY	RZ	MX	MY	MZ
5	1	.000	.000	.000	.000	.000	-4000.000

STRUCTURE	ELEMENT LOAD MULTIPLIERS			
LOAD CASE	A	B	C	D
1	1.000	.000	.000	.000

NODAL LOADS IN BLOCK NUMBER 1

EQUATION NO	LOAD	CASE
		1
1	.00000	
2	−2125.00000	
3	.00000	
4	1384.50000	
5	.00000	
6	2992.50000	
7	.00000	
8	.00000	
9	−4000.00000	

NODE DISPLACEMENTS AND ROTATIONS

NODE	LOAD CASE	X	Y	Z	XX	YY	ZZ
6	1	.000E+00	.000E+00	.000E+00	.00E+00	.00E+00	.00E+00
5	1	.000E+00	−4.191E-00	.000E+00	.00E+00	.00E+00	−3.49E-03
4	1	.000E+00	.000E+00	.000E+00	.00E+00	.00E+00	−6.98E-04
3	1	.000E+00	.000E+00	.000E+00	.00E+00	.00E+00	5.16E-04
2	1	.000E+00	.000E+00	.000E+00	.00E+00	.00E+00	−4.49E-04
1	1	.000E+00	.000E+00	.000E+00	.00E+00	.00E+00	.00E+00

BEAM FORCES AND MOMENTS

BEAM NO.	LOAD NO.	AXIAL R1	SHEAR R2	SHEAR R3	TORSION M1	BENDING M2	BENDING M3
1	1	.000E+00	2.871E+01	.000E+00	.000E+00	.000E+00	1.308E+03
		.000E+00	4.129E+01	.000E+00	.000E+00	.000E+00	−3.511E+03
2	1	.000E+00	5.069E+01	.000E+00	.000E+00	.000E+00	3.511E+03
		.000E+00	4.931E+01	.000E+00	.000E+00	.000E+00	−3.165E+03
3	1	.000E+00	2.525E+01	.000E+00	.000E+00	.000E+00	3.165E+03
		.000E+00	2.975E+01	.000E+00	.000E+00	.000E+00	−4.000E+03
4	1	.000E+00	−7.368E-07	.000E+00	.000E+00	.000E+00	4.000E+03
		.000E+00	7.368E-07	.000E+00	.000E+00	.000E+00	−4.000E+03

Example 14.4

Analyze the continuous beam shown in Fig. 14.4 using the beam and boundary elements of STAP-3D.

Solution

The node and member numbering are shown in the same figure. Nodes 2, 3 and 4 are unrestrained nodes with zero displacement in the vertical direction. The input and part output are shown in EX144.FIL and EX144.DOC

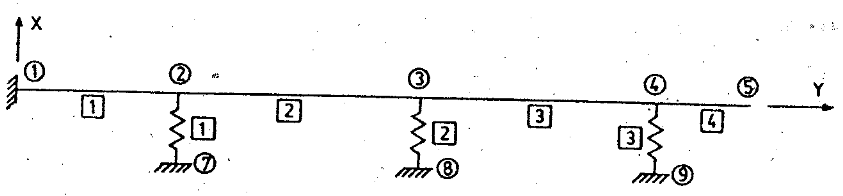


Fig. 14.4

EX144.FIL

```
    1
continuous beam with boundary elements kN and cm  Example 14.4
    9    2    1    0
dead load
    1    1    1    1    1    1    1 0.      0.
    2    0    0    1    1    1    0 350.    0.
    3         0    1    1    1    0 850.    0.
    4         0    1    1    1    0 1350.   0.
    5              1    1    1    0 1550.   0.
    6    1    1    1    1    1    1 0.      100.
    7    1    1    1    1    1    1 350.    -200.
    8    1    1    1    1    1    1 850.    -200.
    9    1    1    1    1    1    1 1350.   -200.
    2    4    1    2    3
    1 21000.   0.        0.        0.
    1  66.7    0.        0.        32.4      537.7     13630 .3
    2 78.46    0.        0.        46.9      622.1     20458 .4
0.        0.        0.        0.
0.        0.        0.        0.
0.        0.        0.        0.
   1 0.        35.       0.        0.        0.        2042.0
               35.                                   -2042 .0
   2 0.        50.       0.        0.        0.        4167.0
               50.                                   -4167 .0
   3 0.        26.5      0.        0.        0.        2782.5
               28.5                                  -2992 .5
   1    1    2    6    1    1    1
   2    2    3    6    1    2    2
   3    3    4    6    1    1    3
   4    4    5    6    1    1
   3    3
   2    7    1    0    0 0.
   3    8    1    0    0 0.
   4    9    1    0    0 0.
   5    1 0.      0.        0.        0.        0.        -4000 .0
   0    0
1.        0.        0.        0.
```

EX144.DOC

****

STRUCTURAL ANALYSIS PROGRAM STAP-3D

continuous beam with boundary elements kN and cm Example 14.4

****

NUMBER OF NODAL POINTS = 9
NUMBER OF ELEMENT TYPES = 2
NUMBER OF LOAD CASES = 1
SOLUTION MODE (MODEX) = 0
EQ.0 : EXECUTION
EQ.1 : DATA CHECK

***

BOUNDARY ELEMENTS
NUMBER OF ELEMENTS = 3

NODE	NODE DEFINING CONSTR. DIR.	CODES			SPECF. DISPL	SPECF. ROTAT	STIFF	CONSTR.
N	NI	KD	KR	KN	D	R	S	NUMBER
2	7	1	0	0	.00E+00	.00E+00	1.00E+10	1
3	8	1	0	0	.00E+00	.00E+00	1.00E+10	2
4	9	1	0	0	.00E+00	.00E+00	1.00E+10	3

***

CONSTRAINT FORCES

NUMBER	LOAD CASE	FORCE	MOMENT
1	1	.91986E+02	.00000E+00
2	1	.74559E+02	.00000E+00
3	1	.29750E+02	.00000E+00

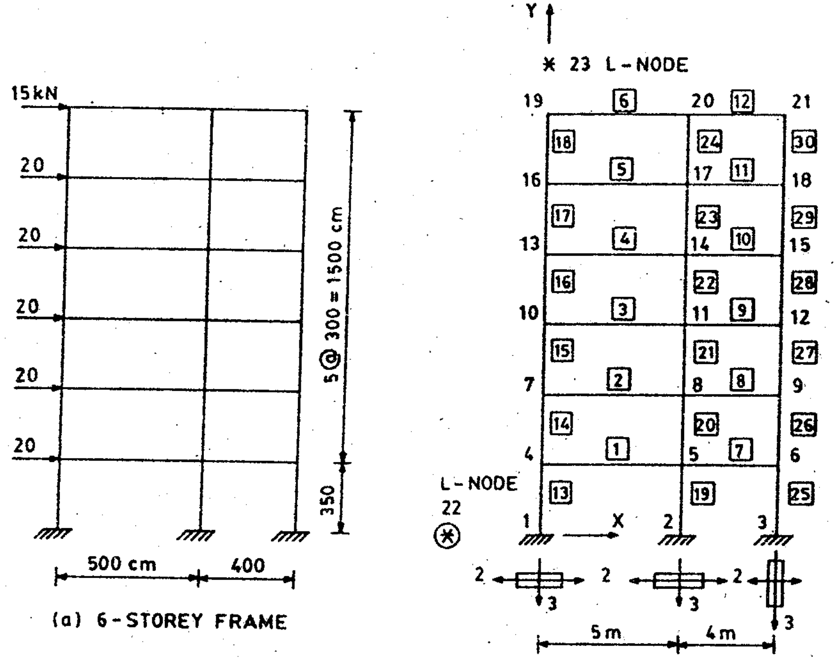
Example 14.5

Analyze a six-storey two bay frame shown in Fig. 14.5a using the beam elements of STAP-3D.

Solution

The node and member numbering are shown in Fig. 14.5b. The columns 13-24 bend about their strong axis and 25-30 bend about their weak axis. The member properties are shown in Table 14.4.

The member axis orientation is specified through L node on the member data line. For beam members, the local 2-2 axis is along the global y-axis, the local 1-1 axis is along the global x-axis, and the axis of bending (local 3-3 axis) is perpendicular to the plane of the paper. Point L is chosen in the x-y plane but not lying on the local 1-1 axis of any beam. Thus, node 23 having (0, 2500.00) coordinates and fully restrained is the L node for all the eight beam elements.



(a) 6-STOREY FRAME

(b) NODE AND MEMBER NUMBERS

Fig. 14.5

For column members (13 –24), the local 2-2 axis is along the global x-axis, the local 1-1 axis is along the global y-axis and the local 3-3 axis is along the global z-axis, that is, perpendicular to the plane of the paper. Node 22 having (–500, 0) coordinates and fully restrained is the L-node for all the above column elements. For columns 25-30, the values of moment of inertia are specified in Table 14.4. The L-node remains same as node 22.

Table 14.4 Member geometric properties

Element	Rectangular section cm	Area cm^2	I_{2-2} cm^4	I_{3-3} cm^4
Beam 1-12	30 × 45	1350	101250	227800
Columns 13-24	30 × 35	1050	78750	107190
Columns 25-30	35 × 30	1050	107190	78750

The dead and live load intensities are 20 kN/m and 10 kN/m, respectively. The fixed end forces for beams of different spans are shown in Table 14.5.

Table 14.5 Fixed end forces

Fixed end force set	Span m	Load	$\frac{wL}{2}$, kN	$\pm\frac{wL^2}{12}$ kN cm
1	5	Dead	50	4166.67
2	5	Live	25	2083.34
3	4	Dead	40	2666..67
4	4	Live	20	1333.34

Clause 5.2.3.1 of IS:456-1978 code specifies the short term modulus of elasticity of concrete at the origin of the stress-strain curve as follows:

$$E_c = 5700\sqrt{\sigma_{ck}} \quad \text{MPa}$$

where σ_{ck} = characteristic strength of concrete at 28-days

Due to creep, the modulus of elasticity gets reduced depending upon the age of concrete at the time of loading. It is, therefore, desirable that the modulus of elasticity should be modified appropriately for creep for use in the structural analysis. The effective modulus of elasticity of M15 grade concrete at 12 months age is given as follows :

$$E_{ce} = \frac{E_c}{1+\theta} = \frac{5700\sqrt{15}}{1+1.1} = 10500 \text{ MPa} \equiv 1050 \text{ kN/cm}^2$$

where θ = creep coefficient

It is interesting to note that modulus of elasticity value affects the frame displacements but not the member forces. In most cases, we are interested in the member forces.

The frame is analyzed for two load cases :

Case 1– Dead load + live load

Case 2– Dead load + 0.5 live load + lateral load

The dead and live loads are specified through the fixed end forces on the element data. Element load case A represents dead load and case B represents live load. The lateral loads are specified through the nodal load data for load case 2.

The structural load data is specified as follows :

Case 1– 1.0 × Element load Case A + 1.0 × Element load Case B

Case 2– 1.0 × Element load Case A + 0.5 × Element load Case B

The nodal loads for case 2 are directly added to the corresponding element loads within the program. The input data and part output are given in EX145.FIL and EX145.DOC.

EX145.FIL

```
   1
Six storey-two bay frame kN-cm  Example 14.5
  23   1   2   0
dead + live load
dead + live load + 0.5 × earthquake
   4   0   0   1   1   1   0 0.      350.
  19           1   1   1     0.      1850.              3
   5   0   0   1   1   1   0 500.    350.
  20           1   1   1     500.    1850.              3.
   6   0   0   1   1   1   0 900.    350.
  21           1   1   1     900.    1850.              3
   1   1   1   1   1   1   1 0.      0.
   2   1   1   1   1   1   1 500.    0.
   3   1   1   1   1   1   1 900.    0.
  22   1   1   1   1   1   1 -500.   0.
  23   1   1   1   1   1   1 0.      2500.
   2  30   1   3   4
   1 1050.   0.    0.
   1 1350.   0.    0.    1.   10125 0.  22780 0.
   2 1050.   0.    0.    1.   78750.    10719 0.
   3 1050.   0.    0.    1.   10719 0.  78750.
0.     0.     0.     0.
0.     0.     0.     0.
0.     0.     0.     0.
 1 0.      50.    0.     0.     0.     4166.67
           50.                         -4166.67
 2 0.      25.    0.     0.     0.     2083.34
           25.                         -2083.34
 3 0.      40.    0.     0.     0.     2666.67
           40.                         -2666.67
 4 0.      20.    0.     0.     0.     1333.34
           20.                         -1333.34
   1   4   5  23   1   1   1   2   0   0   3
   6  19  20  23   1   1   1   2
   7   5   6  23   1   1   3   4           3
  12  20  21  23   1   1   3   4
  13   1   4  22   1   2                   3
  18  16  19  22   1   2
  19   2   5  22   1   2                   3
  24  17  20  22   1   2
  25   3   6  22   1   3                   3
  30  18  21  22   1   3
   4   2 20.
   7   2 20.
```

```
   10     2 20.
   13     2 20.
   16     2 20.
   19     2 15.
    0     0
1.        1.        0.        0.
1.        0.5       0.        0.
```

EX145.DOC

```
*****************************************************************************
STRUCTURAL ANALYSIS PROGRAM STAP-3D
   Six storey- two bay frame  kN-cm   Example 14.5
*****************************************************************************

NUMBER OF NODAL POINTS        =  23
NUMBER OF ELEMENT TYPES       =  1
NUMBER OF LOAD CASES          =  2
SOLUTION MODE (MODEX)         =  0
   EQ.0 : EXECUTION
   EQ.1 : DATA CHECK
LOAD CASE NO  1 IS  dead + live load
LOAD CASE NO  2 IS  dead + live load +  0.5 * earthquake

****
 THREE   DIMENSIONAL   BEAM    ELEMENTS
                          NUMBER OF BEAMS       =    30
                      NUMBER OF MATERIALS       =    1
       NUMBER OF GEOMETRIC PROPERTY SETS        =    3
             NUMBER OF FIXED END FORCE SETS     =    4
```

MATERIAL	YOUNG S MODULUS	POISSON S RATIO	WEIGHT DENSITY
1	1050.	.00000	.00000

BEAM GEOMETRIC PROPERTIES

ELEMENT TYPE	AREA 1	SHEAR AREA 2	SHEAR AREA 3	INERTIA 1	INERTIA 2	INERTIA 3
1	1350.000	.000	.000	1.000	101250.000	227800.000
2	1050.000	.000	.000	1.000	78750.000	107190.000
3	1050.000	.000	.000	1.000	107190.000	78750.000

ELEMENT LOAD MULTIPLIERS FOR GRAVITY LOADS

	A	B	C	D
X-DIR	.000000E+00	.000000E+00	.000000E+00	.000000E+00
Y-DIR	.000000E+00	.000000E+00	.000000E+00	.000000E+00
Z-DIR	.000000E+00	.000000E+00	.000000E+00	.000000E+00

FIXED END FORCES IN LOCAL COORDINATES

TYPE	NODE	FORCE1	FORCE2	FORCE3	MOMENT1	MOMENT2	MOMENT3
1	I	.000	50.000	.000	.000	.000	4166.670
	J	.000	50.000	.000	.000	.000	–4166.670
2	I	.000	25.000	.000	.000	.000	2083.340
	J	.000	25.000	.000	.000	.000	–2083.340
3	I	.000	40.000	.000	.000	.000	2666.670
	J	.000	40.000	.000	.000	.000	–2666.670
4	I	.000	20.000	.000	.000	.000	1333.340
	J	.000	20.000	.000	.000	.000	–1333.340

BEAM	NODES			MATL	GEOM	ELEM LOADS				LENGTH
NO	I	J	L	NO	NO	A	B	C	D	
1	4	5	23	1	1	1	2	0	0	500.000
2	7	8	23	1	1	1	2	0	0	500.000
3	10	11	23	1	1	1	2	0	0	500.000
4	13	14	23	1	1	1	2	0	0	500.000
5	16	17	23	1	1	1	2	0	0	500.000
6	19	20	23	1	1	1	2	0	0	500.000
7	5	6	23	1	1	3	4	0	0	400.000
8	8	9	23	1	1	3	4	0	0	400.000
9	11	12	23	1	1	3	4	0	0	400.000
10	14	15	23	1	1	3	4	0	0	400.000
11	17	18	23	1	1	3	4	0	0	400.000
12	20	21	23	1	1	3	4	0	0	400.000
13	1	4	22	1	2	0	0	0	0	350.000
14	4	7	22	1	2	0	0	0	0	300.000
15	7	10	22	1	2	0	0	0	0	300.000
16	10	13	22	1	2	0	0	0	0	300.000
17	13	16	22	1	2	0	0	0	0	300.000
18	16	19	22	1	2	0	0	0	0	300.000
19	2	5	22	1	2	0	0	0	0	350.000
20	5	8	22	1	2	0	0	0	0	300.000
21	8	11	22	1	2	0	0	0	0	300.000
22	11	14	22	1	2	0	0	0	0	300.000
23	14	17	22	1	2	0	0	0	0	300.000
24	17	20	22	1	2	0	0	0	0	300.000
25	3	6	22	1	3	0	0	0	0	350.000
26	6	9	22	1	3	0	0	0	0	300.000
27	9	12	22	1	3	0	0	0	0	300.000
28	12	15	22	1	3	0	0	0	0	300.000
29	15	18	22	1	3	0	0	0	0	300.000
30	18	21	22	1	3	0	0	0	0	300.000

****

BEAM FORCES AND MOMENTS

BEAM NO.	LOAD NO.	AXIAL R1	SHEAR R2	SHEAR R3	TORSION M1	BENDING M2	BENDING M3
1	1	−1.098E+01	7.147E+01	.000E+00	.000E+00	.000E+00	4.513E+03
		1.098E+01	7.853E+01	.000E+00	.000E+00	.000E+00	−6.278E+03
1	2	−2.177E+00	2.834E+01	.000E+00	.000E+00	.000E+00	−4.890E+03
		2.177E+00	9.666E+01	.000E+00	.000E+00	.000E+00	−1.219E+04
2	1	3.084E-02	7.433E+01	.000E+00	.000E+00	.000E+00	5.339E+03
		−3.084E-02	7.567E+01	.000E+00	.000E+00	.000E+00	−5.672E+03
2	2	1.610E+01	3.688E+01	.000E+00	.000E+00	.000E+00	−2.329E+03
		−1.610E+01	8.812E+01	.000E+00	.000E+00	.000E+00	−1.048E+04
3	1	−1.239E+00	7.573E+01	.000E+00	.000E+00	.000E+00	5.630E+03
		1.239E+00	7.427E+01	.000E+00	.000E+00	.000E+00	−5.267E+03
3	2	1.359E+01	4.369E+01	.000E+00	.000E+00	.000E+00	−5.711E+02
		−1.359E+01	8.131E+01	.000E+00	.000E+00	.000E+00	−8.836E+03
4	1	9.596E-02	7.679E+01	.000E+00	.000E+00	.000E+00	5.857E+03
		−9.596E-02	7.321E+01	.000E+00	.000E+00	.000E+00	−4.964E+03
4	2	1.489E+01	5.031E+01	.000E+00	.000E+00	.000E+00	1.186E+03
		−1.489E+01	7.469E+01	.000E+00	.000E+00	.000E+00	−7.283E+03
5	1	−6.502E+00	7.806E+01	.000E+00	.000E+00	.000E+00	6.254E+03
		6.502E+00	7.194E+01	.000E+00	.000E+00	.000E+00	−4.722E+03
5	2	9.117E+00	5.707E+01	.000E+00	.000E+00	.000E+00	3.074E+03
		−9.117E+00	6.793E+01	.000E+00	.000E+00	.000E+00	−5.790E+03
6	1	2.576E+01	7.314E+01	.000E+00	.000E+00	.000E+00	4.316E+03
		−2.576E+01	7.686E+01	.000E+00	.000E+00	.000E+00	−5.248E+03
6	2	3.222E+01	5.779E+01	.000E+00	.000E+00	.000E+00	2.764E+03
		−3.222E+01	6.721E+01	.000E+00	.000E+00	.000E+00	−5.116E+03

Example 14.6

Analyze a three storey-one bay space frame shown in Fig.14.6a using STAP-3D.

Solution

The node and member numbers are shown in the same figure. The geometric properties of the members are given in Table 14.6.

Table 14.6 Geometric properties

Members	Section cm	β	A cm²	$I_{1\text{-}1}$ cm⁴ = J	$I_{2\text{-}2}$ cm⁴	$I_{3\text{-}3}$ cm⁴
Beams 1-6	30 × 45	0.195	1350	237000	101250	227800
Beams 7-12	30 × 40	0.170	1200	183600	90000	160000
Columns 13-24	30 × 30	0.141	900	114200	67500	67500

Modulus of elasticity $E_{ce} = \dfrac{E_c}{1+\theta} = \dfrac{5700\sqrt{15}}{1+1.1} = 10500 \text{ MPa} \equiv 1050 \text{ kN/cm}^2$

Poisson's ratio $\nu = 0.15$

Torsional inertia $= \beta\, x^3 y$ (where $x < y$)

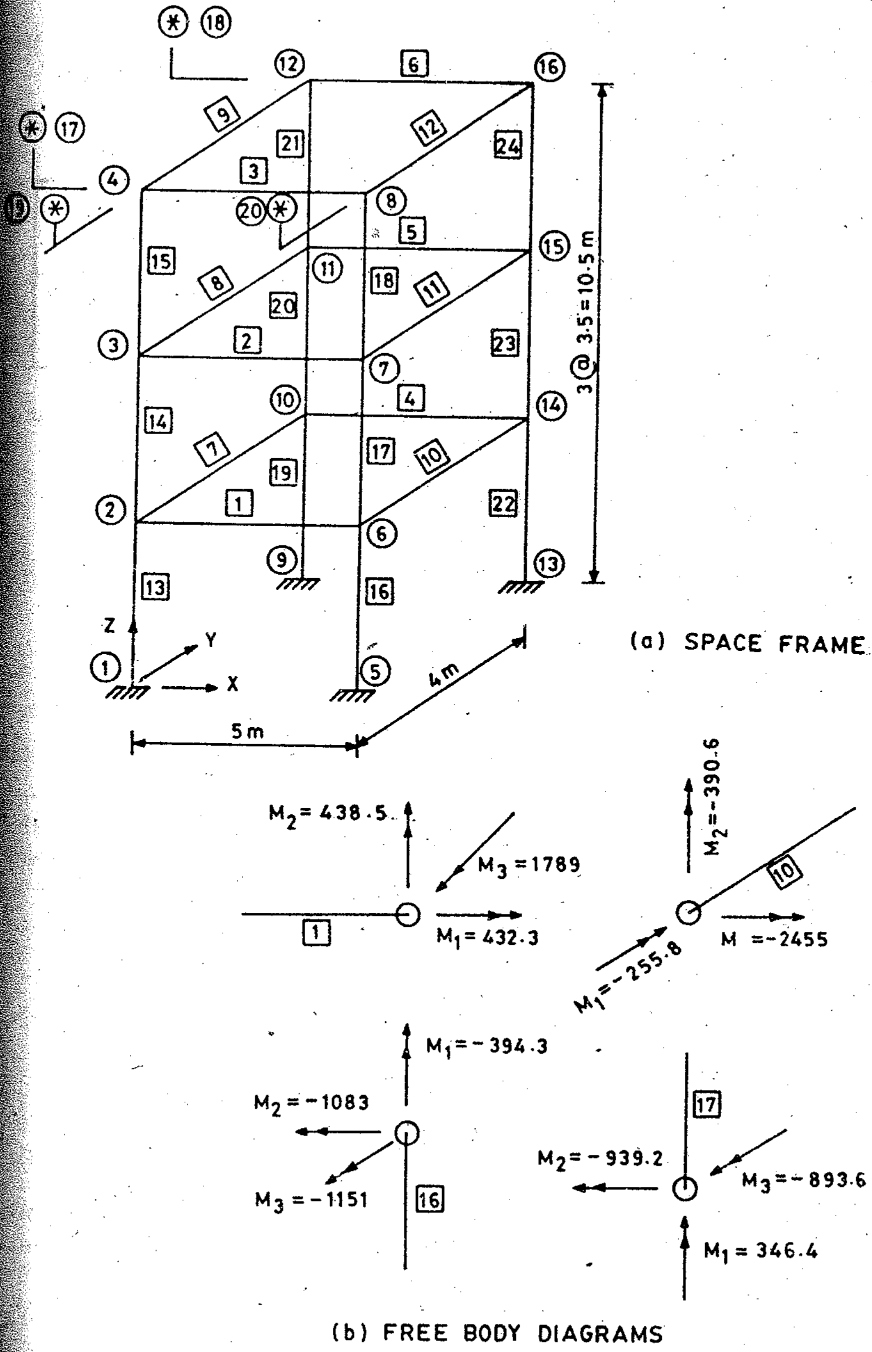


Fig. 14.6

The values of β for different aspect ratios are shown in Table 14.6. The L-node for various members is shown in Table 14.7.

Table 14.7 L-Node for space frame elements

Elements	1-3, 13-15	4-6, 19-21	7-9	10-12	16-18	22-24
L node	17	18	19	20	1	9
(x,y,z) coordinates, meter	(–2, 0, 12)	(–2, 4, 12)	(0, –2, 12)	(5, –2, 12)	(0, 0, 0)	(0, 4, 0)

Thus, a L-node need not be a fully restrained node. The input and partial output is shown in EX146.FIL and EX146.DOC.

EX146.FIL

```
   1
Three storey - one bay three dimensional frame  kN- cm example 14.6
  20    1    1    0
lateral load due to wind
   1    1    1    1    1    1   1 0.       0.        0.
   2                                                 350.
   4                                                 1050.      1
   5    1    1    1    1    1   1 500.               0.
   6                                500.             350.
   8                                500.             1050.      1
   9    1    1    1    1    1   1          400.      0.
  10                                       400.      350.
  12                                       400.      1050.      1
  13    1    1    1    1    1   1 500.    400.      0.
  14                                500.   400.      350.
  16                                500.   400.      1050.      1
  17    1    1    1    1    1   1–200.               1200.
  18    1    1    1    1    1   1–200.   400.        1200.
  19    1    1    1    1    1   1        –200.       1200.
  20    1    1    1    1    1   1500.    –200.       1200.
   2   24    1    3    0
   1 1050.     0.15      0.
   1 1350.                        23700 0.   10125 0.   22780 0.
   2 1200.                        18360 0.    9000 0.   16000 0.
   3  900.                        11420 0.    6750 0.    6750 0.
0.
0.
0.
```



```
   1    2    6   17    1    1    0    0    0    0    1
   3    4    8   17    1    1
   4   10   14   18    1    1                        1
   6   12   16   18    1    1
   7    2   10   19    1    2                        1
   9    4   12   19    1    2
  10    6   14   20    1    2                        1
  12    8   16   20    1    2
  13    1    2   17    1    3                        1
  15    3    4   17    1    3
  16    5    6    1    1    3                        1
  18    7    8    1    1    3
  19    9   10   18    1    3                        1
  21   11   12   18    1    3
  22   13   14    9    1    3                        1
  24   15   16    9    1    3
   2    1 20.
   2    1 0.        25.
   3    1 20.
   3    1 0.        25.
   4    1 20.
   4    1 0.        25.
0.
```

EX146.DOC

```
*******************************************************************************
STRUCTURAL ANALYSIS PROGRAM STAP-3D
Three storey- one bay three dimensional frame  kN-cm  example 14.6
*******************************************************************************

        NUMBER OF NODAL POINTS      =  20
      NUMBER OF ELEMENT TYPES       =  1
          NUMBER OF LOAD CASES      =  1
        SOLUTION MODE (MODEX)       =  0
   EQ.0 : EXECUTION
   EQ.1 : DATA CHECK

LOAD CASE NO 1 IS   lateral load due to wind

****
THREE   DIMENSIONAL   BEAM   ELEMENTS

                        NUMBER OF BEAMS   =  24
                    NUMBER OF MATERIALS   =  1
      NUMBER OF GEOMETRIC PROPERTY SETS   =  3
        NUMBER OF FIXED END FORCE SETS    =  0
```

MATERIAL	YOUNG S MODULUS	POISSON S RATIO	WEIGHT DENSITY
1	1050.	.15000	.00000

BEAM GEOMETRIC PROPERTIES

ELEMENT TYPE	AREA 1	SHEAR AREA 2	SHEAR AREA 3	INERTIA 1	INERTIA 2	INERTIA 3
1	1350.000	000	.000	237000.000	101250.000	227800.000
2	1200.000	000	.000	183600.000	90000.000	160000.000
3	900.000	.000	.000	114200.000	67500.000	67500.000

ELEMENT LOAD MULTIPLIERS FOR GRAVITY LOADS

	A	B	C	D
X-DIR	.000000E+00	.000000E+00	.000000E+00	.000000E+00
Y-DIR	.000000E+00	.000000E+00	.000000E+00	.000000E+00
Z-DIR	.000000E+00	.000000E+00	.000000E+00	.000000E+00

BEAM NO	NODES I	NODES J	NODES L	MATL NO	GEOM NO	ELEM LOADS A	B	C	D	LENGTH
1	2	6	17	1	1	0	0	0	0	500.000
2	3	7	17	1	1	0	0	0	0	500.000
3	4	8	17	1	1	0	0	0	0	500.000
4	10	14	18	1	1	0	0	0	0	500.000
5	11	15	18	1	1	0	0	0	0	500.000
6	12	16	18	1	1	0	0	0	0	500.000
7	2	10	19	1	2	0	0	0	0	400.000
8	3	11	19	1	2	0	0	0	0	400.000
9	4	12	19	1	2	0	0	0	0	400.000
10	6	14	20	1	2	0	0	0	0	400.000
11	7	15	20	1	2	0	0	0	0	400.000
12	8	16	20	1	2	0	0	0	0	400.000
13	1	2	17	1	3	0	0	0	0	350.000
14	2	3	17	1	3	0	0	0	0	350.000
15	3	4	17	1	3	0	0	0	0	350.000
16	5	6	1	1	3	0	0	0	0	350.000
17	6	7	1	1	3	0	0	0	0	350.000
18	7	8	1	1	3	0	0	0	0	350.000
19	9	10	18	1	3	0	0	0	0	350.000
20	10	11	18	1	3	0	0	0	0	350.000
21	11	12	18	1	3	0	0	0	0	350.000
22	13	14	9	1	3	0	0	0	0	350.000
23	14	15	9	1	3	0	0	0	0	350.000
24	15	16	9	1	3	0	0	0	0	350.000

TOTAL NUMBER OF EQUATIONS = 72
BANDWIDTH = 42
NUMBER OF EQUATIONS IN A BLOCK = 72
NUMBER OF BLOCKS = 1

NODAL POINT LOADS

NODE NO.	LOAD CASE	APPLIED LOADS RX	RY	RZ	MX	MY	MZ
2	1	20.000	.000	.000	.000	.000	.000
2	1	.000	25.000	.000	.000	.000	.000
3	1	20.000	.000	.000	.000	.000	.000
3	1	.000	25.000	.000	.000	.000	.000
4	1	20.000	.000	.000	.000	.000	.000
4	1	.000	25.000	.000	.000	.000	.000

STRUCTURE LOAD CASE	ELEMENT LOAD MULTIPLIERS A	B	C	D
1	.000	.000	.000	.000

****

BEAM FORCES AND MOMENTS

BEAM NO.	LOAD NO.	AXIAL R1	SHEAR R2	SHEAR R3	TORSION M1	BENDING M2	BENDING M3
1	1	6.077E-05	7.155E+00	−1.754E+00	−4.323E+02	4.385E+02	1.789E+03
		−6.077E-05	−7.155E+00	1.754E+00	4.323E+02	4.385E+02	1.789E+03
2	1	3.510E-04	4.936E+00	−2.710E+00	−2.684E+02	6.774E+02	1.234E+03
		−3.510E-04	−4.936E+00	2.710E+00	2.684E+02	6.774E+02	1.234E+03
3	1	2.799E-04	1.912E+00	−2.955E+00	−1.047E+02	7.389E+02	4.780E+02
		−2.799E-04	−1.912E+00	2.955E+00	1.047E+02	7.389E+02	4.780E+02
4	1	−1.302E-04	−7.154E+00	−1.741E+00	−4.312E+02	4.352E+02	−1.789E+03
		1.302E-04	7.154E+00	1.741E+00	4.312E+02	4.352E+02	−1.789E+03
5	1	2.612E-04	−4.935E+00	−2.699E+00	−2.685E+02	6.746E+02	−1.234E+03
		−2.612E-04	4.935E+00	2.699E+00	2.685E+02	6.746E+02	−1.234E+03
6	1	4.020E-04	−1.912E+00	−2.945E+00	−1.047E+02	7.361E+02	−4.779E+02
		−4.020E-04	1.912E+00	2.945E+00	1.047E+02	7.361E+02	−4.779E+02
7	1	1.246E+01	−3.701E+01	1.947E+00	−2.558E+02	−3.906E+02	−7.404E+03
		−1.246E+01	3.701E+01	−1.947E+00	2.558E+02	−3.881E+02	−7.400E+03
8	1	1.250E+01	−2.378E+01	2.557E+00	−1.726E+02	−5.128E+02	−4.756E+03
		−1.250E+01	2.378E+01	−2.557E+00	1.726E+02	−5.101E+02	−4.757E+03
9	1	1.249E+01	−9.006E+00	2.778E+00	−6.177E+01	−5.570E+02	−1.801E+03
		−1.249E+01	9.006E+00	−2.778E+00	6.177E+01	−5.543E+02	−1.801E+03
10	1	6.811E-03	−1.227E+01	1.947E+00	−2.558E+02	−3.906E+02	−2.455E+03
		−6.811E-03	1.227E+01	−1.947E+00	2.558E+02	−3.881E+02	−2.454E+03
11	1	5.995E-03	−8.713E+00	2.557E+00	−1.726E+02	−5.128E+02	−1.743E+03
		−5.995E-03	8.713E+00	−2.557E+00	1.726E+02	−5.101E+02	−1.743E+03
12	1	4.791E-03	−3.512E+00	2.778E+00	−6.177E+01	−5.570E+02	−7.025E+02
		−4.791E-03	3.512E+00	−2.778E+00	6.177E+01	−5.543E+02	−7.024E+02
13	1	−5.580E+01	−7.283E+00	3.013E+01	3.943E+02	−5.889E+03	−1.398E+03
		5.580E+01	7.283E+00	−3.013E+01	−3.943E+02	−4.656E+03	−1.151E+03

14	1	−2.594E+01	−5.336E+00	1.934E+01	3.464E+02	−3.181E+03	−8.936E+02
		2.594E+01	5.336E+00	−1.934E+01	−3.464E+02	−3.588E+03	−9.739E+02
15	1	−7.094E+00	−2.778E+00	9.551E+00	1.819E+02	−1.437E+03	−4.326E+02
		7.094E+00	2.778E+00	−9.551E+00	−1.819E+02	−1.906E+03	−5.398E+02
16	1	−3.850E+01	−7.283E+00	7.400E+00	3.943E+02	−1.507E+03	−1.398E+03
		3.850E+01	7.283E+00	−7.400E+00	−3.943E+02	−1.083E+03	−1.151E+03
17	1	−1.907E+01	−5.336E+00	5.653E+00	3.464E+02	−9.392E+02	−8.936E+02
		1.907E+01	5.336E+00	−5.653E+00	−3.464E+02	−1.039E+03	−9.739E+02
18	1	−5.424E+00	−2.778E+00	2.950E+00	1.819E+02	−4.347E+02	−4.326E+02
		5.424E+00	2.778E+00	−2.950E+00	−1.819E+02	−5.978E+02	−5.398E+02
19	1	5.580E+01	7.282E+00	3.007E+01	3.935E+02	−5.877E+03	1.398E+03
		−5.580E+01	−7.282E+00	−3.007E+01	−3.935E+02	−4.647E+03	1.151E+03
20	1	2.594E+01	5.335E+00	1.935E+01	3.464E+02	−3.184E+03	8.935E+02
		−2.594E+01	−5.335E+00	−1.935E+01	−3.464E+02	−3.589E+03	9.739E+02
21	1	7.095E+00	2.778E+00	9.548E+00	1.818E+02	−1.436E+03	4.325E+02
		−7.095E+00	−2.778E+00	−9.548E+00	−1.818E+02	−1.906E+03	5.397E+02
22	1	3.850E+01	7.282E+00	7.402E+00	3.935E+02	−1.507E+03	1.398E+03
		−3.850E+01	−7.282E+00	−7.402E+00	−3.935E+02	−1.084E+03	1.151E+03
23	1	1.907E+01	5.335E+00	5.654E+00	3.464E+02	−9.395E+02	8.935E+02
		−1.907E+01	−5.335E+00	−5.654E+00	−3.464E+02	−1.039E+03	9.739E+02
24	1	5.424E+00	2.778E+00	2.950E+00	1.818E+02	−4.346E+02	4.325E+02
		−5.424E+00	−2.778E+00	−2.950E+00	−1.818E+02	−5.977E+02	5.397E+02

In a 3-D problem, the interpretation of results becomes quite difficult. The static check on moments becomes tedious because the member forces are printed in local coordinates and orientation of local axis may be different for each member meeting at a joint . Let us check the moment equilibrium at node 6. The moments in the members meeting at joint 6 are shown in Fig. 14.6b. The double arrows indicate the positive direction of the moment in local member axes. Net moment along each of the three global axes is as follows :

$$M_x = M_1^1 + M_{10}^3 - M_{16}^2 - M_{17}^2$$
$$= 432.3 - 2455 + 1083 + 939.2 = 0 \qquad \text{O.K.}$$

$$M_y = M_1^3 - M_{10}^1 + M_{16}^3 + M_{17}^3$$
$$= 1789 + 255.8 - 1151 - 893.6 = 0 \qquad \text{O.K.}$$

$$M_z = M_1^2 + M_{10}^2 + M_{16}^1 + M_{17}^1$$
$$= 438.5 - 390.6 - 394.3 + .346.4 = 0 \qquad \text{O.K.}$$

The superscript indicates the local axis, while subscript indicates the member number.

Example 14.7

Illustrate the effect of relative stiffness of beam and column on the moments produced in a frame

Solution

Consider a single storey portal frame as shown in Fig.14.7a. It is subjected to a point load of 10 kN at the mid span of the beam. The relative stiffness of the beam and column is given by

$$k = \frac{(I/L)\ \text{beam}}{(I/L)\ \text{column}}$$

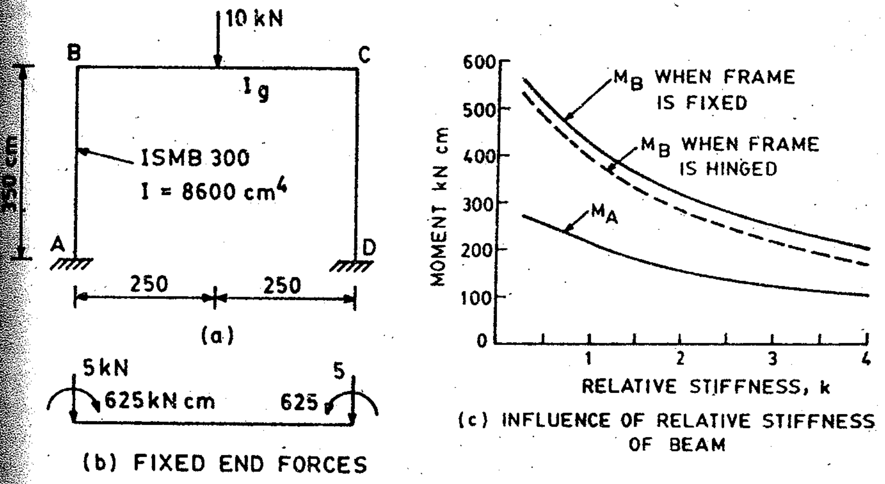


Fig. 14.7

The column has a span of 350 cm and consists of ISMB 300 section. The beam has a span of 500 cm. The moment of inertia of the beam is varied such that the relative stiffness varies as 0.25, 0.5, 1.0, 1.5, 2.0, 2.5, 3, 3.5 and 4.0. The data of the frame is prepared for STAP-3D and the program is executed for different values of relative stiffnesses. The point load of 10 kN is replaced with the equivalent nodal forces as shown in Fig. 14.7b. The values of moment at A and B are plotted in Fig.14.7c for different values of the relative stiffnesses. It is seen that the moment produced in the frame decreases as the relative stiffness of the beam increases.

In any case, the joint equilibrium has to be maintained. The distribution factor at B in the span BC increases with the increase in beam stiffness which results in decrease in the net moment produced in the joint.

The problem is repeated by taking the portal frame as pin-ended. The moment at joint B is plotted in the same figure. It is seen that the moment is consistently lower than that in the frame with fixed ends. The decrease in the moment produced is attributed to the decrease in the frame stiffness.

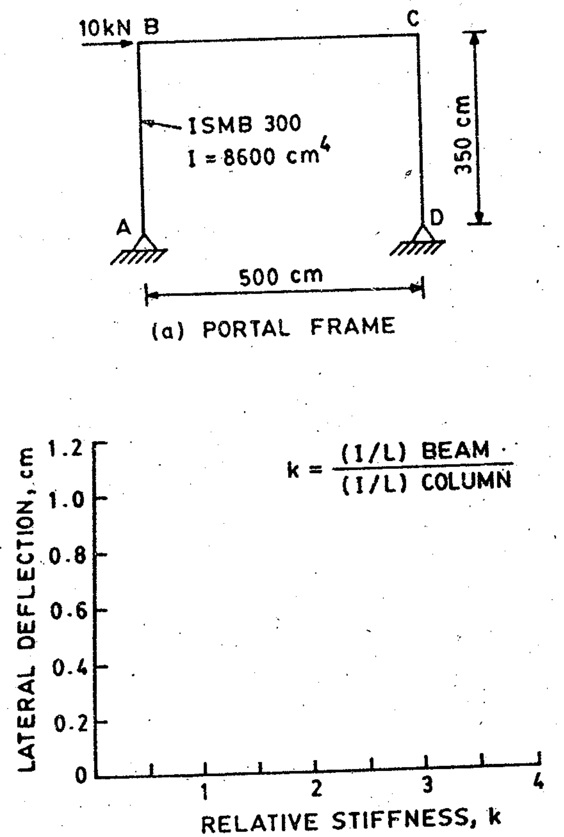
Example 14.8

Illustrate the effect of relative stiffness of beam and column on the lateral deflection in a frame.

Solution

Consider a portal frame shown in Fig. 14.8 which is the same as used in the previous example. The columns are pin-ended. The moment of inertia of the beam is changed such that the relative stiffness varies as 0.25, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5 and 4.0. The lateral deflection of the frame due to a lateral point load of 10 kN is computed using the program STAP-3D. The deflection is plotted in Fig. 14.8b. It can be seen that the lateral deflection decreases with the increase in beam stiffness. However, the decrease in lateral deflection is steeper when the relative beam stiffness increases from 0.25 to 1.0. Beyond a relative stiffness of 2.0, the lateral deflection is nearly constant.

This information is quite useful in controlling storey drift in frames due to wind or earthquake loads.



(b) INFLUENCE OF RELATIVE STIFFNESS OF BEAM

Fig. 14.8

CHAPTER
fifteen

NON-LINEAR ANALYSIS : MATERIAL NON-LINEARITY

15.1 INTRODUCTION

There are two different forms of non-linearity in structural analysis. The first is due to non-linear material behavior and is usually referred to as material non-linearity. The second is geometric non-linearity which is caused by large deformations leading to appreciable changes in the geometry of the structure. In a structure, there may be either material non-linearity, or geometric non-linearity or both. In the previous chapters, it was assumed that the material remains within the elastic limit and that the deformations developed in the structure are small.

If stress-strain relationship of the material is non-linear, the behavior of the structure also becomes non-linear. The displacements may still be considered small so that the kinematic relationships remain linear. In the present chapter, the methods of analysis of material non-linearity in structures consisting of truss and beam elements are discussed. The classical theory of plastic analysis is discussed first to prepare the necessary back ground. The modifications in the stiffness matrices of a 2-D truss and a 2-D beam elements are then suggested along with the solution algorithm. The geometric non-linearity will be discussed in the next chapter.

15.2 STRESS-STRAIN CURVE OF STEEL

The uniaxial stress-strain curve of steel shows three distinct regions : elastic, yield and strain-hardening as shown in Fig. 15.1. At yield the material flows and the strain increases from ϵ_y to ϵ_{st} at nearly constant stress. Beyond ϵ_{st}, strain hardening of the material takes place. For Fe 250 grade steel, the strain values are as follows :

yield stress σ_y = 250 MPa
yield strain ϵ_y = 0.12%
strain hardening ϵ_{st} = 1.5 %
max. strain ϵ_{max} = 25%
modulus of elasticity E = 2×10^5 MPa

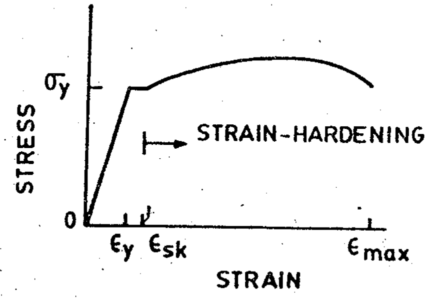


Fig. 15.1 Stress-strain curve of mild steel

The idealized stress-strain curves are shown in Fig. 15.2. Figure 15.2a shows perfectly elastic curve, Fig. 15.2b shows a rigid plastic curve, while Fig. 15.2c shows a perfectly elastic-plastic curve. There is no strain-hardening. The theory of plastic analysis assumes a perfectly elastic-plastic stress-strain curve for steel.

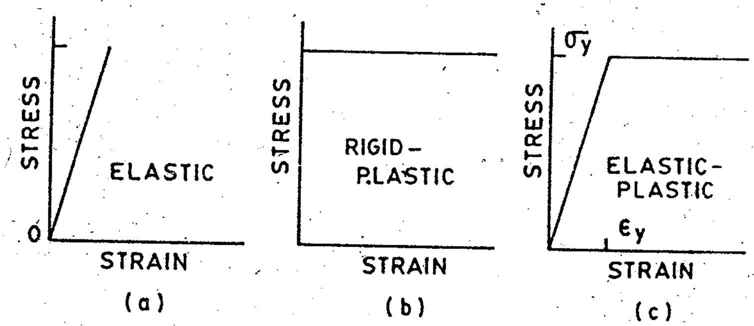


Fig. 15.2 Idealized stress-strain curves

15.3 THEORY OF PLASTIC ANALYSIS

The theory of plastic analysis is based upon the following assumptions :

1. Plane sections remain always plane and normal to the axis of bending.
2. The stress-strain relationship is assumed to be perfectly elastic-plastic as discussed earlier.
3. The deformations are assumed to be small.
4. There is no axial load on the beam.

Let us determine the moments in a simply supported beam under increasing lateral load.

Elastic Stage

The fibre strains as well as the stresses increase linearly with the load as long as the beam remains within the elastic range as shown in Fig. 15.3a for a rectangular section. The moment is given by :

$$\frac{M}{I} = \frac{\sigma}{y} \tag{15.1}$$

or,

$$M = S\sigma \tag{15.2}$$

where, S = elastic section modulus
σ = stress in extreme fibre

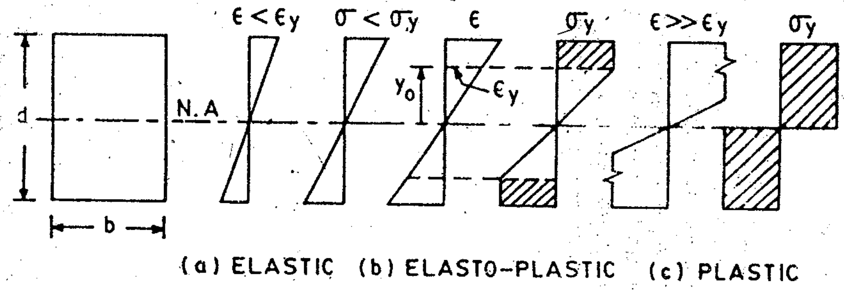


Fig. 15.3 Stress-strain curve in a rectangular section

When ϵ equals ϵ_y, the stress $\sigma = \sigma_y$, and the section develops yield moment, that is,

$$M_y = \sigma_y S \tag{15.3}$$

The moment M_y associated with the first yield is called the yield moment.

Elasto-Plastic Stage

Under increasing moment, yielding will take place at the extreme fibre. As the moment is increased, yielding will progress towards the interior of the beam as shown in Fig.15.3b. Taking moment of forces about the neutral axis :

$$M = 2\left[\sigma_y \frac{bd}{2}\frac{d}{4} - \sigma_y \frac{by_0}{2}\frac{y_0}{3}\right]$$

$$= \sigma_y \frac{bd^2}{6}\left[\frac{3}{2} - \frac{2y_0^2}{d^2}\right] = M_y\left[\frac{3}{2} - \frac{2y_0^2}{d^2}\right] \tag{15.4}$$

Thus, moment capacity M is greater than the yield moment M_y, and its value increases with the decrease in y_0.

Plastic Stage

Yielding will progress towards the interior of the beam until the section is fully yielded as shown in Fig. 15.3c. At this point,

Compressive force $C = \sigma_y A_c$ (15.5a)

Tensile force $T = \sigma_y A_t$ (15.5b)

and because C is equal to T for equilibrium,

$$A_c = A_t \qquad (15.5c)$$

where, A_c and A_t are the compressive and tensile areas, respectively and C and T the resulting compressive and tensile forces, respectively.

The moment M_p associated with yielding over the entire depth of a beam is called the plastic moment.

$$M_p = \text{force} \times \text{lever arm}$$

$$= \sigma_y \frac{bd}{2} \times \frac{d}{2} = \sigma_y \frac{bd^2}{4}$$

$$= \frac{3}{2} \sigma_y \frac{bd^2}{6} = 1.5\, M_y \qquad (15.6)$$

Also, $M_p = Z\, \sigma_y$ (15.7a)

where Z = plastic section modulus

$$= A_c \overline{y_1} + A_t \overline{y_2} \qquad (15.7b)$$

$\overline{y_1}, \overline{y_2}$ = c.g. of the compression and tension areas about the neutral axis

The neutral axis under the plastic condition divides the section into two equal parts, that is, $A_c = A_t$. The ratio of the plastic moment M_p to the yield moment M_y is called the shape factor f, thus

$$f = \frac{M_p}{M_y} \qquad (15.8)$$

15.4 PLASTIC HINGE AND MECHANISM

Consider the simply supported beam of Fig. 15.4a loaded by a concentrated load at the mid span. As the load increases beyond that causes the first yielding, yielding proceeds not only towards the centre of the beam, but it also moves out. The beam will collapse when the centre section becomes fully plastic. This section will keep on rotating at constant moment and the vertical deflection will also keep on increasing. Thus, the beam will transform into a mechanism consisting of two links with a plastic hinge in the middle. In all there are three hinges in a simply supported beam: two elastic hinges, one at each support, and one plastic hinge within the span. Such a plastic hinge has no additional moment resistance. The plastic hinge is not a section but it is the zone of yielding near the section of full plasticity. The shaded portion in Fig. 15.4b shows the plastic zone. The length of plastic zone depends upon the shape factor. Greater is this ratio, the larger will be the length of the plastic zone. For flanged sections, the value of f varies from 1.10 to about 1.16.

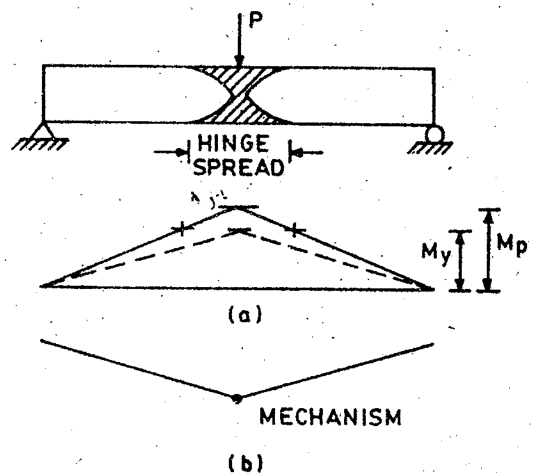


Fig. 15.4 Plastic hinge and a mechanism

A full plastic hinge will develop only under ideal conditions. The factors which influence the formation of the plastic hinge are :

(1) Local buckling
(2) lateral-torsional buckling
(3) axial force, and
(4) shear force.

With the formation of a plastic hinge in a structure, there is a redistribution of internal forces, causing other sections to reach their full strength and develop plastic hinges. This process goes on until the structure forms a collapse mechanism. That is, the deflection keeps on increasing without any increase in the loading. The loading associated with the onset of collapse is called the *ultimate load*. The ratio of ultimate load to working load is called *load factor*. Thus,

$$\text{Load factor} = \frac{\text{ultimate load}}{\text{working load}} \qquad (15.9)$$

A mechanism is a condition of instability. Mathematically the degree of statical indeterminacy of a structure at the verge of a mechanism is negative.

15.5 MOMENT-CURVATURE RELATION

For an elastic section, the curvature ϕ is given by

$$\phi = \frac{1}{R} = \frac{\epsilon}{y} \qquad (15.10a)$$

where, y = distance of the fibre from the neutral axis

$\in$ = strain at the fibre
R = radius of curvature

The moment-curvature relation is given by

$$\frac{M}{M_y} = \frac{\in}{\in_y} = \frac{\phi}{\phi_y} \tag{15.10b}$$

For an elasto-plastic rectangular section

$$M = M_y\left[\frac{3}{2} - \frac{2y_o^2}{d^2}\right] \quad \text{also} \quad \phi_y = \frac{\in_y}{0.5d}$$

setting,

$$\frac{y_0}{d} = \frac{y_0}{\in_y}\frac{\in_y}{d} = \frac{1}{\phi}\frac{\phi_y}{2} = \frac{\phi_y}{2\phi}$$

or,

$$\frac{M}{M_y} = \frac{3}{2}\left[1 - \frac{1}{8}\left(\frac{\phi_y}{\phi}\right)^2\right] \tag{15.10c}$$

The equations 15.10b and 15.10c are plotted in Fig. 15.5. A straight line is obtained in the elastic range, while a curve is obtained in the plastic range which becomes asymptotic to the line when M becomes equal to the plastic moment. This shows that when the section becomes fully plastic, it undergoes infinite curvature. Thus, infinite rotation can occur at a section under the full plastic moment, and it can behave as a plastic hinge without application of any additional moment.

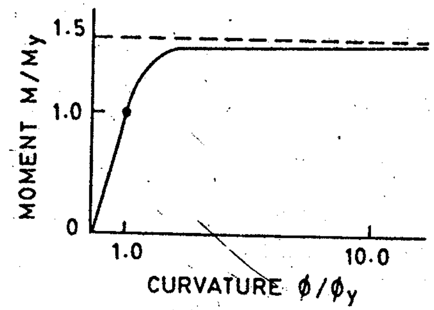


Fig. 15.5 M-ϕ curve for a rectangular section

15.6 PLASTIC ANALYSIS

Any plastic analysis must satisfy three conditions :

1. Static equilibrium condition : The structure is under static equilibrium until collapse.
2. Plastic moment condition : The moment no where must exceed the plastic moment capacity M_p of the section under consideration .
3. Mechanism condition : There are just sufficient hinges when a collapse mechanism forms.

If a structure has a degree of statical indeterminacy equal to α_s ,then it will require α_s hinges to be a statically determinate structure. Therefore, just one more hinge will be sufficient to convert it into a mechanism. Therefore, minimum number of plastic hinges required to convert a statically indeterminate structure into a mechanism is equal to $\alpha_s + 1$.

There are two methods for analyzing beams and frames when using the plastic analysis :

1. Statical or equilibrium method.
2. Mechanism or virtual-work method.

The statical method satisfies the first two conditions; the mechanism method satisfies the last two conditions.

Statical Method

This method consists in constructing a bending moment diagram in which $M \leq M_p$ at each section in the structure, such that a collapse mechanism is formed. There is no need to analyze the structure using the stiffness or flexibility methods. The various steps are as follows :

Step 1 Remove redundant forces such that a statically determinate structure is available.

Step 2 Draw a suitable bending moment diagram at failure due to the applied loads.

Step 3 Draw a suitable bending moment diagram at failure due to the redundant forces.

Step 4 Superimpose the two moments diagrams such that a mechanism is formed.

Step 5 Compute the ultimate load using static equilibrium equations.

Step 6 Check the moments to see that $M \leq M_p$ every where in the structure.

If M exceeds M_p anywhere, repeat steps 2 to 6 so that a plastic hinge is formed at the section of maximum bending moment and the other basic conditions are satisfied. Sometimes there may be more than one possible bending moment diagram. In such a situation draw all possible bending moment diagrams, check various conditions and determine the collapse loads. The minimum collapse load among them is the *true collapse load.* The statical method gives a lower bound on the ultimate load.

Example 15.1

Determine the ultimate load for the fixed beam shown in Fig. 15.6a. The beam is prismatic with plastic moment capacity equal to M_p.

Solution

The degree of statical indeterminacy α_s = 2. Number of hinges required to form a mechanism = 2 + 1 = 3. Two hinges can be assumed to form at each support and one at the mid-span. The bending moment diagrams are shown in Figs. 15.6b and 15.6c. The equilibrium equation at the mid-span section is written as :

$$M_p + M_p = \frac{w_u L^2}{8}$$

or, $$w_u = \frac{16 M_p}{L^2}$$

It is obvious that no where $M > M_p$. The beam mechanism is shown in Fig.15.6c. O. K.

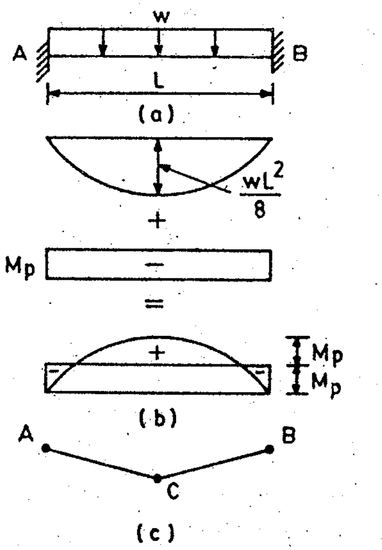


Fig. 15.6 Fixed end beam: statical method

Example 15.2

Consider the continuous beam shown in Fig. 15.7a. It has a plastic moment capacity of M_p over the exterior spans AB and CD, and 1.5 M_p over the interior span BC. Determine the collapse load.

Solution

Since there are three spans, there can be three independent beam mechanisms. The bending moment diagrams due to the applied loads assuming each span to be a simply supported span is shown in Fig. 15.7b. The moment diagram due to the redundant end moments in each span is shown in Fig. 15.7 c. The two moment diagrams are superimposed as shown in Fig. 15.7d.

Mechanism 1 - Span AB

Support A is simply supported. Hence only two plastic hinges are required to form a beam mechanism. One hinge will form just to the left of support B in the span AB since moment capacity of the span BC is higher. Let us assume that the other hinge will form at a distance x from end A and not at the mid span.

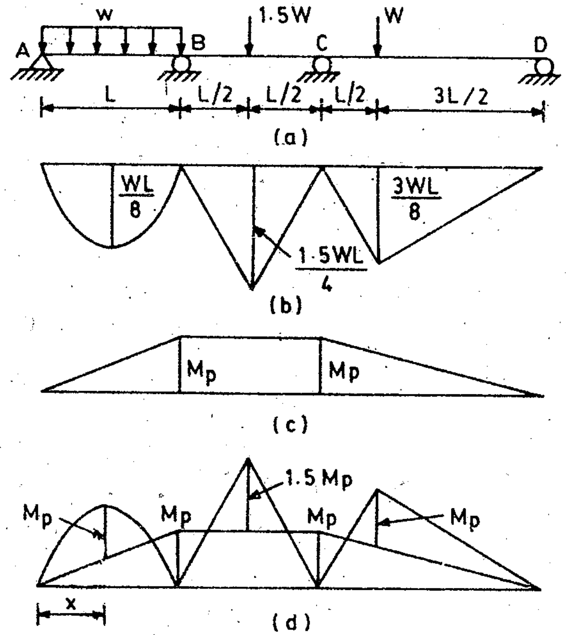


Fig. 15.7 Continuous beam : statical method

Reaction $R_A = \frac{W}{2} - \frac{M_p}{L}$

Shear force at x $= \frac{W}{2} - \frac{M_p}{L} - \frac{W}{L}x$

$= 0$ for maximum moment to occur at x

$$\therefore \quad x = \frac{L}{2} - \frac{M_p}{W}$$

$\therefore$ Maximum moment at x

$$M_{max} = \left(\frac{W}{2} - \frac{M_p}{L}\right)\left(\frac{L}{2} - \frac{M_p}{W}\right) - \frac{W}{2L}\left(\frac{L}{2} - \frac{M_p}{W}\right)^2$$

For plastic hinge to form at x, $M_{max} = M_p$

or, $$M_p = \left(\frac{W}{2} - \frac{M_p}{L} - \frac{W}{4} + \frac{M_p}{2L}\right)\left(\frac{L}{2} - \frac{M_p}{W}\right)$$

$$= \left(\frac{W}{4} - \frac{M_p}{2L}\right)\left(\frac{L}{2} - \frac{M_p}{W}\right)$$

or, $$M_p = \frac{WL}{2}\left(\frac{1}{2} - \frac{M_p}{WL}\right)^2$$

setting $$\frac{M_p}{WL} = m, \quad 2m = \left(\frac{1}{2} - m\right)^2$$

or, $$4m^2 - 12m + 1 = 0 \quad \text{or,} \quad m = 2.914 \text{ or } 0.0858$$

$$\therefore \quad W = W_u = \frac{M_p}{2.914L} = 0.343\frac{M_p}{L}$$

or, $$W_u = \frac{M_p}{0.0858L} = 11.65\frac{M_p}{L}$$

$$\therefore \quad x = 0.5L - 0.0858L = 0.414\,L$$

It will be seen that $m = 2.914$ gives $x = -2.414L$ which is inadmissible. Hence the collapse load

$$W_u = 11.65\frac{M_p}{L} \quad \text{and} \quad x = 0.414L \tag{i}$$

Mechanism 2 - Span BC

It is a continuous span having statical indeterminacy equal to 2. It needs three plastic hinges to form a mechanism. Two plastic hinges will form just outside the supports B and C where the moment capacities are M_p and not $1.5\,M_p$. The other hinge will form at the midspan. The equilibrium equation is given by

$$M_p + 1.5\,M_p = 1.5\frac{WL}{4}$$

or, $$W = W_u = \frac{10}{1.5}\frac{M_p}{L} = 6.67\frac{M_p}{L} \tag{ii}$$

Mechanism 3 - Span CD

This span needs only two plastic hinges to form a mechanism. One hinge will form at end C and the other at the location of the point load. The equation of equilibrium is given as :

$$M_p + \frac{3}{4}M_p = \frac{3WL}{8}$$

or, $$W = W_u = \frac{7 \times 8}{12}\frac{M_p}{L} = 6.67\frac{M_p}{L} \tag{iii}$$

The ultimate load is the smallest of the three collapse loads, that is,

$$W_u = 6.67\frac{M_p}{L}$$

Example 15.3

A portal frame is shown in Fig. 15.8a. Determine its collapse load.

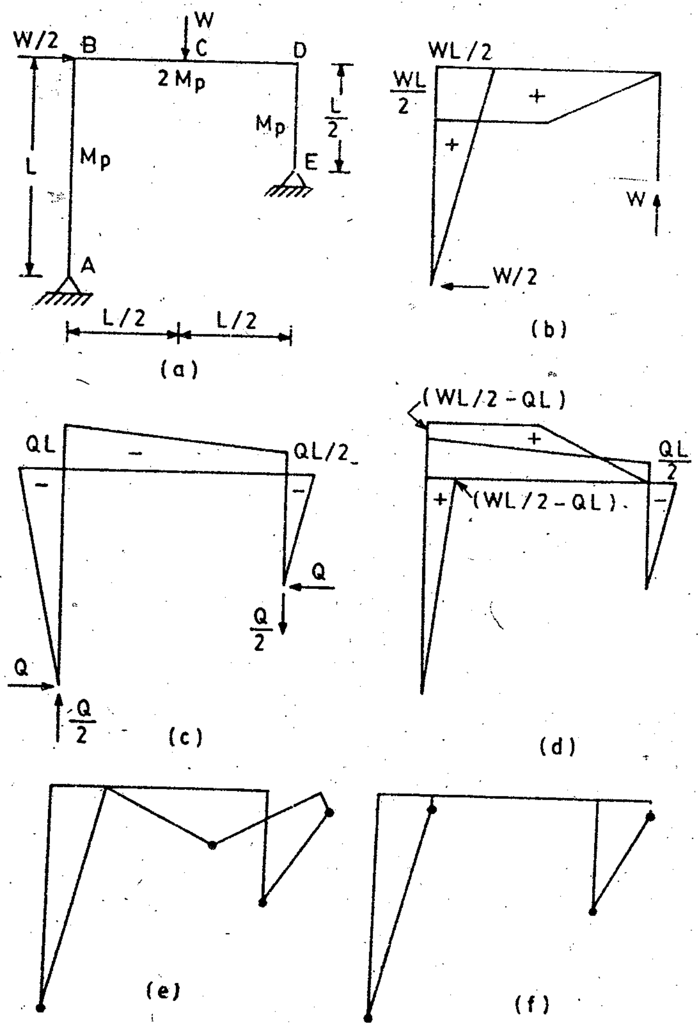


Fig. 15.8 Portal frame : statical method

Solution

The frame is statically indeterminate to a degree 1. Let the horizontal reaction at E be the redundant and equal to Q. The vertical reactions at A and E are given by :

$$R_{Ey} = \left(\frac{WL}{2} + \frac{WL}{2}\right)\frac{1}{L} = W, \quad R_{Ay} = 0, \quad R_{Ax} = \frac{W}{2}$$

There are natural hinges at the supports A and E. The moment diagrams due to the applied loads and the horizontal thrust are shown in Figs. 15.8b and 15.8c. These two diagrams should be so combined that a mechanism is formed. At least two plastic hinges are required to form a mechanism. Let us assume that two plastic hinges form at C and D (Fig. 15.8e). Let us write the equilibrium equations at C and D.

At C $\quad \frac{WL}{2} - \frac{3}{4}QL = 2\,M_p \qquad (1)$

At D $\quad \frac{QL}{2} = M_p \quad$ or, $\quad Q = Q_u = 2\frac{M_p}{L} \qquad (2)$

$\therefore$ Eq. 1 gives,

$$\frac{WL}{2} = 2M_p + \frac{3}{4}(2M_p) = 3.5\,M_p$$

or, $\quad W = W_u = 7\frac{M_p}{L}$

Check

Moment at B, $\quad M_B = \frac{WL}{2} - QL = 3.5\,M_p - 2\,M_p$

$\qquad = 1.5\,M_p < 2\,M_p$

$\qquad > M_p \qquad$ NOT O. K.

Since moment at B can not exceed M_p in the span AB, the assumed mechanism is incorrect.

Let us assume two hinges form at B and D as shown in Fig.15.8f. The equilibrium equations are :

At B $\quad \frac{WL}{2} - QL = M_p \qquad (i)$

or, $\quad W = (QL + M_p)\frac{2}{L}$

At D $\quad \frac{QL}{2} = M_p \quad$ or, $\quad Q = Q_u = 2\frac{M_p}{L} \qquad (ii)$

$\therefore$ Eq. (i) gives,

$$W = W_u = 6\frac{M_p}{L}$$

Check

Moment at C, $\quad M_c = \frac{WL}{2} - \frac{3}{4}QL = 3M_p - 1.5\,M_p$

$\qquad = 1.5\,M_p < 2\,M_p \qquad$ O.K.

Hence, the collapse load is $W_u = 6\frac{M_p}{L}$

Mechanism Method

This method consists in assuming a mechanism and writing the equilibrium equations using the virtual displacement method. These equations are solved for the ultimate load.

This method is very convenient when the structure is highly indeterminate. The various steps are as follows :

Step 1 Determine the degree of statical indeterminacy α_s and the total possible locations of the plastic hinges NPLASTIC.
Therefore, number of independent mechanisms
$= \text{NPLASTIC} - \alpha_s$

Step 2 Select possible independent and combined mechanisms. The independent mechanisms are known as *beam mechanism*, *sway mechanism* or *joint mechanism* as shown in Figs. 15.9a-f. They may occur independent of the failure of the structure as a whole, and may lead to partial failure of the structure.

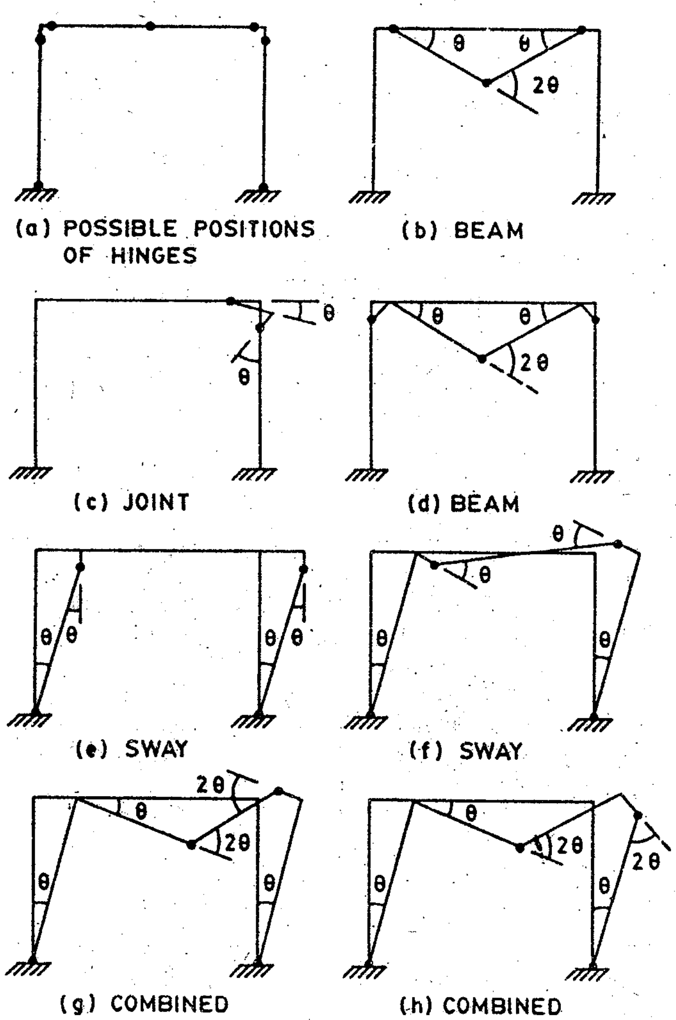


Fig. 15.9 Mechanisms

The *combined mechanism* is formed with the combination or superposition of the beam and sway mechanisms as shown in Fig. 15.9 g, h. The correct mechanism or mechanisms are those which satisfy the other two basic conditions for the plastic analysis.

Step 3 Determine the lowest ultimate load by solving the equilibrium equations.

Step 4 Check the moment to see that $M \leq M_p$ everywhere in the structure.

The mechanism method gives an upper bound on the ultimate load or collapse load.

Sign Convention

The following sign convention is used for the analysis of structures :

1. A dotted line is drawn at the inner side of the structure.
2. All possible mechanisms are drawn individually.
3. Open angles are taken as positive and closing angles are taken as negative.
4. Moment causing tension on the dotted side is taken as positive, and that causing compression is taken as negative.

Example 15.4

Analyze the fixed beam shown in Fig. 15.10a by the mechanism method.

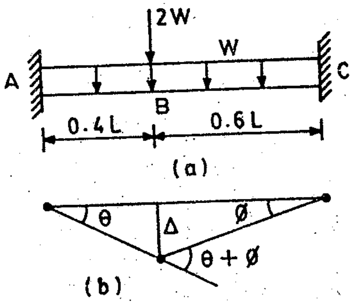


Fig. 15.10 Fixed end beam : mechanism method

Solution

The degree of indeterminacy of the beam is 2. It requires three hinges to form a mechanism. One hinge will form at each of the two supports, while the third hinge will form at B under the point load. Let rotation at A = θ and rotation at C $= \phi$. Then deflection at B

$$\Delta = AB \times \theta = BC \times \phi \qquad (i)$$

$$= 0.4L\,\theta = 0.6L\,\phi$$

$$\text{Hence,} \quad \theta = 1.5\,\phi \qquad (ii)$$

The slope θ at A is negative, and the slope ϕ at C is also negative being closing angles. The slope $(\theta + \phi)$ at B is positive being an opening angle. The moment is taken as positive if it causes tension on the dotted face.

$$\therefore \quad M_A = -M_p \quad \text{(hogging)}$$

$$M_B = M_p \quad \text{(sagging)}$$
$$M_C = -M_p \quad \text{(hogging)}$$

The virtual work equation may now be written as :

$$\text{Internal work done} = M_A(-\theta) + M_B(\theta + \phi) + M_C(-\phi) \qquad (iii)$$
$$= M_p\theta + M_p(\theta + \phi) + M_p\,\phi$$
$$= 1.5\,M_p\,\phi + 2.5\,M_p\,\phi + M_p\,\phi = 5\,M_p\,\phi$$

$$\text{External work done} = 2W\Delta + 0.4\,W\frac{\Delta}{2} + 0.6W\frac{\Delta}{2}$$

$\because$ The average deflection in the span AB is $(0 + \Delta)/2 = \Delta/2$

$\therefore$ External work done $= 2.5\ W\Delta = 2.5\ W(0.6L\phi)$ using Eq.(i)

$\therefore$ Equating internal and external work done,

$$2.5\ W(0.6L\phi) = 5\ M_p\,\phi$$

or

$$W = W_u = \frac{5}{1.5}\frac{M_p}{L} = \frac{10}{3}\frac{M_p}{L}$$

Let us check the moment at the mid span

$$R_A = \frac{W}{2} + 2W \times 0.6 = 1.7\ W \text{ and } R_C = 1.3\ W$$

Hence bending moment at the mid span

$$M_0 = -M_p + 1.3\ W \times 0.5\ L - \frac{W}{2L}\left(\frac{L}{2}\right)^2$$
$$= -M_p + 0.65 \times \frac{10}{3}M_p - \frac{10}{24}M_p$$
$$= 0.75M_p < M_p \qquad \text{O.K.}$$

Thus the collapse load is $W_u = \dfrac{10}{3}\dfrac{M_p}{L}$

Example 15.5

Reanalyze the three span continuous beam of Example 15.2 using the mechanism method.

Solution

Mechanism 1 - span AB (Fig. 15.11b)

Two hinges are required to form a mechanism in the span AB. Let one hinge forms at a distance x from A and the other hinge forms at B in the span AB.

$$x\theta = \Delta = (L - x)\phi \quad \text{by geometry} \qquad (i)$$

or,

$$\phi = \frac{x}{L - x}\theta$$

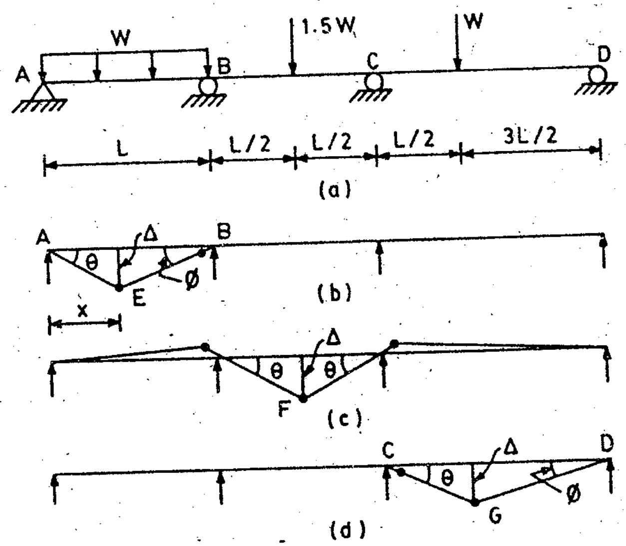


Fig. 15.11 Continuous beam : mechanism method

$$\text{Internal work done} = M_p(\theta + \phi) + M_p\,\phi \qquad \text{(ii)}$$

$$= M_p\left(\frac{L}{L-x}\right)\theta + \frac{M_p\,x}{L-x}\theta = M_p\left(\frac{L+x}{L-x}\right)\theta$$

$$\text{External work done} = \frac{Wx}{L}\left(\frac{\Delta}{2}\right) + \frac{W}{L}(L-x)\frac{\Delta}{2} \qquad \text{(iii)}$$

$$= \frac{W\Delta}{2} = \frac{1}{2}Wx\theta \quad \text{using Eq (i)}$$

Equating internal and external work done,

$$\frac{1}{2}Wx\theta = M_p\left(\frac{L+x}{L-x}\right)\theta$$

or,

$$W = \frac{2(L+x)}{x(L-x)}M_p \qquad \text{(iv)}$$

For minimum collapse load, differentiating W with respect to x and setting it to zero, gives

$$\frac{dW}{dx} = \frac{2[x(L-x)-(L+x)(L-2x)]M_p}{[x(L+x)]^2} = 0$$

or,

$$x^2 + 2Lx^2 - L^2 = 0$$

or,

$$x = -\ 2.414\,L \text{ or } 0.414L \qquad \text{(v)}$$

The first value is inadmissible. The ultimate load is given by

$$W = W_u = \frac{2 \times 1.414\ M_p/L}{0.414 \times 0.586} = 11.65\frac{M_p}{L} \qquad \text{(vi)}$$

Mechanism 2 - span BC (Fig. 15.11c)

$$\text{Internal work done} = M_p\theta + 1.5M_p(\theta + \theta) + M_p\,\theta = 5\,M_p\,\theta$$

$$\text{External work done} = 1.5\ W\Delta = 1.5W\frac{L}{2}\theta$$

Equating internal and external work done,

$$0.75\ WL\,\theta = 5\,M_p\,\theta$$

or,

$$W = W_u = 6.67\frac{M_p}{L} \qquad \text{(vii)}$$

Mechanism 3 - span CD (Fig. 15.11d)

It requires only two plastic hinges to form a mechanism. One hinge will form at C and the other at the section of point load.

$$\frac{L}{2}\theta = \Delta = 3\frac{L}{2}\phi \text{ or,} \qquad \phi = \theta/3 \text{ by geometry}$$

$$\text{Internal work done} = M_p\theta + M_p(\theta + \phi) = \frac{7}{3}M_p\,\theta$$

$$\text{External work done} = W\Delta = W\frac{L}{2}\theta$$

Equating internal and external work done,

$$W\frac{L}{2}\theta = \frac{7}{3}M_p\,\theta$$

or,

$$W = W_u = 4.67\frac{M_p}{L} \qquad \text{(viii)}$$

The collapse load is the smallest of the three ultimate loads, that is,

$$W_u = 4.67\frac{M_p}{L}$$

Example 15.6

Reanalyze the portal frame shown in Fig.15.8a using the mechanism method.

Solution

Degree of static indeterminacy = 1. The locations of possible plastic hinges are three at B, C and D, respectively.

$\therefore$ Number of independent mechanisms $= 3 - 1 = 2$

One is beam mechanism and the other is sway mechanism as shown in Figs. 15.12b and 15.12c. The combined mechanism is shown in Fig. 15.12d. The ultimate loads from the various mechanisms are as follows :

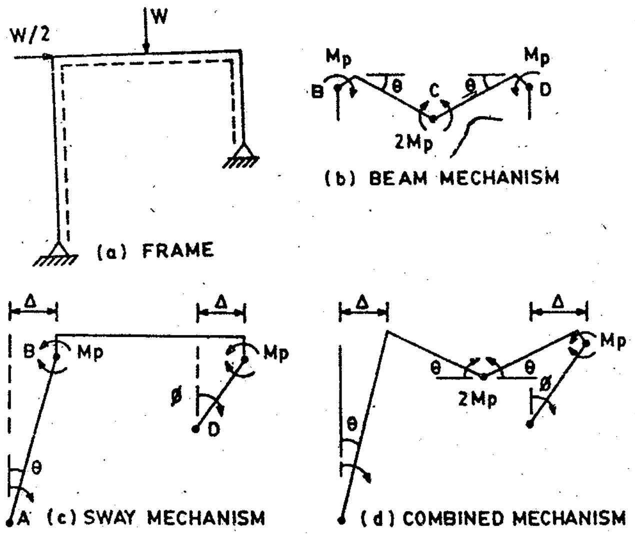


Fig. 15.12 Portal frame : mechanism method

Beam Mechanism

$$\frac{L}{2}\theta = \Delta \quad \text{by geometry}$$

Equating the external and internal work done

$$W\Delta = M_B(-\theta) + M_C(\theta + \theta) + M_D(-\theta)$$

but

$$M_B = -M_p = M_D$$

$$M_C = 2M_p$$

$$\therefore \quad W\frac{L}{2}\theta = M_p\theta + 2M_p(\theta + \theta) + M_p\theta = 6M_p\theta$$

or,

$$W = 12\frac{M_p}{L} \qquad \text{(i)}$$

Sway Mechanism

$$\Delta = L\theta = \frac{L}{2}\phi \quad \text{by geometry}$$

or,

$$\phi = 2\theta$$

Equating the external and internal work done

$$\frac{W}{2}\Delta = M_A(-\theta) + M_B(\theta) + M_D(-\phi) + M_E(\phi)$$

but

$$M_A = 0 = M_E$$

and

$$M_B = M_p$$

$$M_D = -M_p$$

or,

$$\frac{W}{2}L\theta = M_p\theta + M_p\phi = 3M_p\theta$$

or,

$$W = W_u = 6\frac{M_p}{L} \qquad \text{(ii)}$$

Combined Mechanism

$$\Delta = L\theta = \frac{L}{2}\phi \quad \text{by geometry}$$

or,

$$\phi = 2\theta$$

Equating the external and internal work done

$$\frac{W}{2}\Delta + W\Delta' = M_C(\theta + \theta) + M_D(-\theta - \phi)$$

where

Δ = lateral sway displacement at the level of the beam

Δ' = vertical displacement of the beam

and

$M_C = 2M_p$ sagging

$M_D = -M_p$ hogging

$$\therefore \quad \frac{W}{2}\Delta + W\Delta' = 2M_p(\theta + \theta) + M_p(\theta + \phi)$$

or,

$$\frac{WL\theta}{2} + \frac{WL\theta}{2} = 4M_p\theta + 3M_p\theta = 7M_p\theta$$

or,

$$W = W_u = 7\frac{M_p}{L} \qquad \text{(iii)}$$

Hence, the collapse load is equal to the lowest among the three loads :

$$W = W_u = 6\frac{M_p}{L}$$

Example 15.7

Determine the design moment in the two bay frame shown in Fig. 15.13a using the mechanism method.

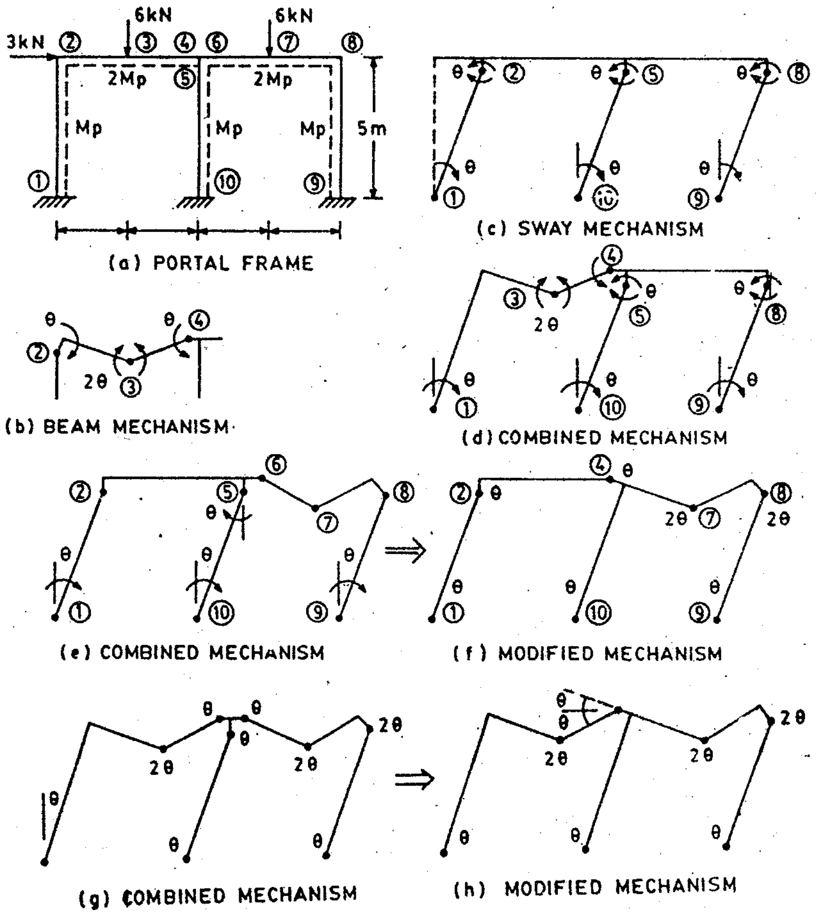


Fig. 15. 13 Two bay portal frame : mechanism method

Solution

Degree of statical indeterminacy = 6, Total no of nodes in the frame = 10

∴ Number of independent mechanisms = 4

These are : Beam mechanisms = 2, Joint mechanisms = 1, Sway mechanism = 1

A plastic hinge can form at any one or more of the ten nodes.

***Mechanism 1*- Beam 2-3-4** (Fig.15.13b)

Equating the external and internal work done.

$$6 \times 2.5\,\theta = M_2(-\theta) + M_3\,(2\,\theta) + M_4\,(-\theta)$$

but $\quad M_2 = -M_p \;\; M_3 = 2M_p, \;\; M_4 = -2M_p$

$$15\theta = 7\,M_p\theta \text{ or } M_p = 2.14 \text{ kNm} \qquad \text{(i)}$$

***Mechanism 2*- Beam 6-7-8**

$$M_p = 2.14 \text{ kNm} \qquad \text{(ii)}$$

***Mechanism 3* - Sway** (Fig. 15.13c)

In Fig. 15.13 c, the arrows represent direction of rotation.

Opening angles are taken as positive and closing angles are taken as negative. Sagging moments are taken as positive, that is

$$M_1 = -M_p = M_{10} = -M_9$$
$$M_2 = M_p = M_5 = -M_8$$

Equating the external and internal work done :

$$3 \times 5\,\theta = M_1(-\theta) + M_2\,(\theta) + M_{10}\,(-\theta) + M_5\,(\theta) + M_9\,(\theta) + M_8\,(-\theta)$$

∴ $\quad 15\,\theta = 6\,M_p\,\theta$

or, $\quad M_p = 2.5 \text{ kNm} \qquad \text{(iii)}$

Let us combine mechanisms 1 and 3 (Fig. 15.13d)

While combining these two mechanisms, the plastic hinge at 2 will disappear.

External work done $= 3 \times 5\theta + 6 \times 2.5\,\theta = 30\,\theta$

Internal work done $= M_1(-\theta) + M_3\,(2\theta) + M_4(-\theta) + M_5\,(\theta) + M_8(-\theta) + M_9\,(\theta) + M_{10}(-\theta)$

$= M_p\theta + 4M_p\theta + 2M_p\theta + M_p\theta + M_p\theta + M_p\theta + M_p\theta$

$= 11\,M_p\,\theta$

or, $\quad M_p = 2.73 \text{ kNm} \qquad \text{(iv)}$

Let us combine mechanisms 2 and 3 (Fig. 15.13e)

While combining these two mechanisms, the plastic hinge at 8 will disappear. There are two plastic hinges at the top middle joint. Let us rotate the middle joint clockwise by an amount θ. The two hinges at 5 and 6 will disappear and a new hinge will form at 4 as shown in Fig. 15.13 f.

External work done $= 3 \times 5\theta + 6 \times 2.5\,\theta = 30\theta$

Internal work done $= 4\,(M_p\,\theta) + 2\,M_p\,\theta + 2M_p(2\theta) + M_p(2\theta)$

$= 12\,M_p\,\theta$

or, $\quad M_p = 2.5 \text{ kNm} \qquad \text{(v)}$

Let us combine mechanisms 1,2 and 3 (Fig. 15.13g)

The combined mechanism is shown in Fig. 15.13g. Let us rotate the middle joint clockwise by an amount θ. The two hinges at 5 and 6 will disappear and only one hinge remains at 4. The rotation at hinge 4 will be 2θ as shown in Fig. 15.13 h.

External work done $= 3 \times 5\theta + 6 \times 2.5\,\theta + 6 \times 2.5\,\theta = 45\,\theta$

Internal work done $= 3(M_p\,\theta) + 2\,M_p\,(2\theta + 2\theta + 2\theta) + M_p(2\theta)$

$$= 17\,M_p\,\theta \qquad \text{(vi)}$$

or, $M_p = 2.6$ kNm

The largest value of $M_p = 2.73$ kNm based on Fig.15.13d.

Check

We need to draw B.M. diagram corresponding to the mechanism shown in Fig. 15.13d to check that the moment at other sections does not exceed their plastic moment capacities. Out of the 10 joints, the moment values are known at 7 locations only, that is, (vii)

$$M_1 = M_{10} = M_8 = -2.73 \text{ kNm}$$

$$M_5 = M_9 = +2.73 \text{ kNm}$$

$$M_3 = 2 \times 2.73 = 5.46 \text{ kNm}$$

$$M_4 = -5.46 \text{ kNm}$$

Let us write the equations of static equilibrium.

Span 6-7-8 : *Beam mechanism*

$$M_6(-\theta) + M_7\,(2\theta) + M_8(-\theta) = 6 \times 2.5\,\theta$$

or, $M_6 - 2\,M_7 + M_8 = -15$

or, $M_6 - 2M_7 = -12.23$ (viii)

Span 2-3-4 : *Beam mechanism*

$$M_2 - 2M_3 + M_4 = -15$$

$$M_2 - 2\,(5.46) + (-5.46) = -15$$

or, $M_2 = 1.38 \text{ kNm} < M_p$ O. K.

Middle Joint : *Joint mechanism*

$$M_4 + M_5 - M_6 = 0$$

∴ $M_6 = -2.73 \text{ kNm} < 2\,M_p$ O. K.

∴ Eq.(viii) gives, $M_7 = \frac{1}{2}(-2.73 + 12.23) = 4.75$ kNm

$< 2\,M_p$ O. K.

Thus, the moment everywhere in the frame is less than or equal to the corresponding moment capacity. Therefore, the design moment is 2.73 kNm.

Example 15.8

Analyze the gable frame shown in Fig. 15.14a subjected to gravity and wind loads and determine the design plastic moment value.

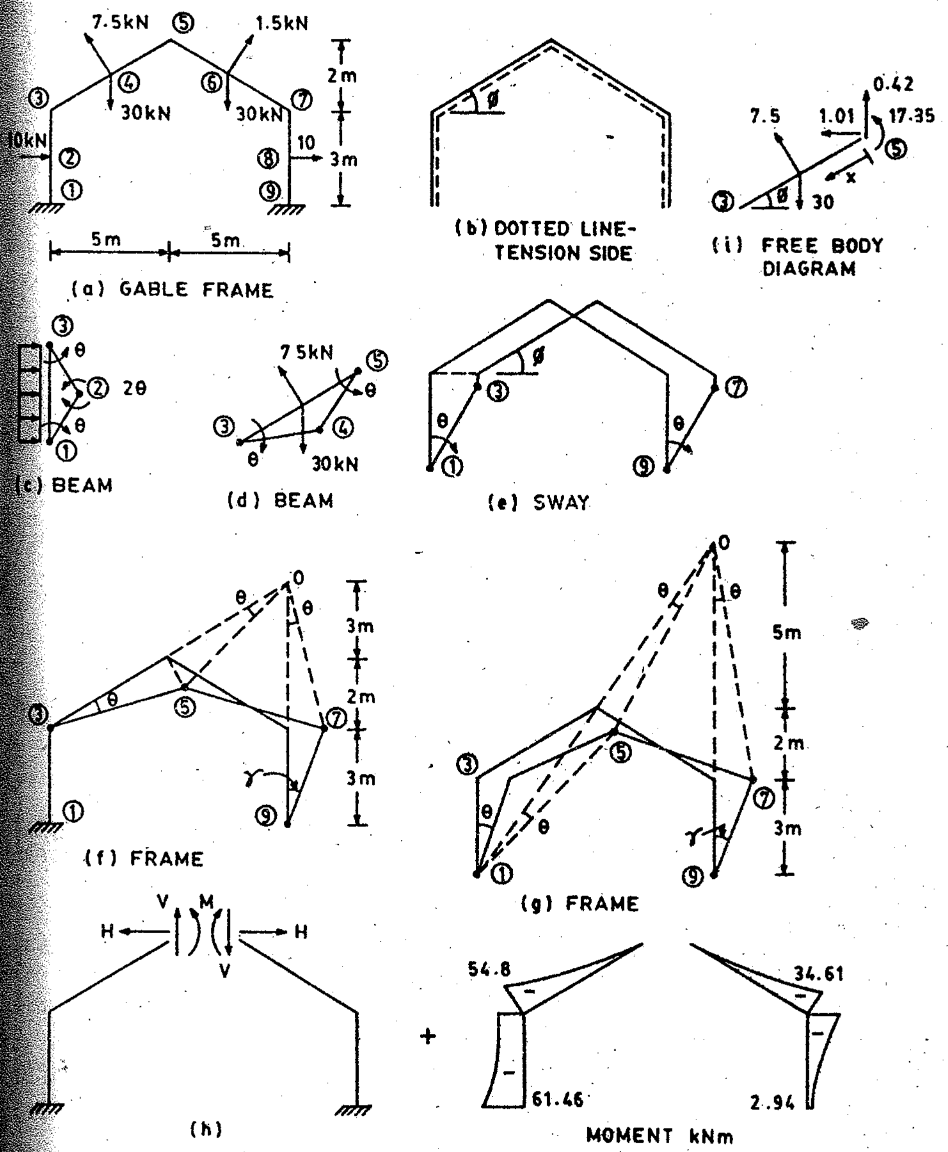


Fig. 15.14 Gable frame : mechanism method

Solution

Degree of statical indeterminacy = 3

No. of locations of possible plastic hinges = 9

$\therefore$ Independent mechanisms = 6

Beam type mechanisms = 4

Sway mechanism = 1

Frame mechanism = 1

The frame is subjected to uniformly distributed loads. The total load on each member is applied at its mid-span. The inclination of members 3-4-5 and 5-6-7 with the horizontal is ϕ, that is,

$$\sin\phi = 0.371 \quad \text{and} \quad \cos\phi = 0.928$$

The inside of the gable frame is shown by a dotted line in Fig.15.14b. It will help in determining positive and negative rotation as well as hogging and sagging moments.

Mechanism 1 - **Beam 1-2-3** (Fig.15.14c)

$$\text{Average displacement} = \frac{1.5\theta}{2}$$

$$\text{External work done} = 10 \times \left(\frac{1.5\theta}{2}\right) = 7.5\,\theta$$

$$\text{Internal work done} = 4M_p\,\theta$$

$$\therefore \quad M_p = 1.875 \text{ kNm} \qquad (1)$$

Mechanism 2 - **Beam 3-4-5** (Fig. 15.14 d)

$$\text{Inclined length 3-5} = 5.385 \text{ m}$$

$$\therefore \quad \text{External work done} = \left(30 \times \frac{5}{2\times 2}\right)\theta - \left(7.5 \times \frac{5.385}{2\times 2}\right)\theta = 27.4\,\theta$$

$$\text{Internal work done} = 4M_p\,\theta$$

$$\therefore \quad M_p = \frac{27.4}{4} = 6.85 \text{ kNm} \qquad (2)$$

Mechanism 3 - **Beam 5-6-7**

$$\text{External work done} = \left(30 \times \frac{5}{4}\right)\theta - \left(15 \times \frac{5.385}{4}\right)\theta = 17.3\,\theta$$

$$\text{Internal work done} = 4M_p\,\theta$$

$$\therefore \quad M_p = \frac{17.3}{4} = 4.325 \text{ kNm} \qquad (3)$$

Mechanism 4 - **Beam 7-8-9** same as beam 1-2-3

or, $$M_p = 1.875 \text{ kNm} \qquad (4)$$

Mechanism 5 - **Sway Mechanism** (Fig.15.14e)

Net horizontal load in top storey = $(15 - 7.5)\sin\phi = 2.785$ kN

$$\text{Sway } \Delta = 3\,\theta$$

$$\text{External work done} = 2 \times 10 \times \frac{3\theta}{2} + 2.785 \times 3\theta = 38.355\,\theta$$

$$\text{Internal work done} = 4\,M_p\,\theta$$

$$\therefore \quad M_p = 9.59 \text{ kNm} \qquad (5)$$

Mechanism 6 - **Frame Mechanism** (Fig.15.14f)

The plastic hinges form at 3, 5, 7 and 9. The centre of rotation 0 is shown in Fig. 15.14f along with the deflected shape. The sway of right leg is given by

$$5\theta = 3\gamma \quad \text{or} \quad \gamma = 1.67\,\theta$$

$$\text{External work done} = 2.785 \times 1\theta - 20.89 \times 2.5\,\theta + 2 \times 30 \times 2.5\,\theta + 10 \times \frac{3\gamma}{2}$$

$$= 125.61\,\theta$$

$$\text{Internal work done} = M_p\,[\theta + 2\theta + (\theta + \gamma) + \gamma] = 7.34\,M_p\,\theta$$

or, $$M_p = 17.1 \text{ kNm} \qquad (6)$$

Let us check the moment at section 1 in Fig. 15.14 e

$$M_1\,(-\theta) + M_3\,(\theta) + M_7\,(-\theta) + M_9\,(\theta) = 38.355\,\theta \quad (\text{Mechanism 5})$$

or, $$M_1 = M_3 - M_7 + M_9 - 38.355$$

$$= -17.1 - (-17.1) + 17.1 - 38.355 = -21.255 \text{ kNm} > 17.1 \text{ kNm} \quad \text{NG}$$

Hence mechanism 6 is not possible.

Mechanism 7 - **Frame Mechanism** (Fig.15.14g)

Let us combine the frame and sway mechanism as shown in Fig. 15.14g. The centre of rotation 0 is shown in the same figure. The sway of right leg is given by

$$(5 + 2)\,\theta = 3\gamma \quad \text{or} \quad \gamma = 7/3\,\theta = 2.34\theta$$

Net vertical load in top storey = $(7.5 + 15)\cos\phi = 20.89$ kN

$$\text{External work done} = 2 \times 30 \times 2.5\,\theta + 20.89 \times (-2.5\,\theta) +$$

$$2.785 \times 1\,\theta + 10 \times \left(\frac{3\theta}{2} + \frac{3\gamma}{2}\right)$$

$$= 150\,\theta - 52.23\,\theta + 2.785\,\theta + 50.1\,\theta = 150.65\,\theta$$

$$\text{Internal work done} = M_p\,[\theta + 2\theta + (\theta + \gamma) + \gamma] = 8.68\,M_p\,\theta$$

or, $$M_p = \frac{150.65}{8.68} = 17.35 \text{ kNm} \qquad (7)$$

Check

Let us examine the moment at section 3 in Fig. 15.14 e

$$M_1(-\theta) + M_3(\theta) + M_7(-\theta) + M_9(\theta) = 38.355\,\theta \quad \text{(sway mechanism 5)}$$

$$M_3 = M_1 + M_7 - M_9 + 38.355$$

$$M_3 = (-17.35) + (-17.35) - (17.35) + 38.355 = -13.695$$

$$< M_p \qquad \text{O. K.}$$

Let us also examine the moments in the member 3-4-5.

Let us introduce a cut at 5 and draw B.M. diagrams. (Fig.15.14h)

$$M_3 = -30 \times 2.5 + 7.5 \times \frac{5.385}{2} = -54.8 \text{ kNm}$$

$$M_1 = -30 \times 2.5 - 10 \times 1.5 + 7.5 \sin\phi\,(4) + 7.5\cos\phi\,(2.5) = -61.46 \text{ kNm}$$

$$M_7 = -30 \times 2.5 + 15 \times \frac{5.385}{2} = -34.61 \text{ kNm}$$

$$M_9 = -30 \times 2.5 + 10 \times 1.5 + 15 \sin\phi \times 4 + 15\cos\phi \times 2.5 = -2.94 \text{ kNm}$$

Continuity analysis gives,

At 1, $5V + 5H + M - 61.46 = -M_p$ (i)

At 7, $-5V + 2H + M - 34.61 = -M_p$ (ii)

At 9, $-5V + 5H + M - 2.94 = M_p$ (iii)

But $M = M_p = 17.35$ kNm

$\therefore$ Eq.(ii) and (iii) give

$$3H + 31.67 = 2M_p = 34.7$$

or, $H = 1.01$ kN and $V = 0.42$ kN

Let us write an expression for M_x in the member 3-4-5 taking the loads as uniformly distributed (Fig. 15.14i) :

$$M_x = 17.35 + 0.42\cos\phi\, x + 1.01\sin\phi\, x + \left(\frac{7.5}{5.385}\right)\frac{x^2}{2} - \left(\frac{30\cos\phi}{5.385}\right)\frac{x^2}{2}$$

$$= 17.35 + 0.389\,x + 0.375x + 0.696\,x^2 - 2.585x^2$$

or, $$M_x = 17.35 + 0.764\,x + 1.888\,x^2$$

For maximum moment

$$\frac{dM_x}{dx} = 0 \text{ or, } x = 0.40 \text{ m and } M_x = 17.35 \text{ kNm} = M_p \qquad \text{O.K.}$$

Thus, the plastic hinge does not form at the vertex (section 5) but at 0.40 m away from section 5 in member 3-4-5.

15.8 NON - LINEAR STIFFNESS MATRIX ANALYSIS

The analysis developed in section 15.6 gives an idea of the collapse load only that too for simple beams and frames under static loading. For a plastic design, it is desirable to know the deflections and rotations in the plastic hinges, in order to get a clear understanding of the behaviour of the structure. Such a non-linear analysis due to material non-linearity can be easily carried out using the direct stiffness method developed earlier in Chapters 12, 13 and 14.

A non-linear analysis of a multistorey structure becomes even more desirable in the event of a severe earthquake. Such a severe force is expected to cause significant structural damage resulting in loss of structural stiffness. A linear analysis is clearly not applicable in such a situation as it cannot account for changes in structural properties, redistribution of forces due to the formation of plastic hinges, nor it can give an idea about the location of plastic hinges and amount of rotations. The main object of a non-linear analysis is to arrive at a satisfactory and economical design which limits the maximum rotations to the rotation capacities of the respective sections and produces a reasonably uniform distribution of plasticity throughout the structure.

The latest codes on the design of steel structures and reinforced concrete structures are based on the *Limit State Design Philosophy*. The main problem is the determination of design forces at the limit state of collapse. Apparently, there are two possibilities :

1. Determine the member forces at the service loads from an elastic analysis, and magnify them by appropriate partial safety factors.
2. Carryout a detailed non-linear analysis.

Obviously, the first approach is very convenient but no-where near the true behaviour. The design codes, however are currently based on the first approach. The second approach is quite involved and needs a very careful evaluation of material properties, preparation of input data and interpretation of the results. That is why, it is confined to only special structures, such as, very tall buildings or unsymmetrical buildings.

There are two basic methods for the solution of a nonlinear problem :

(a) Iterative methods
(b) Incremental methods

These methods make use of either the initial stiffness or the tangent stiffness of the structure as shown in Fig. 15.15. An iterative scheme is illustrated in Fig. 15.16a and b making use of initial stiffness and tangent stiffness, respectively. It is clear that the convergence is faster using the tangent stiffness although it involves more computational efforts in the overall analysis. The incremental method is again an iterative scheme in which the load is applied in small increments as shown in Fig. 15.17. It is a very powerful scheme for nonlinear analysis of structures having material or geometrical non linearity.

Iterative Method with Initial Stiffness

The salient features of the iterative method will be explained with the help of load deflection curves shown in Figs. 15.16a and b. Let us first consider Fig. 15.16a which

represents load-deflection behaviour of a typical strain-softening structure. Corresponding to a load P_0, the correct maximum deflection is Δ_0. Similar load-deflection curves can be generated for each member of the structure knowing the stress-strain behaviour of the material of each member and sectional properties. If the load P_0 is applied in a single step, the elastic linear analysis will give a deflection corresponding to point b_1 as shown in Fig. 15.16a. The correct deflection is obtained iteratively.

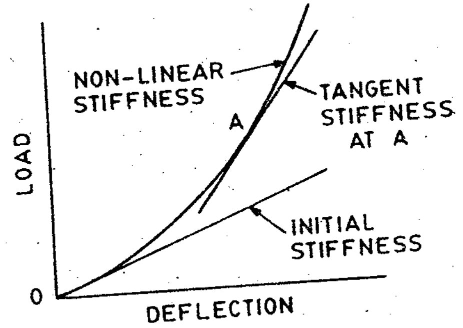


Fig. 15.15 Initial stiffness and tangent stiffness

Step 1 Calculate the elastic stiffness matrix of each member and assemble the global stiffness matrix of the structure as discussed in Chapter 12 or 13. This stiffness matrix is referred to as the initial stiffness matrix of the structure.

Step 2 Assemble the global load vector.

Step 3 Introduce boundary conditions in the stiffness matrix as well as load vector as discussed in Chapter 12.

Step 4 Solve the system of linear simultaneous equations.

$$P_0 = K_I \Delta \tag{i}$$

for deflection Δ at each nodal point in the structure. Since the structure does not remain linear, the displacement so obtained corresponds to b_1 instead of Δ_0 in Fig. 15.16a.

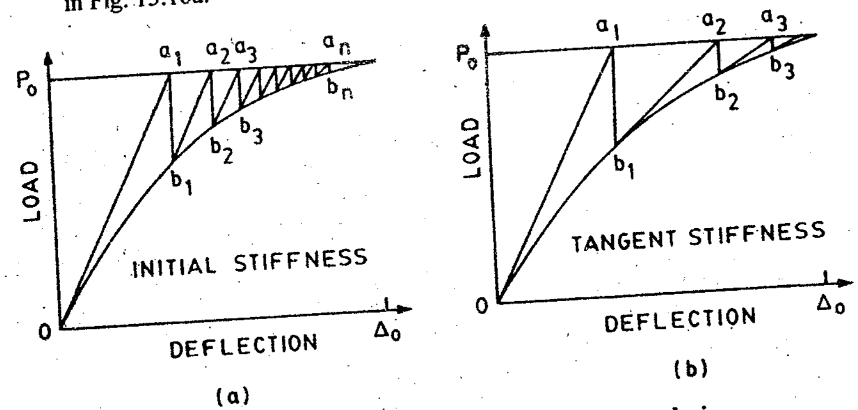


Fig. 15.16 Iterative method for non-linear analysis

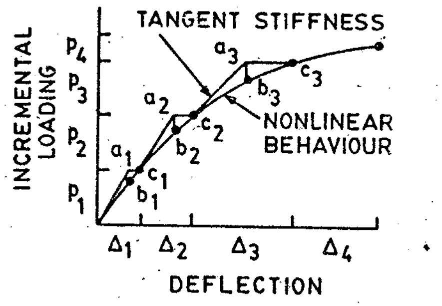


Fig. 15.17 Incremental method for non-linear analysis

Step 5 The ordinate a_1 b_1 represents the error in load for the structure. The correction in the load vector is carried out by considering the load-deflection equilibrium for each member, that is,

$$\mathbf{p_m} = \mathbf{k_m}\ \delta_m \tag{ii}$$

The load ordinate $P_{b1} = p_m$ is obtained by substituting $\Delta_{b1} = \delta_m$ in the equilibrium equations of each member computed using the modified stiffness matrix taking into account formation of one or more plastic hinges in the member.

Thus error $\mathbf{dP_m}^{(1)} = 0$ if member remains elastic. (iii)

$= P_y - P_{b1}$ if member becomes inelastic.

is obtained for each member.

where : P_y = yield load for the member

The load $\mathbf{dP_m}^{(1)}$ is called unbalanced load vector for a member. Similar unbalanced load vectors are calculated for all members of the structure. Finally the unbalanced load vector $dP^{(1)}$ for the structure is assembled.

Step 6 The second iteration is carried out using $dP^{(1)}$ in Eq. (i), that is,

$$\mathbf{dP}^{(1)} = \mathbf{K_I}\, \mathbf{d}\, \Delta^{(1)} \tag{iv}$$

The global stiffness matrix of the structures K_I remains unchanged.

Step 7 The total displacement is now equal to

$$\Delta + d\,\Delta^{(1)}$$

This corresponds to ordinate b_2. The error in load for the structure is given by $a_2\,b_2$

Step 8 Repeat step 5 and obtain $\mathbf{dP}^{(2)}$ for the structure.

Now repeat step 6 and step 7 till the error in load (or resistance) falls within an acceptable level.

In this method a constant stiffness matrix is used for all iterations. Thus, a series of linear analysis is carried out in which only the loads change. The triangularization operation of the stiffness matrix is carried out only once. Here lies the advantage especially in large systems. In this iterative scheme the convergence is very slow. Yet, in certain problems involving mild non-linearities this proves less expensive on the whole.

Iterative Method with Tangent Stiffness

Consider the load-deflection curve of a structures as shown in Fig. 15.16b. Similar curves can be generated for each member of the structure. The correct deflection is obtained by modifying step 6.

Step 6 The second iteration is carried out using the modified global stiffness matrix of the structure. It involves reassembly of the stiffness matrix of the structure using updated stiffness matrix of each element corresponding to a displacement at b_1 . . (iv)

$$dP^{(1)} = K_T \, d\Delta^{(1)}$$

where K_T = tangent stiffness matrix of the structure at b_1 .
Solve for $d\Delta^{(1)}$.

Rest of the analysis remains unchanged. This iterative scheme is also known as the *Newton-Raphson method.* It requires greater computational efforts because of the assembly of tangent stiffness matrix and, therefore, triangularization of such a matrix for each iteration. Nevertheless, convergence is usually faster.

Incremental Method

In the incremental method, load is applied in very small increments and displacement is evaluated. The load step is usually taken to be of equal size although it is not necessary. Rest of the procedure is similar to the iterative method with tangent stiffness. At the beginning of each step the static equilibrium is established. The displacement for the load increment is evaluated assuming that the tangent stiffness remains constant during the load step. The change in stiffness, if any, is considered for each member of the structure for the incremental displacement. The incremental load vector is updated. The modified tangent stiffness is used in the next load step. The unbalanced loads $a_1 b_1$, $a_2 b_2$, $a_3 b_3$, etc. as shown in Fig. 15.17 are very small hence negligible, if the load step are very small. If not, suitable correction is required. Thus, in each load increment, the deflection corresponds to c_1 , c_2 , c_3 etc. instead of b_1 , b_2 , b_3 etc. The procedure is repeated till collapse or total load is applied. In this method, the total number of iterations is equal to the number of load steps.

Incremental Method with Iterations

In this method, the load step may be little larger than that used in the incremental method. The unbalanced load is reduced to zero through iterations before proceeding to the next load step. The method may be seen as a combination of that in Fig. 15.17 and Fig. 15.16b, or Fig. 15.17 and Fig. 15.16a. The former requires updating of the stiffness ... load step leading to greater computational efforts.

simplification is suggested through the iterative scheme shown in Fig. 15.16a which employs a constant stiffness. In a given load increment, the iterations are carried out using the tangent stiffness matrix of the structure computed at the beginning of the load increment. The tangent stiffness matrix is updated at the beginning of next load increment. This is known as the *modified Newton- Raphson method.*

Choice of a Method

Now the obvious question is *which is the best* method of solution of a nonlinear system. The answer cannot be given unequivocably as the algorithm most efficient in one case may be divergent in another. The stability and accuracy of an algorithm depend upon the degree of nonlinearity in a given problem. Nevertheless, the incremental method with tangent stiffness matrix in sufficiently small steps can be recommended. The tangent stiffness matrix is reasonably well defined and can be easily calculated for discrete structural systems. At this stage it is necessary to mention that direct solution algorithms for the solution of linear simultaneous equations are faster than indirect numerical iteration algorithms. Thus a nonlinear problem is solved as a series of linear problems. The problem is further simplified with the existence of powerful linear computer programs which can be conveniently adapted to non-linear solutions.

15.9 HYSTERESIS LOOPS

The elasto-plastic M-ϕ behaviour shown in Fig. 15.18a, b is valid for monotonically increasing loads. When the structure is unloaded, the behaviour is invariably assumed to be elastic until further reverse loading produces plastic yielding. The loading and unloading cycles may be continued on the structure. The force-deformation (or moment-curvature) relation is known as hysteresis loop or model. The area of the hysteresis loop gives the energy dissipated during each cycle. This behaviour is again idealized as an elasto-plastic behaviour for convenience. Typical hysteresis models for a truss element and a beam element are shown in Figs. 15.18a and b, respectively. In Fig. 15.19a, segment 1-0 shows unloading from a maximum stress level of just below the yield level.

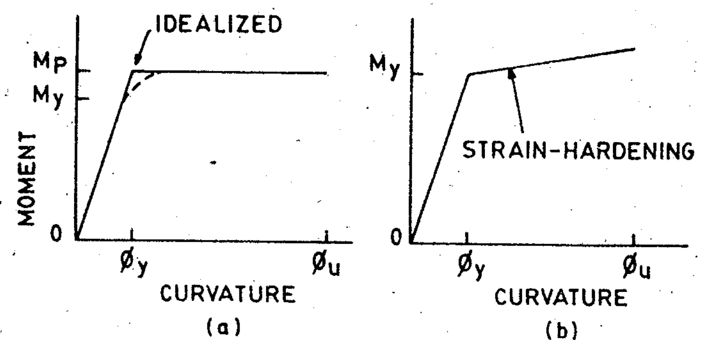
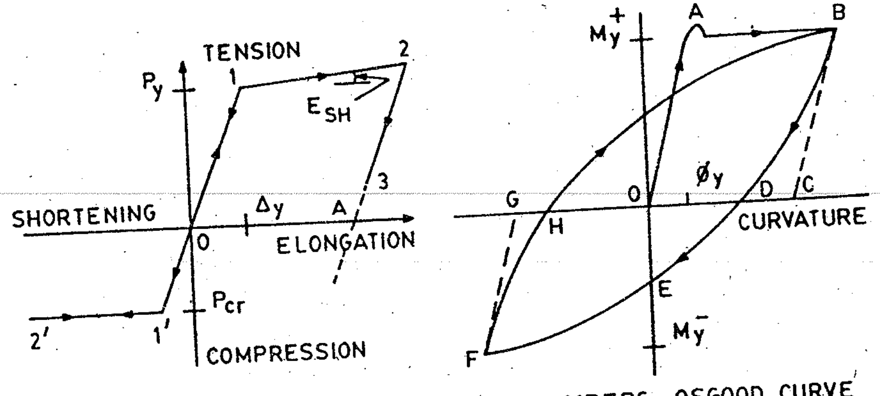


Fig. 15.18 Idealized M-ϕ curves for beams

This unloading traces the same path as during the loading. If, however, unloading takes place from a stress above or beyond the yield level, it is assumed to be elastic as shown by segment 2-3. The residual elongation OA is known as the plastic strain. It occurs only if the unloading takes place from the segment 1-2. Similarly, in Fig. 15.19b; O-D represents plastic strain upon unloading from segment A-B which is beyond the yield level; O-H represents plastic strain upon unloading from F.



(a) TRUSS ELEMENT (b) RAMBERG-OSGOOD CURVE

Fig. 15.19 Typical hysteresis loops - (a) truss element (b) beam element

The hysteresis loop shown in Fig. 15.19 b is known as Ramberg-Osgood curve and is expressed by the equation

$$\frac{\phi}{\phi_y} = \frac{M}{M_y}\left[1+\left|\frac{M}{M_y}\right|^{r-1}\right] \qquad (15.11)$$

For given values of M_y and ϕ_y, the exponent r governs the shape of the curve. With r = 1, a perfectly elastic curve is obtained; with r = ∞, an elasto-plastic curve is obtained. The Ramberg-Osgood curve is reasonably accurate for steel beams although it involves more computational efforts.

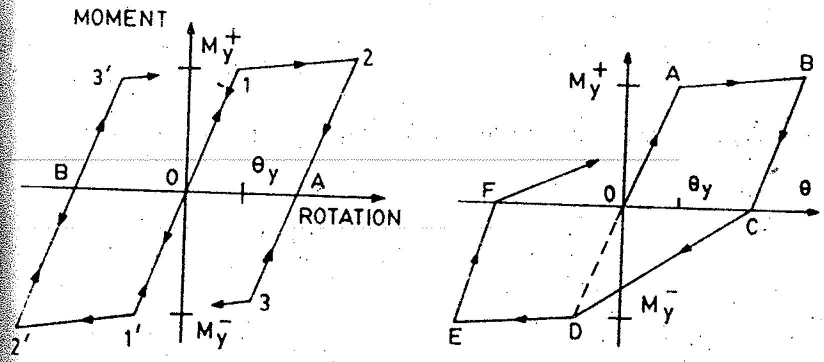
Idealized hysteresis for steel and reinforced concrete beams are shown in Fig. 15. 20a and b. The latter is known as Clough's a stiffness degrading hysteresis model.

In a multistorey building, the inelasticity is confined to beams as far as possible and the columns are treated as elastic. Any inelasticity in columns may trigger a complete collapse because columns are gravity load carrying elements. Inelasticity in beams only leads to a partial or local collapse. Thus, *weak girder-strong column design philosophy* is preferred for building frames. The moment capacity of a column is reduced depending upon the amount of axial force present in it. Typical M-P interaction curves for steel and reinforced concrete columns are shown in Figs. 15.21a and b. For a steel column, the bending moment - axial force relation is given by

$$\frac{M}{M_p} = 1 \quad \text{for} \quad \frac{P}{P_y} \le 0.15 \qquad (15.12)$$

and

$$\frac{M}{1.18M_p} + \frac{P}{P_y} \le 1 \quad \text{for} \quad \frac{P}{P_y} > 0.15$$



(a) STEEL BEAM (b) R.C. BEAM

Fig. 15.20 Idealized hysteresis loops for beams

In non-linear analysis, it is absolutely essential to keep track of the sequence of loading and unloading as the solution is path dependent. The tangent stiffness of a member depends upon its current state of force and deformation. The incremental method of analysis with very small load steps is almost invariably used.

The change in stiffness depends on the axial force-axial deformation characteristics for truss members and moment-curvature or moment-rotation relationship for beam or column members. The moment curvature relation is required to get the moment-rotation relation. For steel beams the moment-curvature relation is similar to that shown in Fig. 15.18a. The small curve near the yield moment makes the derivation of the moment-rotation relation quite complex. Therefore, it is desirable to idealize it without introducing any appreciable errors. The elastic-plastic moment-curvature or the elastic - plastic moment-rotation relation is frequently used in any non-linear structural analysis as shown in Figs. 15.18a and b. The strain-hardening slope in Fig. 15.18b may be taken between 2% to 5% of the elastic slope for steel as well as concrete beams.

For reinforced concrete members, the moment curvature relation exhibits complexities even during the initial loading stages due to cracking of concrete and yielding of reinforcement. The moment-rotation relation is again idealized as elastic-plastic for convenience.

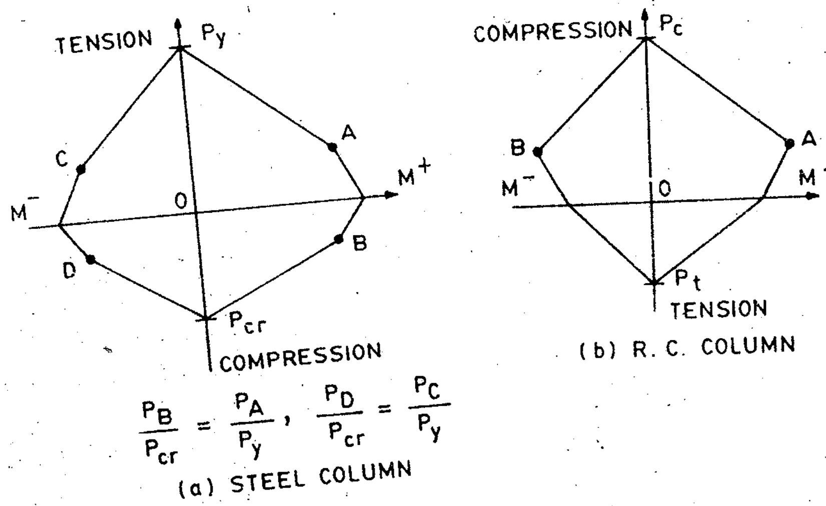


Fig. 15.21 M - P interaction curves for columns

15.10 ASSUMPTIONS

The following assumptions are made for the non-linear analysis of a multistorey framed building. The bare frame is analyzed without any load bearing or partition walls and staircases or lift wells etc.

1. Plane sections remain plane after bending .
2. A plastic hinge can form only at the ends of a beam element. It is a concentrated hinge having a zero length.
3. A plastic hinge can undergo infinite rotation.
4. The moment-curvature relation is bi-linear or elasto-plastic. The moment-rotation relation is also elasto-plastic.
5. The deformations are assumed to be small, that is, p-delta effect is neglected.
6. For a truss element, the axial force-axial displacement relation is given in Fig. 15.19a.

15.11 MEMBER STIFFNESS MATRIX

2-D BEAM ELEMENT

The stiffness matrix of a 2-D prismatic beam element under elastic behaviour (no plastic hinges) is given by Eq.12.4 and is reproduced below :

$$
\mathbf{k}=\frac{EI}{L}\begin{array}{c} \\ \left[\begin{array}{ccc:ccc}
\frac{A}{I} & 0 & 0 & -\frac{A}{I} & 0 & 0 \\
0 & \frac{12}{L^2} & \frac{6}{L} & 0 & -\frac{12}{L^2} & \frac{6}{L} \\
0 & \frac{6}{L} & 4 & 0 & -\frac{6}{L} & 2 \\
\hdashline
-\frac{A}{I} & 0 & 0 & \frac{A}{I} & 0 & 0 \\
0 & -\frac{12}{L^2} & -\frac{6}{L} & 0 & \frac{12}{L^2} & -\frac{6}{L} \\
0 & \frac{6}{L} & 2 & 0 & -\frac{6}{L} & 4
\end{array}\right]\end{array}
\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array}
\qquad \text{columns: } 1\ 2\ 3\ 4\ 5\ 6 \tag{15.13}
$$

When moment at any one end of the beam exceeds the plastic moment capacity of that end, a plastic hinge is formed. The beam has one end fixed and the other end hinged. If the plastic hinge develops at the left end, the stiffness matrix is given below :

$$
\mathbf{k}=\frac{EI}{L}\left[\begin{array}{ccc:ccc}
\frac{A}{I} & 0 & 0 & -\frac{A}{I} & 0 & 0 \\
0 & \frac{3}{L^2} & 0 & 0 & -\frac{3}{L^2} & \frac{3}{L} \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
-\frac{A}{I} & 0 & 0 & \frac{A}{I} & 0 & 0 \\
0 & -\frac{3}{L^2} & 0 & 0 & \frac{3}{L^2} & -\frac{3}{L} \\
0 & \frac{3}{L} & 0 & 0 & -\frac{3}{L} & 3
\end{array}\right]
\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array}
\qquad \text{columns: } 1\ 2\ 3\ 4\ 5\ 6 \tag{15.14}
$$

Analogous to Eq.15.14, the following equation gives the stiffness matrix for a prismatic beam with a plastic hinge at the right end J, that is, rows and coulmns 3 and 6 in Eq. 15.14 must be interchanged,

$$\mathbf{k} = \frac{EI}{L}\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \left[\begin{array}{ccc:ccc} \frac{A}{I} & 0 & 0 & -\frac{A}{I} & 0 & 0 \\ 0 & \frac{3}{L^2} & \frac{3}{L} & 0 & -\frac{3}{L^2} & 0 \\ 0 & \frac{3}{L} & 3 & 0 & -\frac{3}{L} & 0 \\ \hdashline -\frac{A}{I} & 0 & 0 & \frac{A}{I} & 0 & 0 \\ 0 & -\frac{3}{L^2} & -\frac{3}{L} & 0 & \frac{3}{L^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array} \tag{15.15}$$

Finally, if plastic hinges are formed at both ends of the beam the stiffness matrix becomes null.

2-D TRUSS ELEMENT

The axial force-axial deformation behaviour of a truss element is shown in Fig. 15.19a. If it yields in tension, the axial elongation keeps on increasing until the load is reduced or withdrawn or the ultimate strain is reached. The axial stiffness of the truss reduces to zero while it is at the yield level. However, in compression, it is likely to buckle as soon as the critical buckling load is reached unless the member has a very low slenderness ratio. The member stiffness again becomes zero after it buckles. If the member yields in compression instead of bucking, its behaviour is similar to that in tension.

There is a significant difference in the behaviour of a truss element in tension and compression under reversed cyclic loading. The unloading from the tension or compression yield levels is assumed to be elastic and the energy is absorbed, whereas, the unloading from the buckling level is assumed to be along the original path, and no energy is absorbed.

For an elastic prismatic member the stiffness matrix is given by :

$$k_T = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{15.16}$$

If the member yields in tension or buckles in compression its stiffness matrix is a null matrix, that is,

$$k_T = \frac{AE}{L}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{15.17}$$

A null matrix is likely to introduce numerical errors in the solution algorithm. Hence, it is usual to set the post-yield stiffness to a very small value, say

$$k_T = 0.1\% \text{ of } \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{15.18}$$

15.12 MODIFICATION OF THE STRUCTURAL STIFFNESS MATRIX

The global stiffness matrix of a member is given by

$$\mathbf{K'} = \mathbf{R^T kR} \tag{12.12}$$

The structural stiffness matrix can be assembled as discussed in Chapter 12. When a plastic hinge is developed, the stiffness of the structure is reduced. The new stiffness matrix of the member in which the plastic hinge(s) has formed can be generated using Eqs. 15.14 to 15.18. Let the new stiffness matrix of the member be $\mathbf{k'}_n$ and the previous stiffness matrix, that is, prior to the formation of a plastic hinge, be $\mathbf{k'}$ in the global coordinate system. The location vector for this member is already known which gives the locations of the elements of the stiffness matrix of the member under consideration in the structural stiffness matrix. Let the location vector be given by vector $LM_{6 \times 1}$. Now the elements of $\mathbf{k'}$ which were used earlier have to be replaced with the elements of $\mathbf{k'}_n$. Thus,

$$\mathbf{K}_n(I, J) = \mathbf{K}(I, J) + \mathbf{k'}_n(I, J) - \mathbf{k'}(I, J)$$

for $I = LM(1)$ to $LM(6)$

$J = LM(1)$ to $LM(6)$

where, $\mathbf{K}$ = previous structural stiffness matrix.
$\mathbf{K}_n$ = new structural stiffness matrix

Rest of the elements of $\mathbf{K}_n$ are the same as those of $\mathbf{K}$. Thus, there is no need to assemble the entire stiffness matrix for all the elements. Instead, the stiffness matrix is modified for only those members whose stiffness matrix undergoes a change. If more than one member develops plastic hinges, the same procedure is repeated. Again, the same procedure is repeated in each load increment.

The procedure remains the same for a truss element except that four elements of $\mathbf{K}$ will need modifications corresponding to the four d.o.f. of each element undergoing a change of state.

15.13 INCREMENTAL DISPLACEMENT AND LOAD VECTOR

The equilibrium equation at i th load step can be written as

$$\mathbf{K}_i \mathbf{U}_i = \mathbf{P}_i \tag{15.20a}$$

and, at (i+1) th load step, this equation would be

$$(\mathbf{K}_i + \Delta\mathbf{K}_i)(\mathbf{U}_i + \Delta\mathbf{U}) = (\mathbf{P}_i + \Delta\mathbf{P}) \tag{15.20b}$$

Subtracting Eq.15.20a from Eq.15.20b and ignoring very small terms, gives the incremental form of the equilibrium equation:

$$\mathbf{K}_i \,\Delta \underset{\sim}{\mathbf{U}} = \Delta \underset{\sim}{\mathbf{P}} \tag{15.20c}$$

where $\Delta \underset{\sim}{\mathbf{P}}$ = incremental load vector

and $\Delta \underset{\sim}{\mathbf{U}}$ = corresponding incremental displacement vector.

Knowing $\mathbf{K}$ and $\Delta \underset{\sim}{\mathbf{P}}$ in a load step, $\Delta \underset{\sim}{\mathbf{U}}$ can be obtained. Incremental displacement vector for a member in local coordinates $\Delta \underset{\sim}{\delta}$ can be obtained as :

$$\Delta \underset{\sim}{\delta} = \mathbf{R}\,\Delta \underset{\sim}{\mathbf{U}} \tag{15.21}$$

The incremental member forces $\Delta \underset{\sim}{\mathbf{F}}$ can be determined using the stiffness matrix of the member in local coordinates, that is,

$$\begin{aligned} \Delta \underset{\sim}{\mathbf{F}} &= \mathbf{k}\,\Delta \underset{\sim}{\delta} \\ &= \mathbf{k}\,\mathbf{R}\,\Delta \underset{\sim}{\mathbf{U}} \end{aligned} \tag{15.22}$$

The incremental force vector of each member are added at the end of each load step to get the total member force. This force is then compared with the yield capacity to check if a plastic hinge has formed.

15.14 UNBALANCED LOAD VECTOR

Development of a plastic hinge at a certain section is confirmed when the moment at that section exceeds or equals the plastic moment capacity. The section cannot take a moment greater than M_p, and therefore, the difference $|M| - |M_p|$ remains unbalanced. The load vector corresponding to the unbalanced moments in all those members that undergo a change of state in a given load step is applied in the subsegment load vector to restore the equilibrium. The unbalanced load vector p_n for a member can be developed as follows :

Consider a member i-j. The total moment at any load step be M_i and M_j at the two ends. Let the plastic moment capacities be M_{pi} and M_{pj}, respectively at the two ends. The linear analysis in the current load step locates a point B' from point A at end I (say) as shown in Fig. 15.22. However, a plastic hinge develops at point B and the moment cannot exceed M_p. Thus,

$$\begin{aligned} \text{Linear moment increment, } \Delta p_L &= |M_i| - |M_{i-1}| \\ &= p_n + p_0 \end{aligned} \tag{15.23}$$

$$\text{Nonlinear moment increment } p_0 = |M_{pi}| - |M_{i-1}|$$

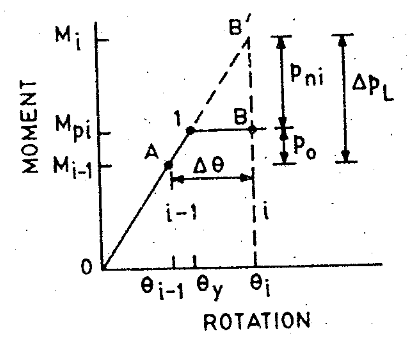


Fig. 15.22 Unbalanced moments in a load step

Instead of predicting a moment increment equal to p_0, it predicts a moment increment of Δp_L. Thus the excess moment p_n is the unbalanced moment. In order to avoid errors which may arise from accumulating equilibrium unbalances over many load steps, a corrective load is applied within the next load step to restore equilibrium. Because of yielding the actual member force is less by p_n. To eliminate this temporary load, a load equal to p_n must be added to the loads for the next load step. The fictitious external load p_n is permitted to act for a short period only.

The unbalanced moments at the ends i and j are given by

$$\begin{aligned} p_{ni} &= |M_i| - |M_{pi}| \\ p_{nj} &= |M_j| - |M_{pj}| \end{aligned} \tag{15.24}$$

The unbalanced moment should have the same sign as the incremental moment. The unbalanced moment so obtained will always have a positive sign. The proper sign is assigned as follows :

$$p_{ni} = p_{ni} \times \frac{M_i}{|M_i|}$$

and

$$p_{nj} = p_{nj} \times \frac{M_j}{|M_j|} \tag{15.25}$$

Unbalanced shear force at end i is given by

$$p_{nsi} = \frac{p_{ni} + p_{nj}}{L}$$

and

$$p_{nsj} = (-)\,\frac{p_{ni} + p_{nj}}{L} \tag{15.26}$$

The unbalanced axial loads are zero, because in flexural members, it is assumed that the axial load carrying capacity is never exceeded. Thus, in local coordinate system,

$$\underset{\sim}{p}_n = \begin{Bmatrix} 0 \\ p_{nsi} \\ p_{ni} \\ 0 \\ p_{nsj} \\ p_{nj} \end{Bmatrix}_{6\times 1} \tag{15.27}$$

This can be transformed into the global coordinate system as follows :

$$\begin{aligned} p'_{n1} &= -\ p_{nsi}\ \sin\theta \\ p'_{n2} &= p_{nsi}\ \cos\theta \\ p'_{n3} &= p_{ni} \\ p'_{n4} &= p_{nsj}\ \sin\theta \\ p'_{n5} &= p_{nsj}\ \cos\theta \\ p'_{n6} &= p_{nj} \end{aligned} \tag{15.28}$$

Now the actual moments developed at the ends of the member are reset as :

$$\begin{aligned} &\text{if} \quad p_{ni} \neq 0,\ \text{set}\ |M_i| = |M_{pi}| \\ &\text{if} \quad p_{nj} \neq 0,\ \text{set}\ |M_j| = |M_{pj}| \end{aligned} \tag{15.29}$$

The sign of M_i and M_j remain unchanged.

The unbalanced load vector for the structure $\mathbf{P}_n$ is assembled by adding tne unbalanced load vectors for all members. The unbalanced load vector is added to the incremental load vector, the stiffness matrix is modified, if plastification has occurred, and the analysis is carried out for the next load step to get the incremental displacement vector.

The procedure to calculate the unbalance load from M-P interaction curve in case of a column is shown in Fig. 15.23a and b. In either situation, the axial force is kept constant and the changes are made in the moment.

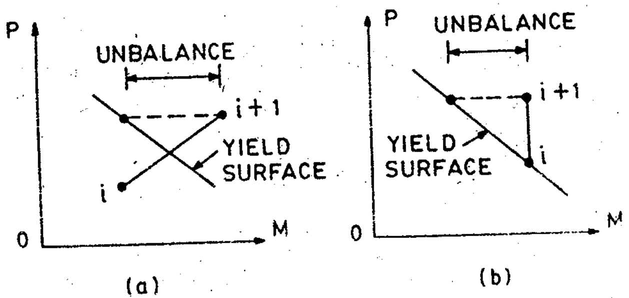


Fig. 15.23 Unbalanced load in M - P interaction curve

15.15 STEP-BY-STEP INCREMENTAL ANALYSIS METHOD

In the incremental method, load is applied in a number of steps. The basic assumption of the process is that the displacement varies linearly during each load increment while the properties of the structure remain constant during this interval. The analysis procedure consists of the following steps:

Step 1 Determine elastic stiffness and equivalent nodal loads for each member and transform in global co-ordinates.

Step 2 Assemble the stiffness matrix and the incremental load vector for the structure.

Step 3 Solve the linear simultaneous equilibrium equations to get the incremental displacements.

Step 4 Determine the incremental and total member forces in local coordinates.

Step 5 Compare the total moment developed at each end of a member with its plastic moment capacity. If plastification occurs any where, calculate the unbalanced load vector.

Step 6 Update the structural stiffness matrix and load vector and repeat steps 3 to 6, till the total load is applied or a mechanism forms.

In case of a truss element, the total axial force is compared with the tension yield load or bucking load depending upon the nature of the axial force. In case, the axial force exceeds P_y or P_{cr}, the unbalanced load vector is computed and rest of the procedure is the same as for a flexural element.

15.16 DUCTILITY

Non-linearity is measured in terms of ductility. It is defined as the ratio of the maximum deformation to the yield deformation. The deformation may be strain, curvature, rotation or displacement. A strain based ductility depends on the material properties, while curvature based ductility depends on the shape and size of the cross-section in addition to the material properties. A rotation based ductility also includes the effect of member length and member end conditions. When the definition of ductility is applied with respect to displacement, the entire configuration of the structure and loading is also taken into account. The various definitions may be written as:

$$\text{Curvature ductility,} \quad \mu_\phi = \frac{|\phi|_{max}}{\phi_y} \tag{15.30}$$

$$\text{Rotational ductility,} \quad \mu_\theta = \frac{|\theta|_{max}}{\theta_y} \tag{15.31}$$

$$= 1 + \frac{|\theta_p|_{max}}{\theta_y} \quad \text{if m-}\theta\text{ relation is bi-linear} \tag{15.32}$$

$$\text{Displacement ductility} \quad \mu_\Delta = \frac{|\Delta|_{max}}{\Delta_y} \tag{15.33}$$

where

ϕ_y = yield curvature
θ_y = yield rotation
θ_p = plastic rotation

$$|\theta|_{max} = |\theta_y| + |\theta_p| \tag{15.34}$$

Δ_y = yield displacement

The yield rotation for a beam is defined as the rotation when the member is subjected to anti symmetric yield or plastic moment M_p as shown in Fig.15.24, and is equal to :

$$\theta_y = \frac{M_P L}{6EI} \tag{15.35}$$

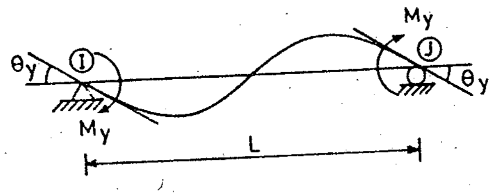


Fig. 15.24 Yield rotation in a beam element

If a structure is ductile, it will tend to deform inelastically. In this process it will redistribute the excess load to elastic parts of the structure. A ductile structure will continue to deform and resist the loads throwing excess load to its elastic members. It signals sufficient warning to the occupants before collapse. On the contrary, a brittle structure has very low ductility and, therefore, a low capacity to deform as shown in Fig. 15.25. It fails suddenly without any warning. A non-linear analysis is able to predict the ductility requirements of a structure to resist the given set of loads. A concrete as well as a steel structure can be detailed to provide sufficient ductility. A ductile structure is highly desirable to resist severe earthquake forces.

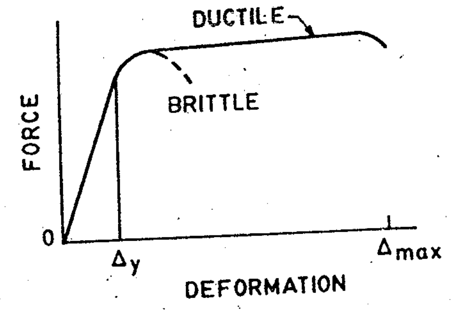


Fig. 15.25 Ductile and brittle behaviours

15.17 ILLUSTRATIVE EXAMPLES

Example 15.9

A single storey portal frame shown in Fig. 15.26 a is subjected to incremental lateral load P. Plot the load-displacement curve and the sequence of plastic hinging. Also, illustrate the effect of weak girder-strong column proportions and the effect of strong girder-weak column proportions on the nonlinear behaviour of the frame.

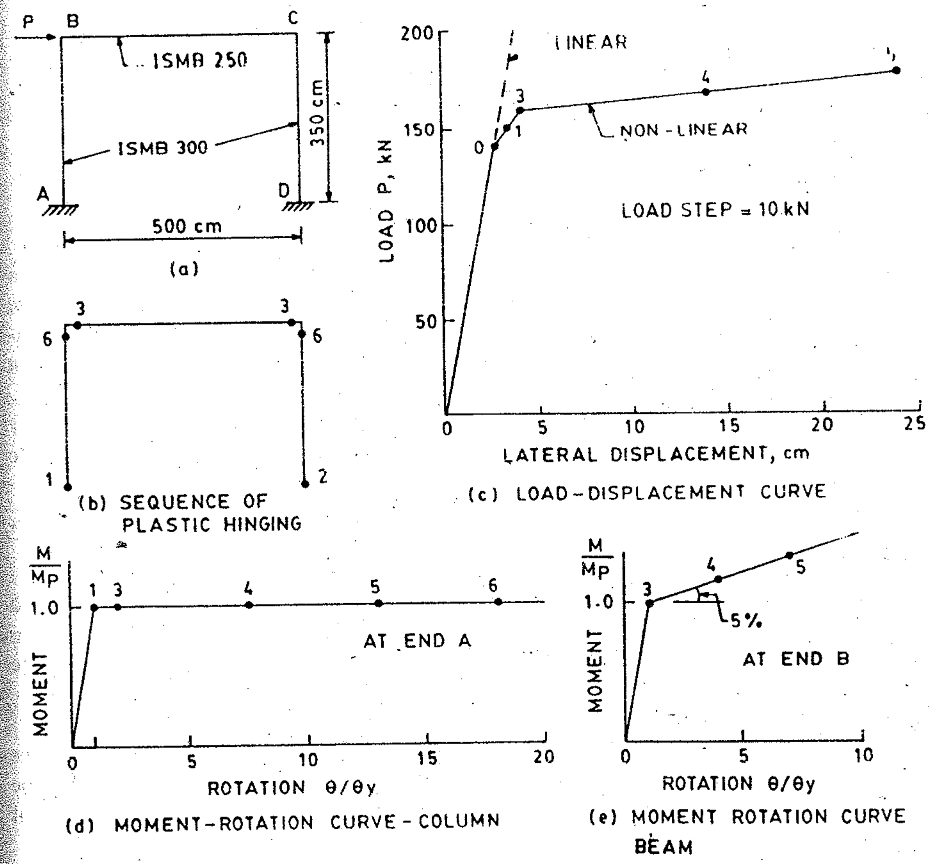


Fig. 15.26

Solution

The geometry and yield properties of the beam and column are shown in Table 15.1. A lateral load in steps of 10 kN is applied at B and the frame is analyzed using the nonlinear analysis program MINA-2D. The program prints lateral displacements, plastic rotations at the ends of each member as well as the location of the plastic hinges. The

sequence of plastic hinge is shown in Fig. 15.26b, and the load-displacement curve in Fig. 15.26c. The frame is linear upto a lateral load of 140 kN and displacement of 2.55 cm. Under increasing load, the frame collapses at a load of 200 kN.

Table 15.1 Member properties

Member	A cm^2	I cm^4	M_p kN-cm	E kN/cm^2	Strain-hardening Ratio
Beam ISMB 250	47.55	5130	11625	21000	0
Column ISMB 300	56.25	8600	16300	21000	0.05

The moment-rotation curve M- θ for end A of the column and for end B of the beam are shown in Figs. 15.26d and 15.26e respectively. The numbers on these curves indicate the corresponding numbers on the load-displacement curve in Fig. 15.26c. The strain-hardening slope is zero in Fig. 15.26d for the column. This is consistent with the input data for these members.

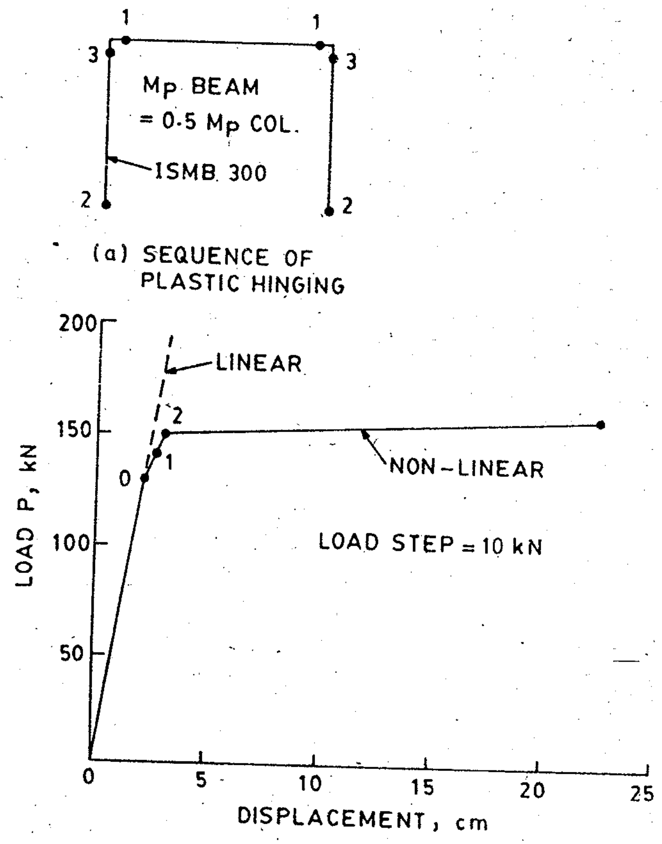
In order to study the behaviour of a weak girder-strong column frame, the plastic moment capacity of the beam is set equal to 50% of that of the column, that is,

$$M_{PB} = 0.50\, M_{PC}$$

The frame is reanalyzed using MINA-2D, and the results are shown in Fig.15.27 a and 15.27b. It remains elastic upto a load of 130 kN and lateral displacement of 2.4 cm. Finally, at a lateral load of 190 kN, it attains a displacement of 51.8 cm and the frame collapses.

In the frame of Fig. 15.26 a the plastic moment capacity of the beam was 0.71 times of that of the column, whereas, in the frame of Fig. 15.27a, it was 0.50 times. It can be seen that there is no significant effect of further reducing the plastic moment capacity of the beam from 71% to 50%. In either case, the frame is a ductile frame.

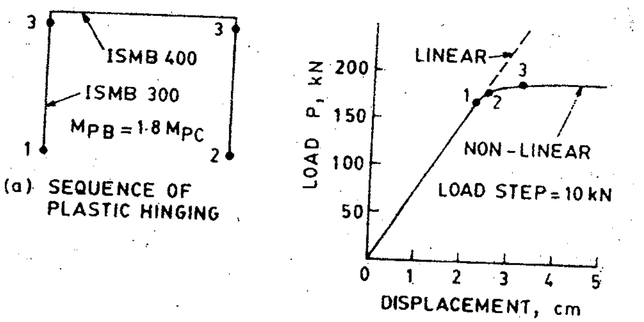
The beam section is increased to ISMB 400 in order to study the effect of strong girder-weak column proportion. The plastic moment capacity of the beam is 1.80 times of that of the column. The frame is analyzed and the results are shown in Fig. 15.28a and 15.28b. It can be seen that frame collapses at a load of 190 kN but at a much smaller displacement of about 3.2 cm. Thus, a strong girder-weak column frame is a brittle frame.



(a) SEQUENCE OF PLASTIC HINGING

(b) LOAD-DISPLACEMENT CURVE

Fig. 15.27



(a) SEQUENCE OF PLASTIC HINGING

(b) LOAD-DISPLACEMENT CURVE

Fig. 15.28

Example 15.10

A two storey-two bay frame is shown in Fig. 15.29a. It is subjected to gravity loads and incremental lateral load as shown. Illustrate the non-linear behaviour of the frame under the incremental lateral load till collapse.

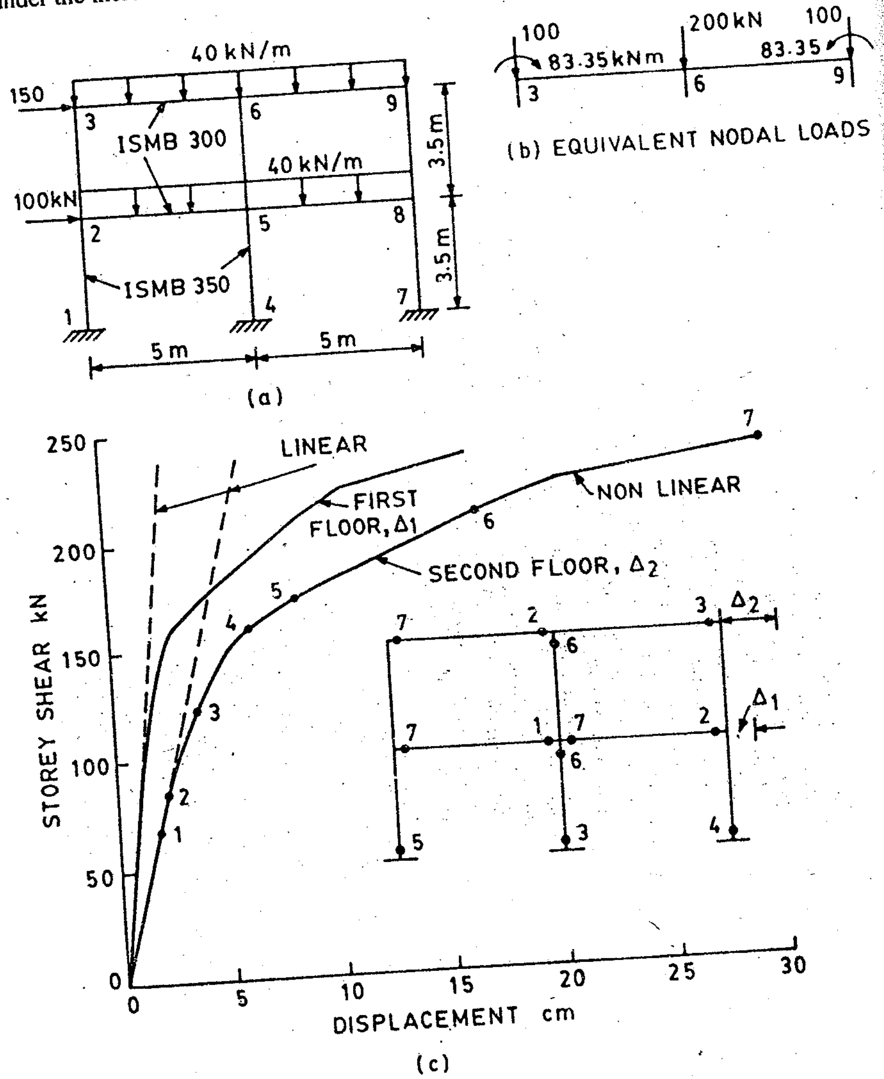


Fig. 15.29

Solution

The geometry and yield properties of the members used in the frame are shown in Table 15.2. The yield properties are calculated in accordance with IS : 800-1984. The gravity and lateral loads acting on the frame are shown in Fig. 15.29a. The uniform gravity load is replaced by equivalent nodal forces as shown in Fig. 15.29b. The lateral load of 100 kN and 150 kN is applied in 100 equal load steps, that is, in increments of 1 kN and 1.5 kN at the first and second floor, respectively. The nonlinear analysis is carried out using the Monotonic INelastic Analysis (MINA-2D) program developed by the author. The salient features of such a non-linear program have already been discussed in sections 15.8 to 15.15. A detailed discussion similar to that for the STAP-3D program is beyond the scope of this book.

The storey shear-lateral displacement behaviour is shown in Fig. 15.29c. The sequence of plastic hinge is also shown in the same figure. It can be seen that the frame is linear upto a base shear of 85 kN. Elastic analysis predicts a maximum lateral displacement of roof equal to 6.25 cm, whereas, the inelastic analysis predicts a displacement equal to 29.75 cm. This is a ductile frame.

Table 15.2 Member properties

Member	Tension yield Load P_y, kN	Compression Buckling Load, P_{cr}, kN	A cm^2	I cm^4	Plastic moment capacity $\pm M_p$, kN-cm
Beam ISMB 300	-	-	56.26	8603	± 16300
Outer Column ISMB 350	1418	–726	66.71	13630	± 22240
Inner Column ISMB 400	1960	–1004	92.27	30390	± 38825

CHAPTER
sixteen

NONLINEAR ANALYSIS: GEOMETRIC NON-LINEARITY

16.1 INTRODUCTION

In the previous chapters discussed so far, it was assumed that the deformations developed in the structure are small. That is why, the principle of superposition was valid. In practical terms, this means that geometry of the elements remains basically unchanged during the loading process. In many cases very large displacements may occur without causing large strains, and the above assumption is no more valid. It may be found that a load is reached where deflections increase more rapidly than predicted by a linear solution; alternatively there may be a situation where deflections increase very slowly than predicted by a linear solution. Such problems are frequently encountered in large span cable roofs and suspension bridges. The classic problem of structural stability where load carrying capacity decreases with continuing deformation also falls within geometrical non-linearity. This aspect, however, is not discussed since specialized relations between stress and strain have to be introduced which is beyond the scope of this book.

As deflection increases with the application of the load, the resulting change in geometry causes the previous linear force-deformation relations to become non-linear. The stiffness matrix depends on the change in geometry as well as on the internal forces. Such a nonlinear analysis can be carried out either using iterative methods or incremental methods. Geometric non-linearity may often be combined with material non-linearity of the type discussed in the previous chapter. This does not involve additional complications and can be easily incorporated in the formulations.

The stiffness matrix of a truss element or a beam element consists of two components:

$$K = K_0 + K_1 \tag{16.1}$$

where, K_0 = elastic stiffness matrix
and K_1 = geometric stiffness matrix

The elastic and geometric stiffness matrices are determined for each element at the beginning of each iteration and assembled to form the system stiffness matrix. Except the solution algorithm, rest of the procedure is the same as discussed earlier in Chapter 12.

16.2 GEOMETRIC STIFFNESS MATRIX - TRUSS ELEMENT

Consider a 3D-truss element shown in Fig. 16.1. Assuming linear elastic behaviour, the strain-displacement equation can be written as :

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \tag{16.2}$$

neglecting the higher order term $\left(\frac{\partial u}{\partial x}\right)^2$ as compared to $\frac{\partial u}{\partial x}$ for a 3-D truss element.

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{16.3}$$

Let us first consider a 2-D truss element for which

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 \qquad \because \; w = 0 \tag{16.4}$$

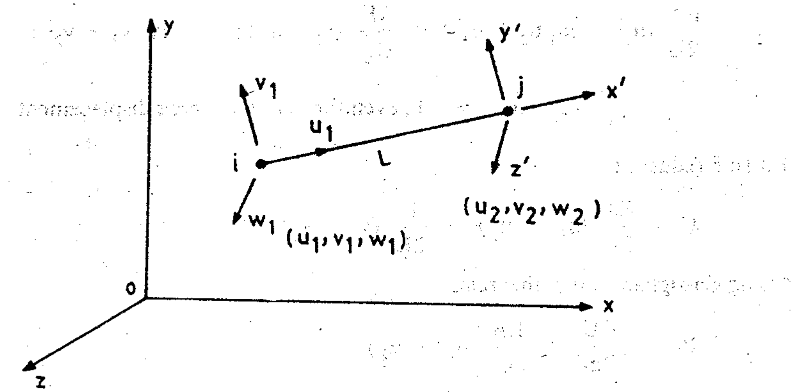


Fig. 16.1 3-D truss element

The displacements u and v vary linearly along the bar length. Let (u_1, v_1) and (u_2, v_2) represent the nodal displacement components for nodes i and j, respectively.

Setting, $\xi = \frac{x}{L}$,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} (1-\xi) & 0 & \xi & 0 \\ 0 & (1-\xi) & 0 & \xi \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \tag{16.5}$$

The displacement derivative can be obtained by differentiation as

$$\frac{\partial u}{\partial x} = \frac{-u_1 + u_2}{L}$$

$$\frac{\partial v}{\partial x} = \frac{-v_1 + v_2}{L} \tag{16.6}$$

The strain energy for a linear elastic material equals

$$U = \frac{1}{2}\int_0^v E\varepsilon_x^2\, dv = \frac{EA}{2}\int_0^L \varepsilon_x^2\, dx$$

$$= \frac{EA}{2}\int_0^L \left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^4\right] dx \tag{16.7}$$

Substituting Eqs. 16.6 in Eq. 16.7, neglecting higher order terms and integrating yields,

$$U = \frac{EA}{2L}(u_1^2 - 2u_1 u_2 + u_2^2) + \frac{AE}{2L^2}(u_2 - u_1)(v_1^2 - 2v_1 v_2 + v_2^2) \tag{16.8}$$

$$\frac{AE}{L}(u_2 - u_1) \approx F, \text{ even for relatively large displacements}$$

Eq.16.8 reduces to

$$U = \frac{EA}{2L}(u_1 - u_2)^2 + \frac{F}{2L}(v_1 - v_2)^2 \tag{16.9}$$

Using Castigliano's first theorem,

$$P_1 = \frac{\partial U}{\partial u_1} = \frac{EA}{L}(u_1 - u_2)$$

$$P_2 = \frac{\partial U}{\partial v_1} = \frac{F}{L}(v_1 - v_2)$$

$$P_3 = \frac{\partial U}{\partial u_2} = \frac{EA}{L}(-u_1 + u_2)$$

$$P_4 = \frac{\partial U}{\partial v_2} = \frac{F}{L}(-v_1 + v_2) \tag{16.10}$$

In matrix notation,

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \left\{ \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{F}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \right\} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \tag{16.11}$$

Thus,

$$\mathbf{K} = \mathbf{K_0} + \mathbf{K_1} \tag{16.12a}$$

$$\mathbf{K_0} = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4} \tag{16.12b}$$

and

$$\mathbf{K_1} = \frac{F}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}_{4\times4} \tag{16.12c}$$

The global stiffness matrix is written as

$$\mathbf{K'} = \mathbf{R^T K R} \tag{16.13a}$$

where

$$\mathbf{R} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}_{4\times4} \tag{16.13b}$$

where, λ is given by Eq. 12.7c.

The direction cosines are determined at the beginning of each iteration and are maintained constant through the iteration.

For a 3-D truss element the corresponding expressions can be written as :

$$\mathbf{K_0} = \frac{AE}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times6}$$

and

$$\mathbf{K}_1 = \frac{F}{L}\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]_{6\times 6} \tag{16.14}$$

The transformation matrix is given by Eqs. 13.19c and 13.20.

16.3 NON-LINEARITY OF CABLE SUSPENSION SYSTEMS

A cable is a truss element, that is, an axial element. However, under compression it becomes *slack* and looses its stiffness. Therefore, cable systems are designed to always remain under tension. Cables do not exhibit linear-elastic behaviour but rather non-linear elastic behaviour. Further non-linearity is caused due to large deformations. Another salient feature of cable suspension systems is the desirability of prestressing the cables. The modulus of elasticity of an unstretched cable varies widely. Under prestress, there is a higher mean tension in the cables which results in a more constant value for the modulus of elasticity.

The equivalent nodal force due to prestress in a truss or a cable element is given by :

$$\mathbf{p}^e = \begin{Bmatrix} -\cos\theta \\ -\sin\theta \\ \cos\theta \\ \sin\theta \end{Bmatrix} p_0 \tag{16.15a}$$

Similarly, the equivalent nodal loads due to thermal expansion is given by :

$$\mathbf{p}^e = \begin{Bmatrix} -\cos\theta \\ -\sin\theta \\ \cos\theta \\ \sin\theta \end{Bmatrix} E\,\alpha\,t\,A \tag{16.15b}$$

where α = coefficient of thermal expansion
t = change in temperature
A = area of cross-section of the element
E = modulus of elasticity of the material

16.4 NON-LINEAR SOLUTION ALGORITHMS

Geometric non-linear problems can be analyzed using one of the several numerical techniques. There are basically three broad categories of the numerical algorithms :

1. Iterative method
2. Incremental method, and
3. Incremental cum iterative method.

Iterative method

In an iterative method, a structure is fully loaded using an initial value for the stiffness. Because of this, the equilibrium is not satisfied; the total unbalanced load is used in the next iteration to compute additional increments of the deflections as shown in Figs. 15.16a and b. For the stiffness, either the initial stiffness or the tangent stiffness at the beginning of each iteration (Fig. 15.15) can be used. In general, the convergence is faster if tangent stiffness is used, although, it increases the programming efforts considerably since the stiffness matrix is computed, transformed and assembled in each iteration.

For geometric non-linear problems the Newton-Raphson iterative technique is very effective. The basic steps with respect to Fig. 16.2 are as follows :

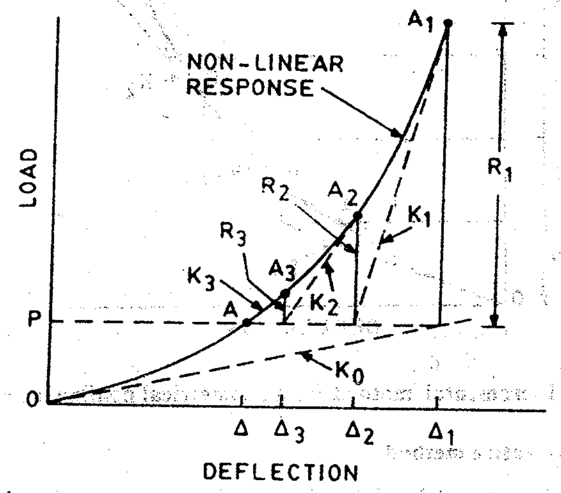


Fig. 16.2 Iterative method for a geometrical nonlinear problem

Step 1 Compute the value of Δ_1 from $K\Delta_1 = P$.
Step 2 Compute the unbalanced load vector R_1 using the initial stiffness.
Step 3 Compute the tangent stiffness K_1 at A_1 and determine Δ_2 using the equilibrium equation

$$\mathbf{K}\,\Delta = \mathbf{P} - \mathbf{R} \tag{16.16}$$

Step 4 Compute the unbalanced load vector R_2 on the basis of values obtained in step 3
Step 5 Repeat steps 3 and 4 until the convergence is achieved, that is, the values of $\mathbf{R}$ or Δ in two successive cycles agree within the desired accuracy.

Incremental method

In this type of algorithm, the total load P is applied in small load increments p as shown in Fig. 16.3. During the application of each increment, the value of stiffness is assumed constant. The solution of each load increment is found as an increment in the deflection Δ. The non-linear problem is thus analyzed as a piecewise linear problem, that is,

$$K_{i-1}\, d\Delta_i = dP_i \tag{16.17}$$

K_{i-1} is the tangent stiffness matrix. The total deflection is given as

$$\Delta_i = \Delta_0 + \sum_{j=1}^{n} d\Delta_j \tag{16.18}$$

where Δ_0 is the initial deflection and n = no. of load steps upto (i-1) load step.

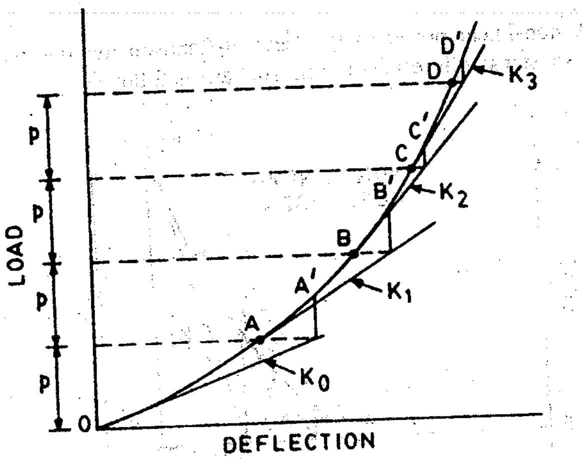


Fig. 16.3 Incremental method for a geometrical nonlinear problem

Incremental cum Iterative method

A combination of incremental and iterative techniques improves the convergence very fast. The Newton-Raphson method is used in increments. Iterations are carried out within each load step till convergence is reached. If the load step is small, the unbalanced load is also small and the convergence is faster.

This method requires greater computational efforts because the stiffness matrix has to be computed in every cycle of iteration. There are, however, two advantages :

(a) Any change in the elastic or physical properties of the structure can be easily accounted for in the stiffness matrix.

The number of load steps depends on the nature of the problem. Therefore, it is not possible to specify an optimum load size in order to achieve maximum computational efficiency. This can only be determined from experience gained by the application of this method to different types of problems.

The basic steps with respect to Fig. 16.3 are as follows :

Step 1 Apply a small load p, assume initial stiffness $\mathbf{K_0}$ and locate point A' on the response curve. This gives the unbalanced load **R**.

Step 2 Iterate using the tangent stiffness matrix at A' in the first iteration till point A is reached.

Step 3 Apply the next load increment, assume tangent stiffness $\mathbf{K_1}$ at point A, and locate point B' on the response curve.

Step 4 Repeat steps 2 and 3, till all the load increments are exhausted.

It is important to note that the load in each incremental step need not be equal.

16.5 CONVERGENCE CRITERIA

In using any iterative scheme, computation is terminated when the desired degree of accuracy has been achieved. If single numbers are used in the iterative scheme, the convergence of a series of numbers, say $x_1, x_2, x_3, \ldots\ldots\ldots, x_k$ to a number x is simply measured by

$$\lim_{k\to\infty} |x_k - x| = 0 \tag{16.19}$$

where k represents the k th iteration.

The rate of convergence C is determined by

$$\lim_{k\to\infty} \frac{|x_{k+1} - x|}{|x_k - x|^p} = C \tag{16.20}$$

where p represents the order of convergence (p > 1).

While working with vectors, the convergence is defined through vector norms. If $x_1, x_2, x_3, \ldots\ldots, x_k$ are the vectors in different iterations, the convergence is said to occur if

$$\lim_{k\to\infty} \|x_k - x\| = 0 \tag{16.21}$$

Order of convergence p, and the rate of convergence C are calculated as :

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x\|}{\|x_k - x\|^p} = C \tag{16.22}$$

Any one of the following three vector norms is generally used :

(a) Infinite-norm $\|x\|_\infty = \max |x_i|$ (16.23a)
that is, the absolute largest value.

(b) First norm

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \tag{16.23b}$$

that is, sum of all the absolute elements in any iteration.

(c) Second norm or Euclidean vector norm

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = \sqrt{x^T x} \tag{16.23c}$$

that is, square root of the sum of the squares of all the elements in any iteration.

In large displacement problems, the convergence criteria is used on the displacement vector. The norm of the displacement vector should be less than a specified percentage say 0.01% or 0.001%.

16.6 CABLE-3D PROGRAM

A computer program is written in FORTRAN to analyze a large displacement problem using 3-D truss elements. Its source listing is available on a floppy. It employs the Newton-Raphson iterative technique with incremental loading. The program was written by Dr. P. N. Godbole, Professor of Civil Engineering, University of Roorkee, and later modified by the author. The program consists of the following subroutines : Main, GDATA, ASSEMB, LDATA, STIFM, GEOMBC, SOLVE, UPDATE and FORCE.

Main Routine

It calls GDATA, ASSEMB, SOLVE, UPDATE and FORCE subroutines for various load steps and iterations for each load step. The data is read from a file and the results are displayed on the terminal as well as written into a file. There are five labeled COMMON blocks : ONE, TWO, THREE, FOUR and FIVE. COMMON block TWO is defined in subroutine GDATA while COMMON block FIVE is defined in subroutine ASSEMB. COMMON block ONE consists of integer control variables. COMMON block TWO consists of dimensioned arrays such as coordinates, connectivity, materials, length, boundary conditions and specified displacements. The remaining COMMON blocks consist of various working variables and arrays. The main routine is a controlling routine for the entire program.

The important variables are as follows :

NP = total number of nodes
NE = total number of elements
NB = total number of boundary nodes
NDF = number of degrees of freedom per node
NLD = number of load steps
NMAT = number of materials
NIT = number of maximum iterations

ILD = current load step
IT = current iteration
NEQ = number of total equations
IMAT = current material
NOPE = element nodal connectivity
IBAND = band width
ALI = member length
PS = initial prestress in a member
PD = prescribed displacement
AK = structural stiffness matrix
EST = element stiffness matrix
R1 = global load vector or displacement vector
R2 = global concentrated load vector
TDIS = total displacement vector
TFORCE = total force vector
FAC = fraction of total load in a load step

Subroutine GDATA This routine reads control data and geometric data including material properties and initial prestress data. The generated data is written into the output file.

Subroutine ASSEMB It assembles the structural stiffness matrix **AK** and load vector **R1** and calls each element one by one. Concentrated loads are read by calling subroutine LDATA. Boundary conditions are also introduced in the stiffness matrix.

Subroutine LDATA It reads the concentrated load data at various nodes, and fraction of the total load to be applied in a given load step. It also updates the load vector when called for each load step. The initial prestress load vector is stored in array **RP** and concentrated load vector is stored in array **R2**.

Subroutine STIFM It generates the stiffness matrix including large deformation component of an element and load vector due to initial prestress in each call. These are generated in global coordinates through necessary transformations. The stiffness matrix is stored in upper triangular transformations. The stiffness matrix is stored in upper triangular matrix in banded form.

Subroutine GEOMBC It modifies the assembled stiffness matrix **AK** and total load vector **R1** for prescribed displacement.

Subroutine SOLVE The main program calls it twice. In the first call, it triangularizes the symmetric banded matrix. In the second call, it performs the back substitution and the incremental displacements are returned in array **R1** which was originally a load array.

Subroutine UPDATE From the incremental displacement, it calculates the total displacement, updates nodal coordinates and prints the results for each iteration of a given load step.

Subroutine FORCE From the incremental displacements it calculates member forces in current load step as well as total member force at the end of the current load step.

16.7 USER'S INSTRUCTIONS

The following questions must be answered while executing the program on a Personal Computer :

1. The names of the input and output files.
2. Please enter print code for more results: 0/1
   0 : do not print ; 1 : print
3. The title of the problem :
   (a) The user may also indicate his/her name, units employed and date etc.
   (b) The program accepts data in any set of consistent units.
4. Maximum number of iterations in a load step.

DATA PREPARATION

The following data must be made available in the input file mentioned earlier.

I. Control Data (6I5)

Columns	1 - 5	Total number of nodes (NP) .LE. 60
	6 - 10	Total number of cable elements (NE) .LE. 40
	11 - 15	Total number of boundary nodes (NB) .LE. 20
	16 - 20	Number of load steps (NLD)
	21 - 25	Number of different materials (NMAT) .LE. 50
	26 - 30	Degree of freedom per node (NDF) 2 : for 2-D analysis 3 : for 3-D analysis

II Material Properties (I5, 2F10.0)

One line is required for each material property.

Columns	1 - 5	Material number
	6 - 15	Area
	16 - 25	Modulus of elasticity

III Nodal Coordinates (I5,3F10.0)

Columns	1 - 5	Node number
	6 - 15	x-coordinate
	16 - 25	y-coordinate
	26 - 35	z-coordinate

NOTES:

1. The coordinate axes correspond to the right hand thumb rule.

IV Member Data (4I5)

Columns	1 - 5	Member number
	6 - 10	Node I
	11 - 15	Node J
	16 - 20	Material identification number

V Prestressing Force data (8F10.0)

This data is required for each member. Provide 8 values per line.

Columns	1 - 10	Prestress force in member 1
	11 - 20	Prestress force in member 2 and so on

VI Boundary Conditions (2I5,3F10.0)

Columns	1 - 5	Boundary node number Boundary condition for displacement 0 : free ; 1 : restrained
		IF NDF =2,
	- 9	x - d.o.f.
	- 10	y - d.o.f.
		IF NDF=3
	- 8	x - d.o.f.
	- 9	y - d.o.f.
	- 10	z - d.o.f.
	11 - 20	Specified displacement in x-direction
	21 - 30	....y-direction
	31 - 40	.... z-direction

VII Load Data (I5, 3F10.0)

Columns	1 - 5	Node number
	6 - 15	Load in x-direction
	16 - 25	Load in y-direction
	26 - 35	Load in z-direction

NOTE :

1. Load data for the last node must be given.

VIII Load Multipliers (8F10.0)

Total load is multiplied by a factor given in each load step.
Give as many values as the number of load steps.

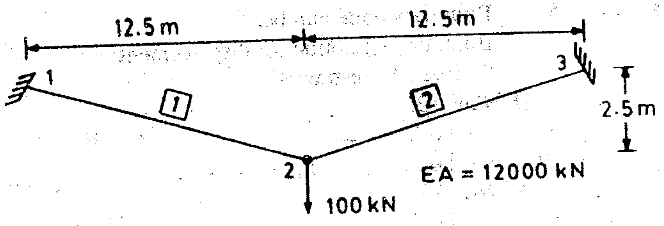
Columns	1 - 10	Multiplier for load step 1
	11 - 20	Multiplier for load step 2
	21 - 30	 3
	31 - 40	 4
	41 - 50	 5
	51 - 60	....
	61 - 70	....
	71 - 80	 8

16.8 ILLUSTRATIVE EXAMPLES

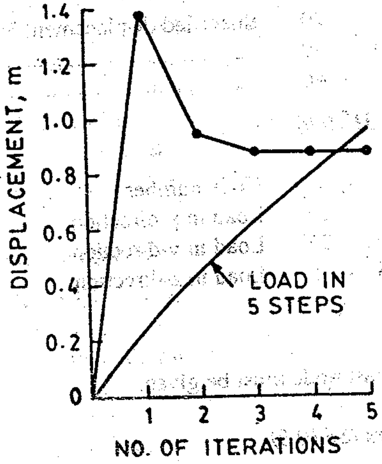
Example 16.1

A cable is suspended across a span of 25 m. It has a sag of 2.5 m and carries a vertical load of 100 kN as shown in Fig. 16.4a. Determine the deflection of the loaded node using the program CABLE-3D. If

(i) the total load is applied in one step
(ii) the total load is applied in five steps.



(a)



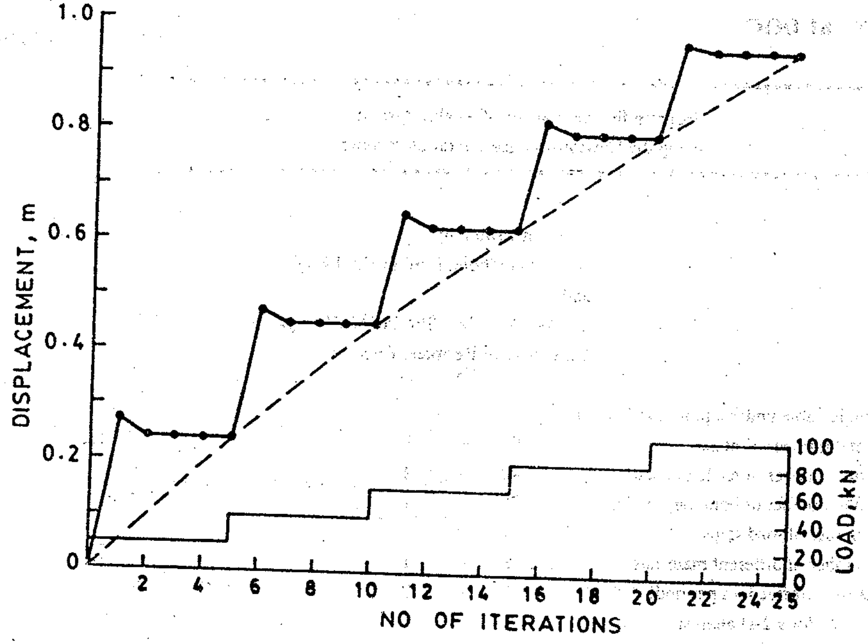
(b)

Fig. 16.4

Solution

(i) Load applied in one step

Five iterations are specified. The value of EA is given as 12000 kN. The area of cross-section of the cable is arbitrarily taken as 1 m², and therefore, modulus of elasticity of the material is taken as 12000 kN/m². Their individual values have no significance. The deflection-iteration curve is shown in Fig. 16.4b. In the first iteration the deflection is 1.37m which reduces in the second and subsequent cycles and converges to 0.88 m in the 5th iteration. The final tension in the cable was 191.2 kN. Input data and output of the program CABLE-3D are shown in EX161.FIL and EX161.DOC



(c)

Fig. 16.4 contd.

INPUT DATA

(Through Key Board)
EX 161.IN EX 161.DOC
single cable under a point load kN-m
5

(*Through File* **EX 161.IN**)

```
 3   2   2   1   1   2
 1 1     12000
 1
 2 12.5  2.5
 3 25.0
 1   1   2   1
 2   2   3   1
0.       0.
 1   11
 3   11
 2       100.
 3       0.
1.0    0.2    0.2    0.2    0.2
```

EX 161.DOC

```
*************************************************************************
            Program for the analysis of a cable system
            using the Newton-Raphson iteration technique
*************************************************************************

                     Program written by
                     Dr. P.N.Godbole, Prof. of Civil Engg.,
                     and
                     Dr. Ashok K. Jain, Prof. of Civil Engg.
                     University of Roorkee, Roorkee
```

single cable under a point load kN-m

Total number of nodes	.LE. 60	=	3
Total number of cable elements	.LE. 40	=	2
Total number of boundary nodes	.LE. 20	=	2
Number of load steps		=	1
Number of different materials	.LE. 5	=	1
Degree of freedom per node :		=	2
2 : for a 2-D analysis			
3 : for a 3-D analysis			
Max. number of iterations		=	5

MATERIAL Number	AREA	MODULUS OF elasticity
1	.100E+01	.120E+05

NODE	X-CORD	Y-CORD	Z-CORD
1	.000	.000	
2	12.500	2.500	
3	25.000	.000	

ELEMENT No.	NODAL I	CONNECTION J	MATERIAL	TYPE
1	1	2	1	
2	2	3	1	

Member	Prestress	Member	Prestress	Members	Prestress	Member	Prestress
1	.0000	2	.0000				

NODE no	BOUNDARY condition	SPECIFIED DISPLACEMENT x-	y-	z-
1	11	.000	.000	
3	11	.000	.000	

SEMIBAND WIDTH IS EQUAL TO 4

Node no	Load x-	y-	z-
2	.000	100.000	
3	.000	.000	

LOAD Step	MULTIPLIER
1	1.000

ADDITIONAL NON ZERO LOAD IN LOAD STEP = 1
2 .000 100.000

UPDATED COORDINATES AND DISPLACEMENTS AT
load step = 1
and iteration = 1

MEMBER	MEMBER forces at the BEGINNING of the current iteration
1	0.0000E+00
2	0.0000E+00

In the beginning of the first iteration, of each load step, it assumes zero member forces

Node no	Total Displacement in current load step x-	y-	z-	Displacement in current iteration x-	y-	z-
1	.0000E+00	.0000E+00		.0000E+00	.0000E+00	
2	.0000E+00	.1381E+01		.0000E+00	.1381E+01	
3	.0000E+00	.0000E+00		.0000E+00	.0000E+00	

Node	Final Coordinates x-	y-	z-
1	.0000E+00	.0000E+00	
2	.1250E+02	.3881E+01	
3	.2500E+02	.0000E+00	

UPDATED COORDINATES AND DISPLACEMENTS AT
load step = 1
and iteration = 2

MEMBER	MEMBER forces at the BEGINNING of the current iteration
1	3.2107E+02
2	3.2107E+02

Node no	Total Displacement in current load step x-	y-	z-	Displacement in current iteration x-	y-	z-
1	.0000E+00	.0000E+00		.0000E+00	.0000E+00	
2	.0000E+00	−.4389E+00		.0000E+00	.9420E+00	
3	.0000E+00	.0000E+00		.0000E+00	.0000E+00	

Node	Final Coordinates x-	y-	z-
1	.0000E+00	.0000E+00	
2	.1250E+02	.3442E+01	
3	.2500E+02	.0000E+00	

UPDATED COORDINATES AND DISPLACEMENTS AT
load step = 1
and iteration = 3

MEMBER	MEMBER forces at the BEGINNING of the current iteration
1	2.0493E+02
2	2.0493E+02

Node no	Total Displacement in current load step x-	y-	z-	Displacement in current iteration x-	y-	z-
1	.0000E+00	.0000E+00		.0000E+00	.0000E+00	
2	.0000E+00	.8869E+00		.0000E+00	−.5513E-01	
3	.0000E+00	.0000E+00		.0000E+00	.0000E+00	

Node	Final Coordinates x-	y-	z-
1	.0000E+00	.0000E+00	
2	.1250E+02	.3387E+01	
3	.2500E+02	.0000E+00	

UPDATED COORDINATES AND DISPLACEMENTS AT
load step = 1
and iteration = 4

MEMBER	MEMBER forces at the BEGINNING of current iteration
1	1.9126E+02
2	1.9126E+02

Node no	Total Displacement in current load step x-	y-	z-	Displacement in current iteration x-	y-	z-
1	.0000E+00	.0000E+00		.0000E+00	.0000E+00	
2	.0000E+00	.8867E+00		.0000E+00	−.2437E-03	
3	.0000E+00	.0000E+00		.0000E+00	.0000E+00	

Node	Final Coordinates x-	y-	z-
1	.0000E+00	.0000E+00	
2	.1250E+02	.3387E+01	
3	.2500E+02	.0000E+00	

UPDATED COORDINATES AND DISPLACEMENTS AT
load step = 1
and iteration = 5

MEMBER	MEMBER forces at the BEGINNING of the current iteration
1	1.9120E+02
2	1.9120E+02

Node no	Total Displacement in current load step x-	y-	z-	Displacement in current iteration x-	y-	z-
1	.0000E+00	.0000E+00		.0000E+00	.0000E+00	
2	.0000E+00	.8867E+00		.0000E+00	.3413E-05	
3	.0000E+00	.0000E+00		.0000E+00	.0000E+00	

Node	Final Coordinates x-	y-	z-
1	.0000E+00	.0000E+00	
2	.1250E+02	.3387E+01	
3	.2500E+02	.0000E+00	

Member	Total Member Forces at the end of current load step	Member Forces due to the current load step only
1	191.199	191.199
2	191.199	191.199

(ii) Load applied in five steps

The load is applied in five equal steps of 20 kN each. The deflection increases with the increase in load. Within each load step, these were five iterations. In the first iteration of each load step, the deflection was largest which reduced and converged in the 3rd iteration shown in Fig. 16.4c. The final deflection was 0.96m. The final tension in the cable was 207.9 kN.

In this problem the Newton-Raphson iterative scheme is more efficient than that with incremental loading. The latter scheme overestimates the deflection by 8%.

Example 16.2

A cable is suspended across a horizontal span of 100 m with unequal supports as shown in Fig. 16.5a. It carries two point loads of 100 kN and 200 kN. Determine the final deflected shape of the cable if it carries an initial prestress of 20 kN, 20 kN and 30 kN in the three segments as shown in the same figure. Take EA = 15000 kN.

Solution -

This problem was very sensitive to

(i) the amount of initial prestress in the three cable segments, and
(ii) maximum number of iterations.

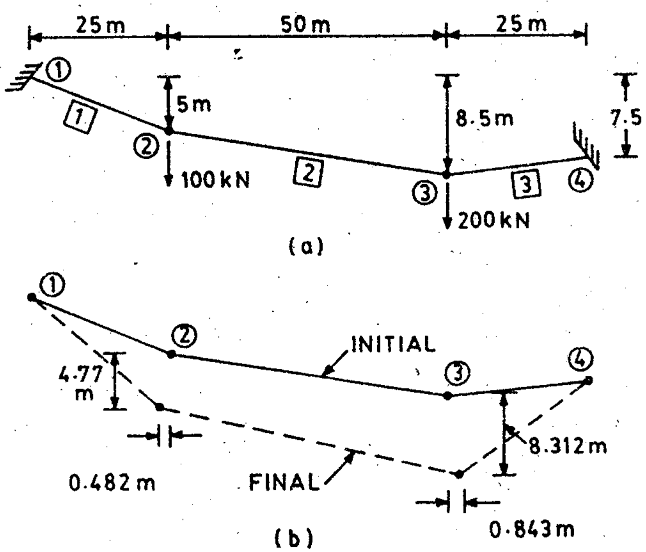


Fig. 16.5

A mechanism is formed if there is no initial prestress in the cable. The solution diverges if the maximum number of iterations are specified as 5 or 6. The final results were obtained by using a maximum of 10 iterations. The total load was applied in five equal steps. The final deflected shape is shown in Fig. 16.5b. The deflections of the nodes 2 and 3 at the end of each step are shown in Table 16.1.

In this problem the Newton-Raphson iterative scheme with incremental loading proved to be quite efficient.

Table 16.1 Deflections at load points

Load Step	Node 2		Node 3	
	u, m	v, m	u, m	v, m
1	– 0.094	1.469	0.170	3.581
2	– 0.131	1.165	0.175	1.669
3	– 0.100	0.854	0.172	1.225
4	– 0.084	0.692	0.166	0.992
5	– 0.0733	0.590	0.160	0.845
Σ Deflections	– 0.482	4.770	0.843	8.312

Example 16.3

A cable net is suspended across a span of 300 m as shown in Fig. 16.6. It is subjected to a load of 2000 kN applied equally at nodes 4, 5, 8 and 9. Each cable is prestressed to 50 kN. Determine the final deflected shape of the net.

Solution

This is a 3-D cable net. There are 12 nodes and 12 elements as shown in circles and boxes, respectively in Fig. 16.6a and b. The area of cross section of cable elements 4, 6, 7 and 9 is 60 cm^2 and that of the rest of the elements is 40 cm^2. The modulus of elasticity is 20000 kN/cm^2. The load is applied in 10 unequal steps. The complete input data and selected output of the program CABLE-3D are shown in EX163.FIL and EX163.DOC.

The solution is quite sensitive to the number of load steps and maximum number of iterations in each load step. The final vertical deflection of node 4 is 1.03 m from the initial position.

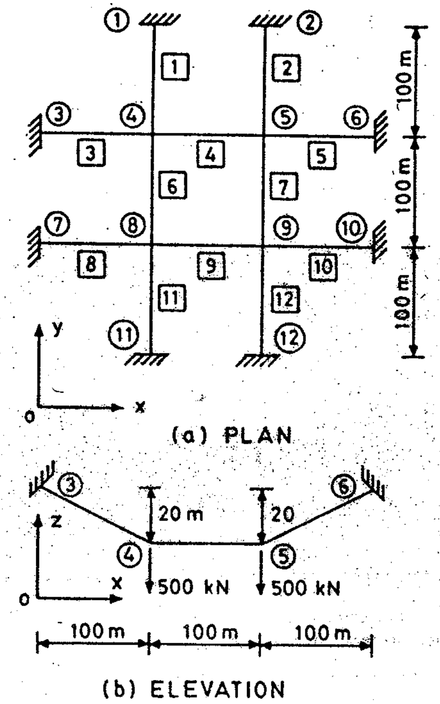


Fig. 16.6

EX 163.FIL

```
12   12    8   10   2   3
 1 0.004   20000 0000.
 2 0.006   20000 0000.
 1 100.    300.
 2 200.    300.
 3 300.    200.
 4 300.    100.
 5 200.
 6 100.
 7         100.
 8         200.
 9 100.    200.       -20.
10 200.    200.       -20.
11 200.    100.       -20.
12 100.    100.       -20.
 1    1    9    1
 2    2   10    1
 3    8    9    1
 4    9   10    2
 5   10    3    1
 6    9   12    2
 7   10   11    2
 8    7   12    1
 9   12   11    2
10   11    4    1
11   12    6    1
12   11    5    1
50.      50.      50.      50.      50.      50.      50.      50.
50.      50.      50.      50.
  1  111
  2  111
  3  111
  4  111
  5  111
  6  111
  7  111
  8  111
  9                -500.
 10                -500.
 11                -500.
 12                -500.
0.05     0.05     0.1      0.15     0.15     0.1      0.1      0.1
0.1      0.1
```

****

Program for the analysis of a cable system
using the Newton-Raphson iteration technique

****

Program written by
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and
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analysis of a cable net kN-m Ex16.3

Total number of nodes	.LE. 60	= 12
Total number of cable elements	.LE. 40	= 12
Total number of boundary nodes	.LE. 20	= 8
Number of load steps		= 10
Number of different materials	.LE. 5	= 2
Degree of freedom per node		= 3
2 : for a 2-D analysis		
3 : for a 3-D analysis		
Max. number of iterations		= 10

MATERIAL Number	AREA	MODULUS OF elasticity
1	.400E-02	.200E+09
2	.600E-02	.200E+09

NODE	X-CORD	Y-CORD	Z-CORD
1	100.000	300.000	.000
2	200.000	300.000	.000
3	300.000	200.000	.000
4	300.000	100.000	.000
5	200.000	.000	.000
6	100.000	.000	.000
7	.000	100.000	.000
8	.000	200.000	.000
9	100.000	200.000	-20.000
10	200.000	200.000	-20.000
11	200.000	100.000	-20.000
12	100.000	100.000	-20.000

ELEMENT No.	NODAL CONNECTION I	J	MATERIAL TYPE
1	1	9	1
2	2	10	1
3	8	9	1
4	9	10	2
5	10	3	1
6	9	12	2
7	10	11	2
8	7	12	1
9	12	11	2
10	11	4	1
11	12	6	1
12	11	5	1

Member	Prestress	Member	Prestress	Member	Prestress	Member	Prestress
1	50.0000	2	50.0000	3	50.0000	4	50.0000
5	50.0000	6	50.0000	7	50.0000	8	50.0000
9	50.0000	10	50.0000	11	50.0000	12	50.0000

NODE no	BOUNDARY condition	SPECIFIED DISPLACEMENT x-	y-	z-
1	111	.000	.000	.000
2	111	.000	.000	.000
3	111	.000	.000	.000
4	111	.000	.000	.000
5	111	.000	.000	.000
6	111	.000	.000	.000
7	111	.000	.000	.000
8	111	.000	.000	.000

SEMIBAND WIDTH IS EQUAL TO 27

Node no	Load x-	y-	z-
9	.000	.000	−500.000
10	.000	.000	−500.000
11	.000	.000	−500.000
12	.000	.000	−500.000

LOAD Step	MULTIPLIER
1	.050
2	.050
3	.100
4	.150
5	.150
6	.100
7	.100
8	.100
9	.100
10	.100

ADDITIONAL NON ZERO LOAD IN LOAD STEP = 1

1	.000	−49.029	−9.806
2	.000	−49.029	−9.806
3	−49.029	.000	−9.806
4	−49.029	.000	−9.806
5	.000	−49.029	−9.806
6	.000	−49.029	−9.806
7	−49.029	.000	−9.806
8	−49.029	.000	−9.806
9	−.971	−.971	−5.388
10	−.971	−.971	−5.388
11	−.971	−.971	−5.388
12	−.971	−.971	−5.388

UPDATED COORDINATES AND DISPLACEMENTS AT
load step = 1
and iteration = 10

Node no	Total Displacement in currrent load step x-	y-	z-	Displacement in current iteration x-	y-	z-
1	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
2	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
3	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
4	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
5	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
6	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
7	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
8	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
9	−.2582E−02	.2619E−02	−.5424E−01	.7960E−05	.4438E−05	.3575E−05
10	.2619E−02	.2619E−02	−.5424E−01	.4414E−05	.4414E−05	.3702E−05
11	.2619E−02	−.2582E−02	−.5424E−01	.4438E−05	.7961E−05	.3577E−05
12	−.2582E−02	−.2582E−02	−.5424E−01	.7985E−05	.7985E−05	.3692E−05

Node	Final Coordinates x-	y-	z-
1	.1000E+03	.3000E+03	.0000E+00
2	.2000E+03	.3000E+03	.0000E+00
3	.3000E+03	.2000E+03	.0000E+00
4	.3000E+03	.1000E+03	.0000E+00
5	.2000E+03	.0000E+00	.0000E+00
6	.1000E+03	.0000E+00	.0000E+00
7	.0000E+00	.1000E+03	.0000E+00
8	.0000E+00	.2000E+03	.0000E+00
9	.1000E+03	.2000E+03	−.2005E+02
10	.2000E+03	.2000E+03	−.2005E+02
11	.2000E+03	.1000E+03	−.2005E+02
12	.1000E+03	.1000E+03	−.2005E+02

Member	Total Member Forces at the end of current load step	Member Forces due to the current load step only
1	63.515	63.515
2	63.515	63.515
3	63.515	63.515
4	62.227	62.227
5	63.515	63.515
6	62.227	62.227
7	62.227	62.227
8	63.515	63.515
9	62.227	62.227
10	63.515	63.515
11	63.515	63.515
12	63.515	63.515

ADDITIONAL NON ZERO LOAD IN LOAD STEP = 2

9	.000	.000	−25.000
10	.000	.000	−25.000
11	.000	.000	−25.000
12	.000	.000	−25.000

UPDATED COORDINATES AND DISPLACEMENTS AT
load step = 2
and iteration = 10

Node no	Total Displacement in currrent load step			Displacement in current iteration		
	x-	y-	z-	x-	y-	z-
1	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
2	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
3	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
4	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
5	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
6	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
7	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
8	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
9	–.2583E–02	.2570E–02	–.5391E–01	–.3148E–05	–.9198E–05	–.6523E–05
10	.2585E–02	.2585E–02	–.5388E–01	–.8951E–05	–.8951E–05	.5365E–04
11	.2570E–02	–.2583E–02	–.5391E–01	–.9198E–05	–.3148E–05	–.6524E–05
12	–.2586E–02	–.2586E–02	–.5395E–01	–.3360E–05	–.3361E–05	–.6898E–04

Node	Final Coordinates		
	x-	y-	z-
1	.1000E+03	.3000E+03	.0000E+00
2	.2000E+03	.3000E+03	.0000E+00
3	.3000E+03	.2000E+03	.0000E+00
4	.3000E+03	.1000E+03	.0000E+00
5	.2000E+03	.0000E+00	.0000E+00
6	.1000E+03	.0000E+00	.0000E+00
7	.0000E+00	.1000E+03	.0000E+00
8	.0000E+00	.2000E+03	.0000E+00
9	.9999E+02	.2000E+03	–.2011E+02
10	.2000E+03	.2000E+03	–.2011E+02
11	.2000E+03	.9999E+02	–.2011E+02
12	.9999E+02	.9999E+02	–.2011E+02

Member	Total Member Forces at the end of current load step	Member Forces due to the current load step only
1	127.029	63.515
2	126.934	63.419
3	126.934	63.419
4	124.168	61.941
5	126.934	63.419
6	124.168	61.941
7	124.168	61.941
8	126.934	63.419
9	124.168	61.941
10	127.029	63.515
11	126.934	63.419
12	126.934	63.419

ADDITIONAL NON ZERO LOAD IN LOAD STEP = 3

9	.000	.000	–50.000
10	.000	.000	–50.000
11	.000	.000	–50.000
12	.000	.000	–50.000

UPDATED COORDINATES AND DISPLACEMENTS AT

load step = 3

and iteration = 10

Node no	Total Displacement in currrent load step			Displacement in current iteration		
	x-	y-	z-	x-	y-	z-
1	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
2	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
3	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
4	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
5	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
6	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
7	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
8	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
9	–.5149E–02	.5151E–02	–.1069E+00	–.5285E–05	–.1927E–04	–.1903E–04
10	.5121E–02	.5121E–02	–.1069E+00	–.1310E–04	–.1311E–04	.1058E–04
11	.5151E–02	–.5149E–02	–.1069E+00	–.1927E–04	–.5287E–05	–.1905E–04
12	–.5164E–02	–.5164E–02	–.1070E+00	–.1130E–04	–.1130E–04	–.4755E–04

Node	Final Coordinates		
	x-	y-	z-
1	.1000E+03	.3000E+03	.0000E+00
2	.2000E+03	.3000E+03	.0000E+00
3	.3000E+03	.2000E+03	.0000E+00
4	.3000E+03	.1000E+03	.0000E+00
5	.2000E+03	.0000E+00	.0000E+00
6	.1000E+03	.0000E+00	.0000E+00
7	.0000E+00	.1000E+03	.0000E+00
8	.0000E+00	.2000E+03	.0000E+00
9	.9999E+02	.2000E+03	–.2022E+02
10	.2000E+03	.2000E+03	–.2022E+02
11	.2000E+03	.9999E+02	–.2022E+02
12	.9999E+02	.9999E+02	–.2022E+02

Member	Total Member Forces at the end of current load step	Member Forces due to the current load step only
1	253.201	126.171
2	253.105	126.171
3	253.105	126.171
4	247.765	123.596
5	253.105	126.171
6	247.765	123.596
7	247.765	123.596
8	253.105	126.171
9	247.765	123.596
10	253.201	126.171
11	253.105	126.171
12	253.105	126.171

ADDITIONAL NON ZERO LOAD IN LOAD STEP = 4

9	.000	.000	–75.000
10	.000	.000	–75.000
11	.000	.000	–75.000
12	.000	.000	–75.000

UPDATED COORDINATES AND DISPLACEMENTS AT

load step = 4

and iteration = 10

Node no	Total Displacement in current load step			Displacement in current iteration		
	x-	y-	z-	x-	y-	z-
1	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
2	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
3	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
4	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
5	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
6	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
7	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
8	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
9	-.7665E-02	.7632E-02	-.1582E+00	.4606E-05	-.4206E-05	-.7622E-06
10	.7675E-02	.7675E-02	-.1582E+00	-.4292E-05	-.4297E-05	-.2271E-05
11	.7632E-02	-.7665E-02	-.1582E+00	-.4198E-05	.4599E-05	-.8247E-06
12	-.7669E-02	-.7669E-02	-.1582E+00	-.6979E-05	-.6984E-05	.5206E-05

Node	Final Coordinates		
	x-	y-	z-
1	.1000E+03	.3000E+03	.0000E+00
2	.2000E+03	.3000E+03	.0000E+00
3	.3000E+03	.2000E+03	.0000E+00
4	.3000E+03	.1000E+03	.0000E+00
5	.2000E+03	.0000E+00	.0000E+00
6	.1000E+03	.0000E+00	.0000E+00
7	.0000E+00	.1000E+03	.0000E+00
8	.0000E+00	.2000E+03	.0000E+00
9	.9998E+02	.2000E+03	-.2037E+02
10	.2000E+03	.2000E+03	-.2037E+02
11	.2000E+03	.9998E+02	-.2037E+02
12	.9998E+02	.9998E+02	-.2037E+02

Member	Total Member Forces at the end of current load step	Member Forces due to the current load step only
1	440.979	187.778
2	440.884	187.778
3	440.979	187.874
4	431.728	183.964
5	440.884	187.778
6	431.871	184.107
7	431.728	183.964
8	440.884	187.778
9	431.871	184.107
10	440.979	187.778
11	440.884	187.778
12	440.979	187.874

ADDITIONAL NON ZERO LOAD IN LOAD STEP = 5

9	.000	.000	-75.000
10	.000	.000	-75.000
11	.000	.000	-75.000
12	.000	.000	-75.000

UPDATED COORDINATES AND DISPLACEMENTS AT
load step = 5
and iteration = 10

Node no	Total Displacement in currrent load step			Displacement in current iteration		
	x-	y-	z-	x-	y-	z-
1	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
2	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
3	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
4	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
5	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
6	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
7	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
8	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
9	-.7593E-02	.7596E-02	-.1558E+00	.1373E-04	-.1034E-05	.1333E-04
10	.7636E-02	.7636E-02	-.1558E+00	.1722E-04	.1722E-04	-.1642E-04
11	.7596E-02	-.7593E-02	-.1558E+00	-.1034E-05	.1373E-04	.1333E-04
12	-.7594E-02	-.7594E-02	-.1558E+00	.7576E-05	.7575E-05	.4516E-04

Node	Final Coordinates		
	x-	y-	z-
1	.1000E+03	.3000E+03	.0000E+00
2	.2000E+03	.3000E+03	.0000E+00
3	.3000E+03	.2000E+03	.0000E+00
4	.3000E+03	.1000E+03	.0000E+00
5	.2000E+03	.0000E+00	.0000E+00
6	.1000E+03	.0000E+00	.0000E+00
7	.0000E+00	.1000E+03	.0000E+00
8	.0000E+00	.2000E+03	.0000E+00
9	.9997E+02	.2000E+03	-.2053E+02
10	.2000E+03	.2000E+03	-.2053E+02
11	.2000E+03	.9997E+02	-.2053E+02
12	.9997E+02	.9997E+02	-.2053E+02

Member	Total Member Forces at the end of current load step	Member Forces due to the current load step only
1	627.327	186.348
2	627.327	186.443
3	627.422	186.443
4	614.262	182.533
5	627.327	186.443
6	614.405	182.533
7	614.262	182.533
8	627.232	186.348
9	614.405	182.533
10	627.327	186.348
11	627.232	186.348
12	627.422	186.443

ADDITIONAL NON ZERO LOAD IN LOAD STEP = 6

9	.000	.000	-50.000
10	.000	.000	-50.000
11	.000	.000	-50.000
12	.000	.000	-50.000

UPDATED COORDINATES AND DISPLACEMENTS AT
load step = 6
and iteration = 10

Node no	Total Displacement in currrent load step			Displacement in current iteration		
	x-	y-	z-	x-	y-	z-
1	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
2	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
3	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
4	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
5	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
6	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
7	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
8	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
9	–.5042E–02	.5043E–02	–.1028E+00	–.3178E–05	.3240E–05	–.1284E–04
10	.5038E–02	.5038E–02	–.1029E+00	.3179E–05	.3179E–05	–.1285E–04
11	.5043E–02	–.5042E–02	–.1028E+00	.3240E–05	.3177E–05	–.1284E–04
12	–.5036E–02	–.5036E–02	–.1028E+00	–.3075E–05	–.3075E–05	–.1204E–04

Node	Final Coordinates		
	x-	y-	z-
1	.1000E+03	.3000E+03	.0000E+00
2	.2000E+03	.3000E+03	.0000E+00
3	.3000E+03	.2000E+03	.0000E+00
4	.3000E+03	.1000E+03	.0000E+00
5	.2000E+03	.0000E+00	.0000E+00
6	.1000E+03	.0000E+00	.0000E+00
7	.0000E+00	.1000E+03	.0000E+00
8	.0000E+00	.2000E+03	.0000E+00
9	.9997E+02	.2000E+03	–.2063E+02
10	.2000E+03	.2000E+03	–.2063E+02
11	.2000E+03	.9997E+02	–.2063E+02
12	.9997E+02	.9997E+02	–.2063E+02

Member	Total Member Forces at the end of current load step	Member Forces due to the current load step only
1	751.019	123.692
2	751.114	123.787
3	751.209	123.787
4	735.283	121.021
5	751.114	123.787
6	735.426	121.021
7	735.283	121.021
8	751.019	123.787
9	735.426	121.021
10	751.019	123.692
11	751.019	123.787
12	751.209	123.787

ADDITIONAL NON ZERO LOAD IN LOAD STEP = 7

9	.000	.000	–50.000
10	.000	.000	–50.000
11	.000	.000	–50.000
12	.000	.000	–50.000

UPDATED COORDINATES AND DISPLACEMENTS AT
load step = 7
and iteration = 10

Node no	Total Displacement in currrent load step			Displacement in current iteration		
	x-	y-	z-	x-	y-	z-
1	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
2	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
3	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
4	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
5	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
6	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
7	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
8	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
9	–.5031E–02	.5032E–02	–.1018E+00	.7290E–05	.5179E–05	–.1197E–04
10	.5008E–02	.5008E–02	–.1018E+00	–.1173E–05	–.1174E–05	–.4146E–04
11	.5032E–02	–.5031E–02	–.1018E+00	.5180E–05	.7290E–05	–.1197E–04
12	–.5020E–02	–.5020E–02	–.1018E+00	.1743E–05	.1743E–05	.2144E–04

Node	Final Coordinates		
	x-	y-	z-
1	.1000E+03	.3000E+03	.0000E+00
2	.2000E+03	.3000E+03	.0000E+00
3	.3000E+03	.2000E+03	.0000E+00
4	.3000E+03	.1000E+03	.0000E+00
5	.2000E+03	.0000E+00	.0000E+00
6	.1000E+03	.0000E+00	.0000E+00
7	.0000E+00	.1000E+03	.0000E+00
8	.0000E+00	.2000E+03	.0000E+00
9	.9996E+02	.2000E+03	–.2073E+02
10	.2000E+03	.2000E+03	–.2073E+02
11	.2000E+03	.9996E+02	–.2073E+02
12	.9996E+02	.9996E+02	–.2073E+02

Member	Total Member Forces at the end of current load step	Member Forces due to the current load step only
1	874.233	123.215
2	874.233	123.119
3	874.329	123.119
4	855.732	120.449
5	874.233	123.119
6	855.875	120.449
7	855.732	120.449
8	874.138	123.119
9	855.875	120.449
10	874.233	123.215
11	874.138	123.119
12	874.329	123.119

ADDITIONAL NON ZERO LOAD IN LOAD STEP = 8

9	.000	.000	–50.000
10	.000	.000	–50.000
11	.000	.000	–50.000
12	.000	.000	–50.000

UPDATED COORDINATES AND DISPLACEMENTS AT
load step = 8
and iteration = 10

Node no	Total Displacement in currrent load step x-	y-	z-	Displacement in current iteration x-	y-	z-
1	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
2	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
3	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
4	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
5	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
6	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
7	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
8	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
9	−.4998E−02	.5011E−02	−.1008E+00	−.4293E−06	.1281E−04	.1391E−04
10	.5006E−02	.5005E−02	−.1009E+00	.1097E−05	.1100E−05	.1068E−04
11	.5011E−02	−.4998E−02	−.1008E+00	.1280E−04	−.4250E−06	.1394E−04
12	−.4993E−02	−.4993E−02	−.1008E+00	.1124E−04	.1125E−04	.1000E−04

Node	Final Coordinates x-	y-	z-
1	.1000E+03	.3000E+03	.0000E+00
2	.2000E+03	.3000E+03	.0000E+00
3	.3000E+03	.2000E+03	.0000E+00
4	.3000E+03	.1000E+03	.0000E+00
5	.2000E+03	.0000E+00	.0000E+00
6	.1000E+03	.0000E+00	.0000E+00
7	.0000E+00	.1000E+03	.0000E+00
8	.0000E+00	.2000E+03	.0000E+00
9	.9996E+02	.2000E+03	−.2083E+02
10	.2000E+03	.2000E+03	−.2083E+02
11	.2000E+03	.9996E+02	−.2083E+02
12	.9996E+02	.9996E+02	−.2083E+02

Member	Total Member Forces at the end of current load step	Member Forces due to the current load step only
1	996.685	122.452
2	996.685	122.452
3	996.876	122.547
4	975.609	119.877
5	996.685	122.452
6	975.752	119.877
7	975.609	119.877
8	996.590	122.452
9	975.752	119.877
10	996.685	122.452
11	996.590	122.452
12	996.876	122.547

ADDITIONAL NON ZERO LOAD IN LOAD STEP = 9

9	.000	.000	−50.000
10	.000	.000	−50.000
11	.000	.000	−50.000
12	.000	.000	−50.000

UPDATED COORDINATES AND DISPLACEMENTS AT
load step = 9
and iteration = 10

Node no	Total Displacement in currrent load step x-	y-	z-	Displacement in current iteration x-	y-	z-
1	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
2	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
3	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
4	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
5	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
6	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
7	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
8	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
9	−.4980E−02	.5004E−02	−.9991E−01	−.4915E−06	.4487E−06	.9736E−05
10	.4966E−02	.4966E−02	−.9990E−01	−.1188E−04	−.1188E−04	.9375E−05
11	.5004E−02	−.4980E−02	−.9991E−01	.4444E−06	−.4871E−06	.9769E−05
12	−.4966E−02	−.4966E−02	−.9991E−01	−.1200E−06	−.1174E−06	.1131E−04

Node	Final Coordinates x-	y-	z-
1	.1000E+03	.3000E+03	.0000E+00
2	.2000E+03	.3000E+03	.0000E+00
3	.3000E+03	.2000E+03	.0000E+00
4	.3000E+03	.1000E+03	.0000E+00
5	.2000E+03	.0000E+00	.0000E+00
6	.1000E+03	.0000E+00	.0000E+00
7	.0000E+00	.1000E+03	.0000E+00
8	.0000E+00	.2000E+03	.0000E+00
9	.9995E+02	.2000E+03	−.2093E+02
10	.2000E+03	.2000E+03	−.2093E+02
11	.2000E+03	.9995E+02	−.2093E+02
12	.9995E+02	.9995E+02	−.2093E+02

Member	Total Member Forces at the end of current load step	Member Forces due to the current load step only
1	1118.660	121.975
2	1118.660	121.975
3	1118.851	121.975
4	1094.770	119.162
5	1118.660	121.975
6	1095.057	119.305
7	1094.770	119.162
8	1118.565	121.975
9	1095.057	119.305
10	1118.660	121.975
11	1118.565	121.975
12	1118.851	121.975

ADDITIONAL NON ZERO LOAD IN LOAD STEP = 10

9	.000	.000	−50.000
10	.000	.000	−50.000
11	.000	.000	−50.000
12	.000	.000	−50.000

UPDATED COORDINATES AND DISPLACEMENTS AT
load step = 10
and iteration = 10

Node no	Total Displacement in current load step			Displacement in current iteration		
	x-	y-	z-	x-	y-	z-
1	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
2	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
3	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
4	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
5	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
6	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
7	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
8	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
9	−.4947E−02	.4947E−02	−.9899E−01	−.3387E−05	−.9088E−05	−.1446E−04
10	.4978E−02	.4978E−02	−.9898E−01	.9002E−05	.9008E−05	−.4112E−04
11	.4947E−02	−.4947E−02	−.9899E−01	−.9098E−05	−.3377E−05	−.1438E−04
12	−.4967E−02	−.4966E−02	−.9899E−01	.2360E−05	.2366E−05	.1271E−04

Node	Final Coordinates		
	x-	y-	z-
1	.1000E+03	.3000E+03	.0000E+00
2	.2000E+03	.3000E+03	.0000E+00
3	.3000E+03	.2000E+03	.0000E+00
4	.3000E+03	.1000E+03	.0000E+00
5	.2000E+03	.0000E+00	.0000E+00
6	.1000E+03	.0000E+00	.0000E+00
7	.0000E+00	.1000E+03	.0000E+00
8	.0000E+00	.2000E+03	.0000E+00
9	.9995E+02	.2001E+03	−.2103E+02
10	.2001E+03	.2001E+03	−.2103E+02
11	.2001E+03	.9995E+02	−.2103E+02
12	.9995E+02	.9995E+02	−.2103E+02

Member	Total Member Forces at the end of current load step	Member Forces due to the current load step only
1	1240.158	121.498
2	1240.063	121.403
3	1240.254	121.403
4	1213.646	118.876
5	1240.063	121.403
6	1213.646	118.589
7	1213.646	118.876
8	1239.967	121.403
9	1213.646	118.589
10	1240.158	121.498
11	1239.967	121.403
12	1240.254	121.403

Appendix A

PROGRAMS FOR SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS

A.1 GAUSS ELIMINATION METHOD

```
C*********************************************************************
C
C        GAUSS ELIMINATION METHOD FOR SOLUTION OF AX=B
C                 WITH PARTIAL PIVOTING
C        written by Dr. Ashok K. Jain, University of Roorkee, Roorkee
C
C*********************************************************************
C
C     N IS SIZE OF MATRIX TO BE INVERTED OR SOLVED
C     M IS NUMBER OF R.H.S. VECTORS
C     [A] IS THE COEFFICIENT MATRIX, SIZE = (N+M) * N
C     X IS SOLUTION VECTOR
C     B IS R.H.S. MATRIX
C
      CHARACTER *24 DATFIL
      COMMON A(50,60)
C
      WRITE (*,*) ' Please enter name of the output file'
      READ (*,50) DATFIL
      OPEN (UNIT=6, FILE= DATFIL , STATUS='NEW')
C
      WRITE (*,100)
      WRITE (*,150)
      READ (*,*) N,M
      WRITE (6,100)
      WRITE (6,200) N,M
      N1=N+1
      NM=N+M
      WRITE (*,*) ' Please enter coefficient matrix row-wise '
      READ (*,*) ((A(I,J),J=1,N),I=1,N)
      WRITE (6, 300)
      DO 10 I=1,N
      WRITE (6, 400) (A(I,J),J=1,N)
10    CONTINUE
      WRITE (*,*) ' Please enter R.H.S. vectors column-wise '
      READ (*,*) ((A(I,J),I=1,N),J=N1,NM)
      WRITE (6, 500)
      DO 20 I=1,N
      WRITE (6, 400) (A(I,J),J=N1,NM)
20    CONTINUE
      CALL GAUSS (N,M)
      WRITE (6, 600)
      DO 30 I=1,N
      WRITE (6, 400) (A(I,J),J=N1,NM)
30    CONTINUE
C
```

```
50    FORMAT (A)
100   FORMAT (' Program for the solution of equation Ax = B '/
     *        ' using Gauss-Elimination method with partial pivoting '/
     *        ' written by Dr. Ashok K. Jain, University of Roorkee'/)
150   FORMAT (' Please enter matrix size : .LE. 50 '/
     *        ' No. of R.H.S. vectors    : .LE. 10 '/)
200   FORMAT (   ' SIZE OF COEFFICIENT MATRIX        =',I5,/
     *           ' NO OF R.H.S. VECTORS              =',I5,/)
300   FORMAT (/5X,' COEFFICIENT MATRIX A '/)
400   FORMAT (8E10.4)
500   FORMAT (/5X,' R. H. S. MATRIX B '/)
600   FORMAT (/5X,' SOLUTION MATRIX X '/)
C
      STOP
      END
C
      SUBROUTINE GAUSS (N,M)
      COMMON A(50,60)
C
C     Pivot selection
C
      NM=N+M
      DO 100 I=1,N
      IP1=I+1
      AA=ABS(A(I,I))
      K=I
      DO 200 J=I,N
      IF (ABS(A(J,I)). GT. AA ) THEN K=J
      AA=ABS(A(J,I))
      END IF
200   CONTINUE
C
C     Row interchange
C
      IF (K. NE. I) THEN
      DO 300 J=I,NM
      AA=A(I,J)
      A(I,J)=A(K,J)
      A(K,J)=AA
      300 CONTINUE
      END IF
      IF (ABS(A(I,I)) .LT. 1E-15) THEN
      WRITE (*,1100)
1100  FORMAT (' DIVISION BY ZERO : EXECUTION TERMINATED '/)
      STOP
      END IF
C
C     Tri-angularization
C
      AA=1.0/A(I,I)
      DO 400 J=I,NM
      A(I,J)=A(I,J)*AA
400   CONTINUE
      DO 500 J=IP1,N
      AA=A(J,I)
      DO 600 K=IP1,NM
      A(J,K)=A(J,K)-AA*A(I,K)
```



```
600   CONTINUE
500   CONTINUE
100   CONTINUE
C
C     Back-substitution
C
      DO 700 I=1,M
      DO 800 L=2,N
      NPI=N+I
      II=N - L + 1
      IP1=II+1
      SUM=A(II,NPI)
      DO 900 J=IP1,N
      SUM=SUM-A(II,J)*A(J,NPI)
900   CONTINUE
      A(II,NPI)=SUM
800   CONTINUE
700   CONTINUE
      RETURN
      END
```

A.2 GAUSS - JORDAN METHOD

```
C***********************************************************************
C
C          GAUSS-JORDAN METHOD FOR SOLUTION OF AX = B
C                AND/OR INVERSION OF MATRIX A
C          written by Dr. Ashok K. Jain, University of Roorkee, Roorkee
C
C***********************************************************************
C
C     N IS SIZE OF MATRIX TO BE INVERTED OR SOLVED
C     M IS NUMBER OF R.H.S. VECTORS
C     [A] IS THE COEFFICIENT MATRIX, ITS DIMENSIONS MUST BE CALCULATED
C     FROM THE RELATION N * (N+M+N)
C     X IS SOLUTION VECTOR
C     B IS R.H.S. MATRIX
C     [C] CONTAINS INVERSION OF [A]
C     IF ONLY INVERSION IS DESIRED, GIVE M=0
C         INV= -1, PROGRAM COMPUTES SOLUTION OF R.H.S. ONLY
C         INV=  0, PROGRAM COMPUTES INVERSION ONLY
C         INV=  1, PROGRAM COMPUTES SOLUTION OF R.H.S. AND INVERSION BOTH
C
      DIMENSION A(20,30)
      CHARACTER *24 DATFIL
      WRITE (*,*) ' Please enter name of the output file '
      READ (*,50) DATFIL
      OPEN (UNIT=6,FILE = DATFIL , STATUS = 'NEW')
C
      WRITE (*,2000)
      READ (*,*) N,M,INV
      WRITE (6,450)
      WRITE (6,500) N,M,INV
      N1=N+1
      NM=N+M
      NM1=NM+1
      NMN=NM+N
```

```
C
C     READ R.H.S. MATRIX IF SOLUTION IS DESIRED
C
      IF (INV.EQ.0) GO TO 1000
      WRITE (*,*) ' PLEASE ENTER R.H.S. VECTORS ROW-WISE '
      IF (INV.NE.0) READ (*,*) ((A(I,J),J=N1,NM),I=1,N)
      WRITE (6, 2200)
      DO 600 I=1,N
      WRITE (6, 550) (A(I,J),J=N1,NM)
600   CONTINUE
C
C     READ COEFFICIENT MATRIX
C
1000  CONTINUE
      WRITE (*,*) ' PLEASE ENTER COEFFICIENT MATRIX ROW-WISE '
      READ (*,*) ((A(I,J),J=1,N),I=1,N)
      WRITE (6, 2400)
      DO 650 I=1,N
      WRITE (6, 550) (A(I,J),J=1,N)
650   CONTINUE
C
      IF (INV.GE.0) THEN
C
C     CONSTRUCT DIAGONAL MATRIX
C
      DO 100 I=1,N
      NMI=NM+I
      DO 200 J=NM1,NMN
200   A(I,J)=0.
100   A(I,NMI)=1.
      ENDIF
C
      CALL GASJOR (A,N,NMN,INV)
C
      IF (INV.NE.0) THEN
      WRITE (6, 2600)
      DO 700 I=1,N
      WRITE (6, 550) (A(I,J),J=N1,NM)
700   CONTINUE
      ENDIF
      IF (INV.LT.0) THEN
      STOP
      ENDIF
      WRITE (6, 2800)
      DO 750 I=1,N
      WRITE (6, 550) (A(I,J),J=NM1,NMN)
750   CONTINUE
C
      CLOSE (UNIT=6)
50    FORMAT (A)
450   FORMAT (' Program for the solution of equation Ax=B ',/
     *        ' using Gauss-Jordan method '/
     *        ' written by Dr. Ashok K. Jain, University of Roorkee'/)
500   FORMAT ( ' SIZE OF COEFFICIENT MATRIX        = ',I5,/
     *         ' NO OF R.H.S. VECTORS              = ',I5,/
     *         ' INVERSION CODE                    = ',I5,/
     *         '    = -1, ONLY SOLUTION './
```



```
     *         '    = 0, ONLY INVERSION './
     *         '    = 1, SOLUTION AND INVERSION BOTH ',/)
550   FORMAT (8F10.3)
2000  FORMAT (' Program for the solution of equations Ax=B'/
     *        ' by Gauss Jordan Method '/
     *        ' written by Dr. Ashok K. Jain, University of Roorkee'/
     *        ' Please enter size of coefficient matrix : .LE. 20 '/
     *        '     no. of R.H.S. vectors   : .LE. 10 '/
     *        '     indicator INV           : -1 Solu. only '/
     *        '                             : 0 inversion only'/
     *        '                             : 1 Solu. and inversion '/)
2200  FORMAT (/5X,' INPUT R. H. S. MATRIX B '/)
2400  FORMAT (/5X, ' INPUT COEFFICIENT MATRIX '/)
2600  FORMAT (/5X,' SOLUTION MATRIX X '/)
2800  FORMAT (/5X.' INVERTED MATRIX C '/)
C
      STOP
      END
      SUBROUTINE GASJOR (A,N,NMN,INV)
      DIMENSION A(20,30)
C
      NM=NMN-N
      NM1=NM+1
      JJ=NMN
      IF (INV.LT.0) JJ=NM
C
      DO 500 K=1,N
      RAKK=1./A(K,K)
      K1=K+1
      DO 600 J=K1,JJ
600   A(K,J) = A(K,J)*RAKK
C
      A(K,K)=1.
      DO 700 I=1,N
      IF ((I-K).NE.0) THEN
      SAIK = -A(I,K)
      DO 800 J=K1,JJ
800   A(I,J) = A(I,J)+SAIK*A(K,J)
      ENDIF
700   CONTINUE
500   CONTINUE
C
      RETURN
      END
```

A.3 CHOLESKY METHOD

```
C**********************************************************************
C
C                CHOLESKY METHOD FOR SOLUTION OF Ax = b
C         written by Dr. Ashok K. Jain, University of Roorkee, Roorkee
C
C**********************************************************************
C
C     SOLUTION OF POSITIVE DEFINITE EQUATIONS
```

```
C     N = NUMBER OF EQUATIONS
C     M = NUMBER OF MAXIMUM OFF-DIAGONAL NON-ZERO TERMS PLUS 1
C     A = POSITIVE DEFINITE MATRIX, HALF BANDED UPPER TRIANGULAR
C         SYMMETRIC MATRIX
C     B = R.H.S. VECTOR
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(100,25),B(100),RA(100)
      CHARACTER *24 DATFIL
C
      WRITE (*,*) ' Please enter name of the output file '
      READ (*,40) DATFIL
      OPEN (UNIT=6,FILE= DATFIL, STATUS = 'NEW')
C
      WRITE (*,45)
      WRITE (*,50)
      READ (*,*) N,M,IND
      WRITE (*,60)
60    FORMAT (    ' Please enter coefficient matrix A , row-wise '/
     *            ' If size of matrix is 6 and semi band width is 3'/
     *            ' then feed the data as follows :'/
     *    '    * * * '/
     *    '    * * * '/
     *    '    * * * '/
     *    '    * * * '/
     *    '    * * 0 '/
     *    '    * 0 0 '/)
      READ (*,*) ((A(I,J),J=1,M),I=1,N)
      WRITE (*,*) ' Please enter right hand side vector b '
      READ (*,*) (B(I),I=1,N)
      WRITE (6,5) N,M,IND
C
      WRITE (6,10)
      DO 15 I=1,N
      WRITE (6,*)  (A(I,J),J=1,M)
15    CONTINUE
      WRITE (6,20)
      WRITE (6,*) (B(I),I=1,N)
C
C     DECOMPOSE MATRIX A INTO L*D*L TRANSPOSE FORM
C
      CALL SOLVE1 (N,M,A,RA)
      WRITE (6,25)
      DO 30 I=1,N
      WRITE (6,*) (A(I,J),J=1,M)
```



```
C
      IF (IND.EQ.1) STOP
C
C     SOLUTION OF EQUATIONS
C
      CALL SOLVE2 (N,M,A,RA,B)
C
      WRITE (6,35)
      WRITE (6,*)  (B(I),I=1,N)
      CLOSE (UNIT=6)
C
5     FORMAT (//  10X,' Cholesky method for the solution of positive'/
     *            10X,' definite symmetric banded matrix Ax = b'/
     *            10X,' written by Dr. Ashok K. Jain, Univ. of Roorkee'/
     *            10X,' NUMBER OF ROWS IN A MATRIX        =',I5//
     *            10X,' HALF BAND WIDTH                   =',I5//
     *            10X,' INDICATOR FOR SOLUTION            =',I5//
     *            10X,'  EQ. 1 : DECOMPOSE A=L*D*U ',//
     *            10X,'  EQ. 0 : SOLUTION OF EQUATION ',//)
10    FORMAT(/25X, ' A MATRIX ')
20    FORMAT(/25X, ' b vector ')
25    FORMAT (/25X,'D*L TRANSPOSE MATRIX ')
35    FORMAT(/25X,' Solution vector ')
40    FORMAT (A)
45    FORMAT (   ' Program for the solution of positive definite and '/
     *           ' symmetric banded matrix  Ax = b using Cholesky method'/
     *           ' Written by Dr. Ashok K. Jain, University of Roorkee'/
     *           ' Only the upper triangular matrix need be supplied '/
     *           ' by shifting the diagonal elements on to the first'/
     *           ' column in each row. The bottom right triangle '/
     *           ' should be filled with zeroes. '/)
50    FORMAT (   ' Please enter no. of rows : .LE. 100'/
     *           ' semi-band width      : .LE. 25 '/
     *           ' indicator for solution :        '/
     *           '      EQ. 1 : Decompose A = L*D*U '/
     *           '      EQ. 0 : solution of equations '/)
      STOP
      END
      SUBROUTINE SOLVE1 (N,NSBW,A,RA)
C
C     SUBROUTINE SOLVE1 REDUCES THE A MATRIX INTO L,L TRANSPOSE FORM
C     VECTOR RA IS A LOCAL VECTOR
C     NSBW IS NUMBER OF OFF-DIAGONAL TERMS + 1
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(100,25),RA(100)
```

```
C
      NA=N-1
      NW=NSBW-1
      DO 50 I=1,NA
      LA=I+1
      LB=I+NW
      CONS=1./A(I,1)
      RA(I)=CONS
      M=1
      IF ((LB-N).GT.0) THEN
      LB=N
      ENDIF
      DO 50 K=LA,LB
      M=M+1
      JC=1
      FUN=-CONS*A(I,M)
      JA=M
      DO 50 J=K,LB
      A(K,JC)=A(K,JC)+A(I,JA)*FUN
      JA=JA+1
      JC=JC+1
50    CONTINUE
C
      RA(N)=1./A(N,1)
C
      RETURN
      END
C
      SUBROUTINE SOLVE2 (N,NSBW,A,RA,C)
C
C     SUBROUTINE SOLVE2 SOLVES THE EQUATIONS FOR THE GIVEN RIGHT HAND
C     SIDE IN THE FORM OF MATRIX C.  ANSWER IS RETURNED IN MATRIX C.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(100,25),RA(100),C(100)
C
C     FORWARD SUBSTITUTION
C
      NA=N-1
      NW=NSBW-1
C
      DO 90 I=1,NA
      LA=I+1
      CI=C(I)/A(I,1)
      C(I)=CI
      CI=-CI
```



```
      JA=2
      LB=I+NW
      IF ((LB-N).GT.0) THEN
      LB=N
      ENDIF
      DO 90 J=LA,LB
      C(J)=C(J)+A(I,JA)*CI
      JA=JA+1
90    CONTINUE
      C(N)=C(N)/A(N,1)
C
C     BACKWARD SUBSTITUTION
C
      NB=N+1
      LA=NW
      DO 75 L=1,NA
      I=NB-L
      CI=-C(I)
      M=1
      LB=I-1
      IF ((I-NW).LE.0) THEN
      LA=LB
      ENDIF
      DO 75 J=1,LA
      M=M+1
      C(LB)=C(LB)+CI*A(LB,M)*RA(LB)
      LB=LB-1
75    CONTINUE
C
      RETURN
      END
```

A.4 SUCCESSIVE OVER RELAXATION METHOD

```
C*************************************************************
C         SUCCESSIVE OVER RELAXATION METHOD FOR SOLUTION
C         OF SIMULTANEOUS EQUATIONS
C         by
C         Dr. Ashok K. Jain, University of Roorkee, Roorkee
C*************************************************************
C     DESIGNED TO SOLVE A MAXIMUM OF 80 SIMULTANEOUS EQUATIONS
C     N = SIZE OF MATRIX
C     A = COEFFICIENT MATRIX
C     B = R.H.S. VECTOR
C     X = SOLUTION VECTOR
C     EPSLON = ABS. MAX. ERROR
```

```
C     W = OMEGA = OVER RELAXATION FACTOR
C     MAXIT = MAX. NO. OF ITERATIONS
C
      CHARACTER*10 DATFIL
      INTEGER CNVRGE
      COMMON /A/ A(80.81),X(80),N,MAXIT,EPSLON,W
      LOGICAL FLAG
      WRITE (*,*) ' Please enter name of the output file '
      READ (*,10) DATFIL
      OPEN (UNIT=6,FILE=DATFIL,STATUS='NEW')
      WRITE (*,50)
      WRITE (6,50)
C
      FLAG=.FALSE.
      CALL READAT (FLAG)
      CNVRGE = 1
      CALL SOLVER (CNVRGE,ITERS)
      IF (CNVRGE .NE. 1) GO TO 130
      ITERS=ITERS-1
      WRITE (*,40) ITERS,(X(I),I=1, N)
      CALL PRNT (ITERS,FLAG)
150   WRITE (*,100)
100   FORMAT(   ' Please enter code for calculations '/
     *          '   1 : recalculate '/
     *          '   2 : stop calculations '/)
      READ (*,*) ICH
      IF (ICH .EQ.1) THEN
      FLAG=.TRUE.
      CALL READAT (FLAG)
      DO 120 I=1,N
120   X(I)=0.0
      CNVRGE=1
      CALL SOLVER (CNVRGE,ITERS)
      IF (CNVRGE .NE. 1) GO TO 130
      ITERS=ITERS-1
      WRITE (*,40) ITERS,(X(I),I=1,N)
      CALL PRNT (ITERS,FLAG)
      GO TO 150
      ELSE
      GO TO 160
      ENDIF
130   CONTINUE
      WRITE (*,20) MAXIT
      WRITE (6,20) MAXIT
      GO TO 150
```

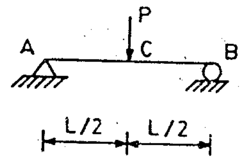
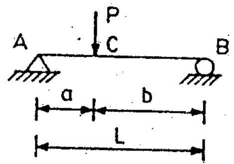
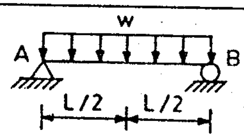
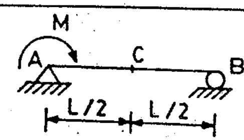
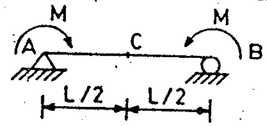
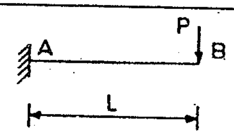
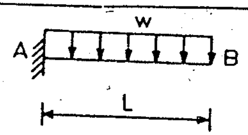
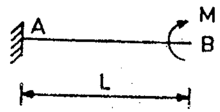


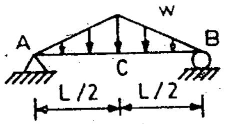
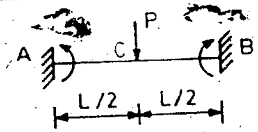
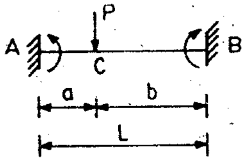
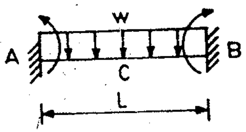
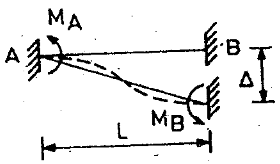
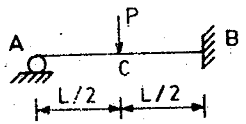
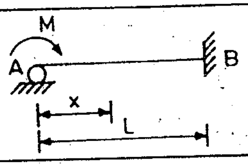
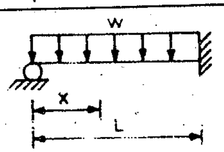
```
C
10    FORMAT (A)
20    FORMAT (5X/'SOLUTION DID NOT CONVERGE IN ',I4,' ITERATIONS')
40    FORMAT (5X,'NO. OF ITERATIONS =',I5//(1PE14.6))
50    FORMAT (   ' Solution of linear simultaneous equations using'/
     *           ' Successive Over Relaxation method '/
     *           ' by Dr Ashok K. Jain, University of Roorkee '/)
      CLOSE (UNIT=6)
      STOP
      END
C
      SUBROUTINE READAT (FLAG)
      COMMON /A/ A(80,81),X(80),N,MAXIT,EPSLON,W LOGICAL FLAG
C
      IF (FLAG) GO TO 20
      WRITE (*,*) ' Please enter size of matrix '
      READ (*,*) N
20    WRITE (*,10)
10    FORMAT (   ' Please enter the following data :'/
     *           ' 1. maximum number of iterations '/
     *           ' 2. maximum absolute error '/
     *           ' 3. over relaxation parameter '/
     *           '    w = 1 means Gauss-Seidal method'/)
      READ (*,*) MAXIT, EPSLON, W
      NPLUS1=N+1
      IF (FLAG) GO TO 30
      WRITE (*,*) ' Please enter the coefficient matrix row wise'
      READ (*,*) ((A(I,J),J=1,N),I=1,N)
      WRITE (*,*) ' Please enter r.h.s. vector B'
      READ (*,*) (A(I,NPLUS1),I=1,N)
C
30    CONTINUE
      RETURN
      END
C
      SUBROUTINE SOLVER (CNVRGE,ITERS)
      COMMON /A/ A(80.81),X(80),N,MAXIT,EPSLON,W
C
      NPLUS1= N+1
      ITERS = 1
10    CONTINUE
      RESID = 0.0
      DO 20 I= 1, N
      SUM = 0.0
      DO 30 J = 1,N
```

```
30     CONTINUE
       TEMP = ((1-W)*X(I))+(W *((A(I,NPLUS1) - SUM) / A(I,I)))
       RESID = AMAX1 (RESID, ABS(X(I) - TEMP))
       X(I) = TEMP
20     CONTINUE
       ITERS = ITERS +1
       IF (RESID .GE. EPSLON .AND. ITERS .LE. MAXIT) GO TO 10
       IF (ITERS .GT. MAXIT) CNVRGE = 0
       RETURN
       END
C
       SUBROUTINE PRNT (ITERS,FLAG)
       COMMON /A/ A(80,81),X(80),N,MAXIT,EPSLON,W
       LOGICAL FLAG
C
       IF (FLAG) GO TO 40
       WRITE (6,20) ((A(I,J),J=1, N),I=1, N)
20     FORMAT (/1X,'MATRIX A'//(5F12.4))
       WRITE (6,30) (A(I,N+1),I=1,N)
30     FORMAT (/1X,'MATRIX B'//(5F12.4))
40     CONTINUE
       WRITE (6,50) EPSLON,MAXIT,W,ITERS
50     FORMAT (/
     *          15X,'MAX. ABS. ACCURACY DESIRED          =',F10.6/
     *          15X,'MAXIMUM ITERATIONS DESIRED          =',I5/
     *          15X,'OVER RELAXATION PARAMETER           =',F8.5/
     *          15X,'ACTUAL NO. OF ITERATIONS            =',I5/)
       WRITE (6,60) (X(I),I=1,N)
60     FORMAT (1X,'Solution vector X '/(E15.4))
C
       RETURN
       END
```

Appendix B

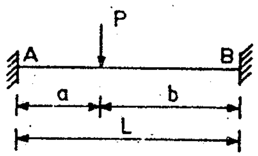
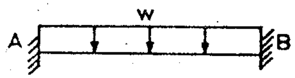
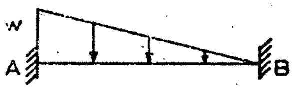
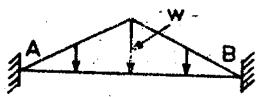
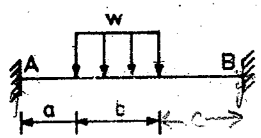
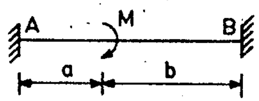
SLOPES AND DEFLECTIONS

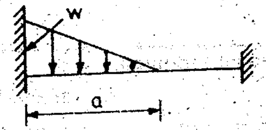
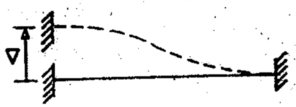
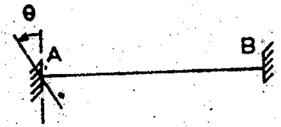
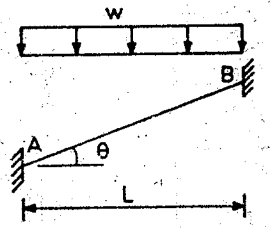
S. No.	Case	Slope or Deflection
1		$\theta_A = \frac{PL^2}{16EI} = \theta_B,\ \Delta_C = \frac{PL^3}{48EI}$
2		$\theta_A = \frac{Pab(2L-a)}{6EIL},\ \theta_B = \frac{Pa(L^2-a^2)}{6EIL}$ $\Delta_C = \frac{Pa^2b^2}{3EIL}$
3		$\theta_A = \theta_B = \frac{wL^3}{24EI},\ \Delta_C = \frac{5wL^4}{384EI}$
4		$\theta_A = \frac{ML}{3EI},\ \theta_B = \frac{ML}{6EI}$
5		$\theta_A = \frac{ML}{2EI} = \theta_B,\ \Delta_C = \frac{ML^2}{8EI}$
6		$\theta_B = \frac{PL^2}{2EI},\ \Delta_B = \frac{PL^3}{3EI}$
7		$\theta_B = \frac{wL^3}{6EI},\ \Delta_B = \frac{wL^4}{8EI}$
8		$\theta_B = \frac{ML}{EI},\ \Delta_B = \frac{ML^2}{2EI}$

S. No.	Case	Slope or Deflection
9		$\theta_A = \frac{5WL^2}{96EI} = \theta_B,\ \Delta_C = \frac{WL^3}{60EI}$ (where W = total load)
10		$\Delta_C = \frac{PL^3}{192EI}$
11		$\Delta_C = \frac{Pa^3b^3}{3EIL^3}$
12		$\Delta_C = \frac{wL^4}{384EI}$
13		$\Delta = \frac{M_AL^2}{6EI} = \frac{M_BL^2}{6EI} = \frac{R_AL^3}{12EI}$
14		$\theta_A = \frac{PL^2}{32EI},\ \Delta_C = \frac{7PL^3}{768EI}$
15		$\theta_A = \frac{ML}{4EI},\ \Delta_x = \frac{Mx}{4EIL}(L-x)^2$
16		$\theta_A = \frac{wL^3}{48EI},\ \Delta_x = \frac{wx}{48EI}\left(L^3 - 3Lx^2 + 2x^2\right)$

Appendix C

FIXED END MOMENTS

S. No	Case	Fixed end moment	
		M_A at end A	M_B at end B
1		$-\frac{Pab^2}{L^2}$	$\frac{Pa^2b}{L^2}$
2		$-\frac{wL^2}{12}$	$\frac{wL^2}{12}$
3		$-\frac{wL^2}{20}$	$\frac{wL^2}{30}$
4		$-\frac{5wL^2}{96}$	$\frac{5wL^2}{96}$
5		$-\frac{w}{12L^2}\left[e^3(4L-3e)-c^3(4L-3c)\right]$ where a+b = d, b + c = e,	$\frac{w}{12L^2}\left[d^3(4L-3d)-a^3(4L-3a)\right]$ a+b+c = L
6		$M\left[1-4\frac{a}{L}+3\left(\frac{a}{L}\right)^2\right]$	$M\frac{a}{L}\left[2-3\left(\frac{a}{L}\right)\right]$

S. No	Case	Fixed end moment M_A at end A	Fixed end moment M_B at end B
7		$(-)\frac{wa^2}{60}\left[10-10\frac{a}{L}+3\left(\frac{a}{L}\right)^2\right]$	$\frac{wa^3}{60L}\left[5-3\left(\frac{a}{L}\right)\right]$
8		$-\frac{6EI\Delta}{L^2}$	$-\frac{6EI\Delta}{L^2}$
9		$\frac{4EI\theta}{L}$	$-\frac{2EI\theta}{L}$
10		$-\frac{wL^2}{12}$	$\frac{wL^2}{12}$

Note: A negative moment is anti-clockwise, and a positive moment is clockwise

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